Study of transformations

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Abstract

This paper covers a first approach study of the angles and modulo of vectors in spaces of Hilbert considering a riemannian metric where, instead of taking the usual scalar product on space of Hilbert, this will be extended by the tensor of the geometry g. As far as I know, there is no a study covering space of Hilbert with riemannian metric. It will be shown how to get the angle and modulo on Hilbert spaces with a tensor metric, as well as vector product, symmetry and rotations. A section of variationals shows a system of differential equations for a riemennian metric.

Contents

1 Elements

On a Hilbert space, the scalar product of 2 functions is given by:

$< f, g >= \int_X f^*(x)g(x)dx$, where $f^*$is the complex conjugated of f

Now, considering a scalar product function g, let's define the scalar product of e and f by the product function g:

$< e\,|\tilde{g}|\, f >= \int_x e^*(x)\tilde{g}f(x)dx$

Having this, the angle of 2 functions is defined by:

$$cos(e,f) = \frac{< e\,|\tilde{g}|\, f >}{\|e\|\,\|f\|} = \frac{\int_x e^*(x)\tilde{g}f(x)dx}{\sqrt{\int_x e^*(x)\tilde{g}e(x)dx}\sqrt{\int_x f^*(x)\tilde{g}f(x)dx}}$$

Discrete form:

$$g_{kl}(y) = \sum \frac{\partial x^i}{\partial y^k}\frac{\partial x^i}{\partial y^l} = \sum_{i=l}^{n}\sum_{m,p}\frac{\partial x^i}{\partial z^m}\frac{\partial z^m}{\partial y^k}\frac{\partial x^i}{\partial z^p}\frac{\partial z^p}{\partial y^l} = \sum_{m,p}\frac{\partial z^m}{\partial y^k}\left(\sum_{i=l}^{n}\frac{\partial x^i}{\partial z^m}\frac{\partial x^i}{\partial z^p}\right)\frac{\partial z^p}{\partial y^l} = \sum_{m,p}\frac{\partial z^m}{\partial y^k}g_{m,l}(z)\frac{\partial z^p}{\partial y^l}$$

On continuous form:

$$g_{k,l}(y) = \int \frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}dg_{m,l}(z)$$

The "angle" between 2 functions can be defined as the scalar product of those functions:

$$cos(e,f) = \frac{\int_X\int e^*(x)\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}f(x)dg_{m,l}(z)dx}{\sqrt{\int_X\int e^*(x)\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}e(x)dg_{m,l}(z)dx}\sqrt{\int_X\int f^*(x)\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}f(x)dg_{m,l}(z)dx}}$$

Taking as particular case $\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l} = \delta_k^m\delta_l^p, \int \frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}dg_{m,l}(z) = \int \delta_k^m\delta_l^p dg_{m,l} = \mathbb{I}$, so

$$cos(e,f) = \frac{\int e^*(x)f(x)dx}{\sqrt{\int e^*(x)e(x)dx}\sqrt{\int f^*(x)f(x)dx}}$$,which matchs with the Hilbert's formula.

The modulo can be defined as:

$$\|f\|^2 = \int_X f^*(x)\tilde{g}f(x)dx = \int_X\int f^*(x)\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}f(x)dg_{m,l}(z)dx$$

Whence:

$$\|f\| = \sqrt{\|f\|^2} = \sqrt{\int_X\int f^*(x)\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}f(x)dg_{m,l}(z)dx}$$

Following a similar way, taking as particular case $\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l} = \delta_k^m\delta_l^p, \int \frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}dg_{m,l}(z) = \int \delta_k^m\delta_l^p dg_{m,l} = \mathbb{I}$, so

$$\|f\| = \sqrt{< f, f >} = \sqrt{\int f^*(x)f(x)dx}$$

The distance between 2 functions $f_1$and f_2 will be given by:

$$d = \sqrt{\|f_1 - f_2\|^2} = \sqrt{\int_X\int (f_1(x) - f_2(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_1(x) - f_2(x))\,dg_{m,l}(z)dx}$$

Let's see the Minkowski's inequality:

$$\|f_1 + f_2\|^2 = \int_X\int (f_1(x) - f_2(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_1(x) - f_2(x))\,dg_{m,l}(z)dx =$$

$$= \int_X\int (f_1(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_1(x))\,dg_{m,l}(z)dx + \int_X\int (f_2(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_2(x))\,dg_{m,l}(z)dx +$$

$$\int_X\int (f_1(x)^*f_2(x) - f_1(x)f_2(x)^*)^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}dg_{m,l}(z)dx \le \int_X\int (f_1(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_1(x))\,dg_{m,l}(z)dx$$

$$+\int_X\int (f_2(x))^*\frac{\partial z^m}{\partial y^k}\frac{\partial z^p}{\partial y^l}(f_2(x))\,dg_{m,l}(z)dx = \|f_1\|^2 + \|f_2\|^2 \le (\|f_1\| + \|f_2\|)^2$$

So,$\|f_1 + f_2\| \le \|f_1\| + \|f_2\|$

Theorem: Let g_{ij}be a metric on M^n.Then there exists a unique symmetric affine connection compatible with $g_{ij}$and such that

$T^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha} \right)$

So, let's calculate the square modulo of a function in this way:

$\|f\|^2 = \left| \left\langle f \left| \widetilde{T} \right| f \right\rangle \right| = \int_X f^*(x) \widetilde{T} f(x) = \int_X f_i(x) T^k_{i,j} f_k(x) dx^j = \frac{1}{2} \int_X f_i g^{i\alpha} \left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha} \right) f^k dx^j$

So,

$\|f\|^2 = \frac{1}{2} \int_X f_i g^{i\alpha} \left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha} \right) f^k dx^j$

Let's now consider an operator A. In order to calculate the norm of the vector associated to the operator, we will follow a similar way:

$|\langle f |A| f \rangle|^2 = \int_X f^*(x) \tilde{g}(x) A(x) f(x) dx$

Let's see some examples:

1) Polar coordinates

$d\psi = \begin{pmatrix} cos\varphi & sin\varphi \\ -rsin\varphi & rcos\varphi \end{pmatrix}^T$,so

$G(r,\varphi) = (d\psi)^T (d\psi) = \begin{pmatrix} cos\varphi & sin\varphi \\ -rsin\varphi & rcos\varphi \end{pmatrix} \begin{pmatrix} cos\varphi & -rsin\varphi \\ sin\varphi & rcos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

So,

$L(\gamma) = \int_a^b dt \sqrt{\left\langle \frac{d\gamma}{dt} |G| \frac{d\gamma}{dt} \right\rangle} = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\varphi}{dt}\right)^2} dt$

Now, taking the modulo of a function, considering the Hilbert space:

$\|f\|^2 = \int f^*(t) G f(t) dt = \displaystyle\int f^*(t) \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\varphi}{dt}\right)^2} f(t) dt$

2) Cartesian coordinates

In this case, $G(r,\varphi) = (d\psi)^T (d\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We consider that t=x, so

$\|f\|^2 = \int f^*(x) G f(x) = \displaystyle\int f^*(x) \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} f(x) dx$

As x and y are independent, $\frac{dy}{dx} = 0$, so

$\|f\|^2 = \int f^*(x) G f(x) = \int f^*(x) f(x) dx$, which matches with the usual scalar product.

1.1 Length of a curve in a curvilinear coordinate system

Let's consider an arbitrary curvilinear coordinate system in a domain γ. Denoting the curvilinear coordinates, the law of differentiation of a composite function:

$\frac{dx^i(t)}{dt} = \sum\limits_{(k)} \frac{dx^i}{dz^k} \frac{dz^k}{dt}$

The length of a curve will be given by:

$$L(\gamma) = \int_a^b \sqrt{\sum \left(\frac{dx^i}{dt}\right)^2} dt = \int_a^b \sqrt{\sum_{m,l}^{n-1} g_{m,l} \frac{dx^m}{dt} \frac{dx^l}{dt}} dt$$

Considering a density of scalar product, the length of a curve will be given by:

$$L(\gamma) = \int_a^b \sqrt{\int_X dg_{m,l} \frac{dx^m}{dt} \frac{dx^l}{dt}} dt$$

1.2 The first fundamental form

The first fundamental form of a hypersurface $V^{n-1}$is the form $ds^2\big|_V = \sum\limits_{m,p} g_{m,p} dz^m dz^p$.

The differential of space will have the following form:

$ds^2\big|_V = \sum\limits_{m,p} g_{m,p} dz^m dz^p = \sum\limits_{i=1}^{n-1} \left(dx^i\right)^2 + \sum\limits_{k,p=1}^{n-1} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^p} dx^k dx^p = \sum\limits_{k,p=1}^{n-1} \left(\delta_{k,p} + \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^p}\right) dx^k dx^p$

Considering a density $dg_{m,l}$, the differential can be written like this:

$ds^2\big|_V = \int dg_{m,p} dz^m dz^p = \displaystyle\int dg_{m,l} (dz')^2 + \int dg_{m,l} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^p} dx^k dx^p$

So,

$$ds^2\big|_V = \int dg_{m,l} \left\{ \delta_{k,p} + \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^p} \right\} dx^k dx^p = g_{k,p} dx^k dx^p$$

Let now V^{n-1}be given as an implicit function. Then, $F\left(x^1,..,x^n\right)=0$has the solution $x^n=f\left(x^1,..,x^{n-1}\right)$, with $\frac{\partial f}{\partial x^1}=-\frac{\frac{\partial F}{\partial x^1}}{\frac{\partial F}{\partial x^n}}$.Substituting $\frac{\partial f}{\partial x^\alpha}$ for f_{x^α}, we get:

$g_{k,p}=\left\{\left(\frac{\partial F}{\partial x^k}\frac{\partial F}{\partial x^p}\right)\left(\frac{\partial F}{\partial x^n}\right)^{-2}\right\}+\delta_{k.p}$

1.3 Vector product

Let's define the vector product of 2 functions like this:

$h=e\otimes f=\int_X e^*\otimes f dx=\int_X \epsilon_{ij}^k e^{*,j}(x)f_k(x)dx$, $\epsilon_{ij}^k=g^{kk'}\epsilon_{ijk'}$, where ϵ_{ijk} is the levy-civita tensor:

$$\epsilon_{ijk}=\begin{cases}0 & 2\,labels\,are\,the\,same\\ 1 & even\,pem\,1,2,3\\ -1 & odd\,pem\,1,2,3\end{cases}$$

Properties:

1) $e\otimes(f_1+f_2)=e\otimes f_1+e\otimes f_2$

Prof: $h=e\otimes(f_1+f_2)=\int_X e^*\otimes(f_1+f_2)\,dx=\int_X \epsilon_{ij}^k e^{*,j}(x)\left(f_{k1}(x)+f_{k2}(x)\right)dx=\int_X \epsilon_{ij}^k e^{*,j}(x)f_{k1}(x)dx+\int_X \epsilon_{ij}^k e^{*,j}(x)f_{k2}(x)dx=e\otimes f_1+e\otimes f_2$

2)$f\otimes f=0$

Prof: $f\otimes f=\int_X f\otimes f dx=\int_X \epsilon_{ij}^k f^{*,j}(x)f_j dx=\int_X \epsilon_{ij}^k f^{*,j}(x)\delta_j^k f^k dx=\int_X \epsilon_{ij}^k\delta_j^k f^{*,j}(x)f^k dx=\int_X \epsilon_{ik}^k f^{*,j}(x)f^k dx=0$

3) $e\otimes f=-\left(f\otimes e\right)^*$

Prof: $e\otimes f=\int_X e^*\otimes f dx=\int_X \epsilon_{ij}^k e^{*,j}(x)f_k(x)dx=-\int_X \epsilon_{ik}^j e^{*k}(x)f_j(x)dx=-\left(\int_X \epsilon_{ik}^j e^k(x)f_j^*(x)dx\right)^*=-\left(\int_X \epsilon_{ik}^j f_j^*(x)e^k(x)dx\right)^*=-\left(f\otimes e\right)^*$

1.4 Mixed Product

From the usual geometry, the mixed product of 3 vectors, is given by $P=\overrightarrow{a}\bullet\left(\overrightarrow{b}\otimes\overrightarrow{c}\right)$.Let's define the mixed product of 3 functions on Hilbert space. Having defined the vector product:

$h=b\otimes c=\int_X b^*\otimes c dx=\int_X \epsilon_{ij}^k b^{*,j}(x)c_k(x)dx$

Now, taking in consideration the scalar product formula: $<e\left|\tilde{g}\right|f>=\int_x e^*(x)\tilde{g}f(x)dx$

$<a\left|\tilde{g}\right|b\otimes c>=\int_x a^*(x)\tilde{g}\left(b\otimes c\right)dx=\int_X dx a^*(x)\tilde{g}(x)\left\{\int_X \epsilon_{ij}^k b^{*,j}(x')c_k(x')dx'\right\}=\int_X\int_{X'} a^{i*}\tilde{g}(x)\epsilon_{ij}^k b^{*,j}(x')c_k(x')dxdx'$

1.5 Definition of tangent vector

On differential geometry, the tangent vector follows this definition:

Definition: Let M be a smooth n-dimentional manifold and $P_0\in M$an arbitrary point. A tangent vector ξat the point $P_0$to the manifold satisfies the following relation for each pair of local coordinate systems:

$\xi_i^k=\sum_{l=1}^n\frac{dx_i^h}{dx_j^i}(P_0)\xi_j^l$

In order to extend this on a differential system, let's take infinitesimals on each member of the equation:

$d\xi_i^k=\frac{dx_i^h}{dx_j^i}(P_0)d\xi_j^l$

So, $\xi_i^k=\displaystyle\int_X\frac{dx_i^h}{dx_j^i}(P_0)d\xi_j^l$

In this case, the tangent will be a curve defined by ξ_i^k. This relation is the tensor law of the curve transformation.

Let's call $T_{P_0}(M)$ the set of all the tangent vectors to a manifold M at a fixed point P_0. In order to define the tangent vector we need to find its coordinates in any local coordinates.$\xi_i^k=\sum_{l=1}^n\frac{dx_i^h}{dx_j^i}(P_0)\xi_j^l$.

$\xi_i^k=\sum_{l=1}^n\frac{dx_i^h}{dx_{i_0}^i}(P_0)\xi_j^l.=\sum_{l=1}^n\frac{dx_i^h}{dx_j^s}(P_0)\sum_{s=1}^n\frac{dx_j^s}{dx_{i_0}^i}(P_0)\xi_j^l=\sum_{l=1}^n\left(\sum_{s=1}^n\frac{dx_i^h}{dx_j^s}(P_0)\frac{dx_j^s}{dx_{i_0}^i}(P_0)\right)\xi_j^l$

When considering a differential:

$d\xi_i^k=\sum_{l=1}^n\left(\sum_{s=1}^n\frac{dx_i^h}{dx_j^s}(P_0)\frac{dx_j^s}{dx_{i_0}^i}(P_0)\right)d\xi_j^l$

So, taking a density (in order to consider functions):

$df_k=d\xi_i^k=\sum_{l=1}^n\left(\sum_{s=1}^n\frac{dx_i^h}{dx_j^s}(P_0)\frac{dx_j^s}{dx_{i_0}^i}(P_0)\right)d\xi_j^l$

$\xi_i^k=\displaystyle\int_X\frac{dx_i^h}{dx_j^s}(P_0)\frac{dx_j^s}{dx_{i_0}^i}(P_0)d\xi_j^l$

As the space $T_{P_0}(M)$ can be identified with the vector space $\mathbb{R}^n$, it can be associated with a linear space.The tensor law of coordinate transformation can be used in order to identify arithmetic spaces of the coordinates of tangent vector in any local coordinate system:

$$\begin{pmatrix} \xi_i^1 \\ \\ \\ \xi_i^n \end{pmatrix} = \begin{pmatrix} \frac{dx_i^1}{dx_j^1} & & \frac{dx_i^1}{dx_j^n} \\ & & \\ & & \\ \frac{dx_i^n}{dx_j^1} & & \frac{dx_i^n}{dx_j^n} \end{pmatrix} \begin{pmatrix} \xi_j^1 \\ \\ \\ \xi_j^n \end{pmatrix}$$

On a Hilbert space, this will be defined according to the formula $\xi_i^k = \int_X \frac{dx_i^h}{dx_j^s}(P_0)\frac{dx_j^s}{dx_{i_0}^i}(P_0)d\xi_j^l$, where the coordinates have been mapped by a density.

1.6 External differential

Differential calculus of exterior differential form can be calculated by a gradient of an exterior differential form. In the local coordinate system $\{x^1, ..., x^n\}$the differential form will have the components $\{\omega_1, ...\omega_k\}$. The gradient will be:

$(d\omega)_{j_1 ... j_{k+1}} = \sum\limits_\sigma (-1)^{|\sigma|} \nabla_{\sigma(j_{k'1})} \omega_\sigma = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\partial \omega_\sigma}{\partial x^\sigma} - \sum\limits_\sigma \sum\limits_s (-1)^{|\sigma|} T^\alpha_{\sigma(j_{k'1})\sigma(j_s) ... \sigma(j_k)}$

The second term vanishes because, for fixed s and α, exists 2 permutations of indices $j_1 ... j_{k+1}\sigma$ and σ' such that $\sigma(j_i) = \sigma'(j_i)$. Also, as the Christoffel symbols are symmetric in the lower indices, the permutations σ and σ' are canceled. So:

$(d\omega)_{j_1 ... j_{k+1}} = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\partial \omega_\sigma}{\partial x^\sigma}$

Considering Hilbert spaces with the scalar product. Let $\widetilde{g}$ the orthonormal base. So, the components of x_i and ω_i will be given by:

$x_i = \int \widetilde{g}(x) f(x) dx$ and

$\omega_i = \int \widetilde{g}(x)\omega(x) dx$

Putting in differential form:

$dx_\sigma = \widetilde{g}(x) f(x) dx$

$d\omega_i = \widetilde{g}(x)\omega(x) dx$

'So,

$(d\omega)_{j_1 ... j_{k+1}} = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\partial \omega_\sigma}{\partial x^\sigma} = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\widetilde{g}(x)\omega(x)dx}{\widetilde{g}(x)f(x)dx} = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\omega(x)}{f(x)}$

Let's calculate the differential of a product:

$d(\omega_1 \wedge \omega_2) = \sum\limits_\sigma (-1)^{|\sigma|} \nabla_\sigma (\omega_1 \wedge \omega_2)_K = \sum\limits_\sigma (-1)^{|\sigma(i)|} \sum\limits_I (-1)^{|\sigma(I)|} \frac{\partial \omega_\sigma}{\partial x^i} = \sum\limits_K (-1)^{|\sigma(I)|} \frac{\partial(\omega_{1},I\omega_{2},J)}{\partial x^i}$

$= \sum\limits_{K = I\bigcup J \bigcup\{i\}} (-1)^{|\sigma(I)|} \frac{\partial(\omega_{1},,I)}{\partial x^i} \omega_{2,J} + \sum\limits_{K = I\bigcup J \bigcup\{i\}} (-1)^{|\sigma(I)|} \omega_{1,I} \frac{\partial(\omega_2,J)}{\partial x^i} = (d\omega_1 \wedge \omega_2)_K + (-1)^{deg\omega_2} (\omega_1 \wedge d\omega_2)_K$

As $(d\omega)_{j_1 ... j_{k+1}} = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\omega(x)}{f(x)}$and $\omega_i = \int \widetilde{g}(x)\omega(x)dx$,

$d(\omega_1 \wedge \omega_2)_K = \sum\limits_\sigma (-1)^{|\sigma|} \frac{\omega_1(x)}{f(x)} \wedge \int \widetilde{g}(x)\omega_2(x)dx + (-1)^{deg\omega_2} \left(\left(\int \widetilde{g}(x)\omega(x)dx\right) \wedge \sum\limits_\sigma (-1)^{|\sigma|} \frac{\omega(x)}{f(x)} \right)_K$

2 Symmetry

Let's see how can we define a symmetry of a function on a Hilbert space. Given a line, the equation is given by the function y=ax. Let's call $tg\alpha = a$, so,on a Euclidean metric, the symmetry is calculated by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = G_\alpha S' G_{-\alpha} P = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} cos\alpha & -sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cos2\alpha & -sin2\alpha \\ sin2\alpha & -cos2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}$$

But, when considering a general Hilbert space, a matrix cannot be used. In order to find a symmetric function, let's use the Hilbert formula:

$cos(f, f_G) = \frac{\int f^*(x) f_G(x) dx}{\sqrt{\int f^*(x) f(x) dx} \sqrt{\int f_G^*(x) f_G(x) dx}}$,

The symmetry will be $f * f_G = |f|\,|f_G|\, cos(2\alpha)$

In order to simplify, let's consider that the functions f and $f_G$are on a orthonormal base:

$\overrightarrow{f} = \sum f_i \overrightarrow{e}_i$

$\overrightarrow{f_G} = \sum f_{G,i} \overrightarrow{e}_i$, where $e_i e_j = \delta^i_j$, so $\sqrt{\int f_G^*(x) f_G(x) dx} = \sqrt{\int f^*(x) f(x) dx} = 1$

So,

$cos(f, f_G) = \sum\limits_i \int f_i f_{G_i} dx$

Example: Let's f(x)=x. Let's calculate the symmetric with the scalar product $< f, g >= \int_0^1 f^* g dx$ versus the function g . Let's consider $g(x) = x^2$

$cos(f, g) = \frac{<f,g>}{\|f\| \|g\|} = \frac{\int_0^1 x * x^2 dx}{\sqrt{\int_0^1 x^2 dx} \sqrt{\int_0^1 x^4 dx}} = \frac{\sqrt{15}}{4}$

$cos(f, f_G) = \frac{<f, f_G>}{\|f\| \|f_G\|} = cos(2 < f, g >)$

Let's consider $f_G = Ax^\beta$

$\|f_G\|^2 = \int_0^1 A^2 x^{2\beta} dx = \frac{A^2}{2\beta+1}$

$< f, f_G >= \int_0^1 Ax^{\beta+1} dx = \frac{A}{\beta+2}$

$\|f\|^2 = \int_0^1 x^2 dx = \frac{1}{3}$

$cos(f, f_G) = \frac{<f, f_G>}{\|f\| \|f_G\|} = cos\left(2\frac{\sqrt{15}}{4}\right) = cos\left(\frac{\sqrt{15}}{2}\right)$

So, $< f, f_G >= \|f\|\, \|f_G\|\, cos\left(\frac{\sqrt{15}}{2}\right)$

$\frac{A}{\beta+2} = \sqrt{\frac{1}{3}} \frac{A}{\sqrt{2\beta+1}} cos\left(\frac{\sqrt{15}}{2}\right)$

$\frac{1}{\beta+2} = \sqrt{\frac{1}{3}} \frac{1}{\sqrt{2\beta+1}} cos\left(\frac{\sqrt{15}}{2}\right) = \frac{K}{\sqrt{2\beta+1}}$. so $\beta_+$and β_-will be the roots of the equation $2\beta + 1 = K^2 (\beta + 2)^2$

Solving the equation:

$K^2\beta^2 + (4K^2 - 2)\beta + 4K^2 - 1 = 0$,

$\beta = \frac{1-2K^2 \pm \sqrt{1-3K^2}}{K^2}$

So, getting the root β_+, in order to normalize, $\|f_G\|^2 = \int_0^1 A^2 x^{2\beta} dx = \frac{A^2}{2\beta+1} = 1$, so $A = \sqrt{2\beta_+ + 1}$.

$f_G = \sqrt{2\beta_+ + 1} x^{\beta_+}$

3 Rotations

Considering a Euclidean two-dimensional plane, the condition that the metric $dx^2 + dy^2, g_{ij} = \delta_{ij}$ is invariant can be written as $E = AA^T$, where A is a linear transformation. In this case, the orthogonal groups will be defined by the matrices:

* $\begin{pmatrix} cos\varphi & sin\varphi \\ -sin\varphi & cos\varphi \end{pmatrix}$(proper rotations)
* $\begin{pmatrix} cos\varphi & sin\varphi \\ sin\varphi & -cos\varphi \end{pmatrix}$(reflections)

Let's consider indefinite metrics. In this case, let's consider the metric $-dx^2 + dy^2$, which transforms a 2-dimensional space into a pseudo-Euclidean plane. The matrix will be $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. As the rotations are orthogonal transformation, $B = ABA^T$. Let's find the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -a & -c \\ b & d \end{pmatrix} = \begin{pmatrix} -a^2+b^2 & -ac+bd \\ -ac+bd & -c^2+d^2 \end{pmatrix}$$

Where ac=bd. Solving the equations:

$$\begin{cases} -a^2 + b^2 = -1 \\ -c^2 + d^2 = 1 \\ ac = bd \end{cases}$$

$a = d = cosh\alpha$
$b = c = sinh\alpha$

So the matrix will be like this: $(G_{H_\alpha}) = \begin{pmatrix} \pm cosh\alpha & \pm sinh\alpha \\ \pm sinh\alpha & \pm cosh\alpha \end{pmatrix}$,with the metric $-dx^2 + dy^2$. The combinations matching the relation $|G_{H_\alpha}| = \pm 1$(rotations or reflections) are:

$$\left\{\begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} - & - \\ - & - \end{pmatrix}, \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \begin{pmatrix} - & + \\ - & + \end{pmatrix}\right\}$$

On a hyperbolic plane:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = G_{H_\alpha} S G_{H_{-\alpha}} P = \begin{pmatrix} cosh\alpha & sinh\alpha \\ sinh\alpha & cosh\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} cosh\alpha & -sinh\alpha \\ -sinh\alpha & cosh\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cosh(2\alpha) & -sinh(2\alpha) \\ -sinh(2\alpha) & cosh(2\alpha) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} cosh(2\alpha) & -sinh(2\alpha) \\ -sinh(2\alpha) & cosh(2\alpha) \end{vmatrix} = -1$$

Let's see if there a way to find the angle of 2 functions on a hyperbolic metric. In this case, let's take the previous formula:

$$cos(f,g) = \frac{\int f^*(x)g(x)dx}{\sqrt{\int f^*(x)f(x)dx}\sqrt{\int g^*(x)g(x)dx}},$$

In order to find a formula for cosh, let's add a weight function in order to define the cosh as precedent:

$$cosh(f,g) = \frac{\int f^*(x)g(x)G_{H_\alpha}dx}{\sqrt{\int f^*(x)f(x)G_{H_\alpha}dx}\sqrt{\int g^*(x)g(x)G_{H_\alpha}dx}},$$

The weight function G_{H_α} will define the transformation of Cartesian coordinates to hyperbolic. Let's calculate the first term:

$$\int f^*(x)g(x)G_{H_\alpha}dx = \int f^*(x)g(x)\frac{\partial(u,v)}{\partial(x,y)}dx$$

On hyperbolic coordinates:

$x = ve^u$ and $y = ve^{-u}$, so $u = ln\sqrt{\frac{x}{y}}$ and $v = \sqrt{xy}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2x} & -\frac{1}{2y} \\ \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} \end{vmatrix} = \frac{1}{4}\left(\frac{1}{x}\sqrt{\frac{x}{y}} + \frac{1}{y}\sqrt{\frac{y}{x}}\right) = \frac{1}{2\sqrt{xy}}$$

$$cosh(f,g) = \int f^*(u)g(u)\frac{\partial(u,v)}{\partial(x,y)}dx = \int f^*(x)g(x)\frac{dx}{2\sqrt{xy}}$$

Now, taking the change: $x = Rcosh\alpha$and $y = Rsinh\alpha$, as it's considered the variable x (single variable), let's fix R, $x = x(\alpha)$and $y = y(\alpha)$, so $dx = Rsinh\alpha d\alpha$

$$\int f^*(x)g(x)G_{H_\alpha}dx = \int f^*(\alpha)g(\alpha)\frac{Rsinh(\alpha)dx}{2R\sqrt{cosh(\alpha)sinh(\alpha)}} = \frac{1}{2}\int f^*(\alpha)g(\alpha)\sqrt{tgh(\alpha)}d\alpha$$

Following the same procedure, the cosh for those functions is given by:

$$cosh(f,g) = \frac{\int f^*(\alpha)g(\alpha)\sqrt{tgh\,(\alpha)}d\alpha}{\sqrt{\int f^*(\alpha)f(\alpha)\sqrt{tgh\,(\alpha)}d\alpha}\sqrt{\int g^*(\alpha)g(\alpha)\sqrt{tgh\,(\alpha)}d\alpha}}$$

4 Variationals

In this section it will be shown a method to find the extremal (stationary) functions for a functional J when considering a g depending on coordinates. For a Riemann manifold a geodesic is defined as a trajectory where the translation preserves the velocity field of the trajectory.

The functional$\mathcal{L}$of the length of the trajectory $\gamma(t)$is given by:

$$\mathcal{L} = \int_0^1 \sqrt{g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}}dt$$

Let's call the Lagrangian $\mathcal{L} = \mathcal{L}(g_{ij}, ..., x_i, .., t)$, where the components of the tensor g_{ij}will be functions of x_i. Following the Lagrange formulation for continuous systems, and applying the Hamilton's principle, let's see how to find the differential equations:

$\delta I = \delta \int \mathcal{L} dx = 0$

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2} dx \left\{ \sum_{i,j} \frac{\partial \mathcal{L}}{\partial g_{i,j}} \frac{\partial g_{i,j}}{\partial \alpha} + \sum_{i,j} \frac{\partial \mathcal{L}}{\partial x^i} \frac{\partial x^i}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \frac{\partial}{\partial \alpha} \left(\frac{\partial g_{i,j}}{\partial x_i}\right) \right\}$$

$$\frac{dI}{d\alpha} = \sum_{i,j} \int_{x_1}^{x_2} dx \left\{ \frac{\partial \mathcal{L}}{\partial g_{i,j}} \frac{\partial g_{i,j}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial x^i} \frac{\partial x^i}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \frac{\partial}{\partial \alpha} \left(\frac{\partial g_{i,j}}{\partial x_i}\right) \right\} = 0$$

Integrating by parts:

$$\int_{x_1}^{x_2} dx \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \frac{\partial}{\partial \alpha} \left(\frac{\partial g_{i,j}}{\partial x_i}\right) = - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \right) \frac{\partial g_{i,j}}{\partial \alpha} dx$$

So,

$$\frac{dI}{d\alpha} = \sum_{i,j} \int_{x_1}^{x_2} dx \left\{ \frac{\partial \mathcal{L}}{\partial g_{i,j}} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \right) \right\} \frac{\partial g_{i,j}}{\partial \alpha} = 0$$

As it must be 0 for any choose of $x_1$and x_2, the equation of extremals of the function $\mathcal{L}$ considering the functions g_{ij} will be given by:

$$\frac{\partial \mathcal{L}}{\partial g_{i,j}} - \frac{d}{dx_i} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial g_{i,j}}{\partial x_i}\right)} \right) = 0$$

The above system of differential equations is called the Euler's equations for a differential.

5 Bibliography

References

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