Measuring Fuzziness of Z-numbers and Its Application in Sensor Data Fusion

Yangxue Li[a], Yong Deng[a,*]

[a]*Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China*

Abstract

Real-world information is often characterized by fuzziness due to the uncertainty. Z-numbers is an ordered pair of fuzzy numbers and is widely used as a flexible and efficient model to deal with the fuzziness information. This paper extends the fuzziness measure to continuous fuzzy number. Then, a new fuzziness measure of discrete Z-numbers and continuous Z-numbers is proposed: simple addition of fuzziness measures of two fuzzy numbers of a Z-number. It can be used to obtain a fused Z-number with the best information quality in sensor fusion applications based on Z-numbers. Some numerical examples and the application in sensor fusion are illustrated to show the efficiency of the proposed fuzziness measure of Z-numbers.

Keywords: fuzziness measure, information quality, Z-numbers, fuzzy sets, sensor data fusion.

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1. Introduction

The information of real world is imperfect, often with fuzziness and part of reliability. There are many methods to model real-world information, such as probability theory (Feller 2008), Dempster-Shafer evidence theory (Dempster 1967; Shafer et al 1976; Xu and Deng 2018; Zheng and Deng 2017; Liu et al 2017a; Deng and Deng 2018; Li et al 2017a; Liu

*Corresponding author, Yong Deng, Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China

Email address: `dengentropy@uestc.edu.cn` (Yong Deng)

et al 2017c; Gong et al 2017; Yager et al 2017; Bi et al 2017; Ye et al 2016; Zhao et al 2016; Ma et al 2016; Xiao 2017b; Li et al 2016; Xiao 2017a), fuzzy sets (Zadeh 1996; Yager 2014; Zhang et al 2017a; Collotta et al 2017; Liu et al 2018; Yager 2016a; Fei et al 2017), rough sets (Pawlak et al 1995), D numbers (Deng 2012; Xiao 2016; Bian et al 2018; Chatterjee et al 2018) and so on (Zhang et al 2018, 2017b). But any estimation of information, be it precise or fuzzy, depends on the degree of trust in the source of information (Li and Mahadevan 2016b; Zhang et al 2017c; Yuan et al 2016; Zhang et al 2017d; Meng et al 2016; Liu et al 2017b; Zhang and Mahadevan 2017b; Huynh et al 2006; Fu et al 2015; Li and Mahadevan 2016a; Song et al 2015; Zhang and Mahadevan 2017a; Yin and Deng 2018; Sabahi 2016; Zheng and Deng 2018). In order to take this fact into account, the concept of Z-number was introduced by Zadeh (Zadeh 2011) to describe not only the fuzziness but also partial reliability of real-world information. A Z-number is an ordered pair $Z = (A, B)$ of fuzzy numbers, where A is an inexact constraint on values of X and B is an inexact estimation of reliability of A and is considered to ba a probability measure of A (Zadeh 2011).

Z-numbers play an important role in many fields because of their strong capability to model the incomplete and partial reliable information (Kang et al 2018; Aliev 2017; Khan et al 2017; Aliev and Salimov 2017a; Huang et al 2017; Banerjee and Pal 2017; Aliev et al 2016; Yaakob and Gegov 2016; Banerjee and Pal 2015; Aliev et al 2015). For example, many authors have studied sensor data fusion system (Kamath et al 2017; Li et al 2017b; Sulistyo et al 2017; Gravina et al 2017; Ng et al 2017; Li et al 2018). The information of sensors usually is uncertain, random, fuzzy and partial reliable in sensor data fusion system. So the Z-number can be used to model the fuzziness and reliability of the sensor data (Jiang et al 2016). Fusing Z-numbers provided by multi-sensors can improve the quality of the information to decision making (Elmore et al 2014). In probability theory, the entropy is used to measure the uncertainty associated with the probability distribution, The greater the value of entropy the greater the uncertainty (**??**). The smaller the value of entropy the more information conveyed by a probability distribution. Yager used the

Gini entropy to measure uncertainty of probability distribution and obtain high quality fused results from multiple sources of probability distribution (Yager and Petry 2016). Different entropy is proposed to measure the uncertainty of different math tools for model uncertain information (Jiang and Wang 2017; Song et al 2016; Deng 2016; Abelln 2017).

Similarly, the more information conveyed by a Z-number the smaller the uncertainty. Before introducing the fuzziness measure of a Z-number, reviewing the measures of fuzziness for discrete fuzzy sets. A fuzzy set is characterized by a membership function which assigns a grade of membership between zero and one (Zadeh 1996) for each object. The uncertainty of the crisp set is minimum because the crisp set is certain. When the membership functions of all elements is 0.5, the uncertainty of this fuzzy set is maximum. The first attempt to quantify the uncertainty of a fuzzy set have been made by Zadeh (Zadeh 1968). A entropy incorporates both probability and fuzzy uncertainties have been defined. The definition of entropy of fuzzy sets without reference to probabilities was proposed by Deluca and Termini (Luca and Termini 1972). They defined the fuzziness measure using Shannon's functional. Ebanks (Ebanks 1983) lists the properties which be required of a measure of fuzziness. There are many approaches to measure the uncertainty of a fuzzy set are presented (Yager 2016b,a). This paper will introduce serval well known and frequently-used fuzziness measures.

Fuzzy sets are divided into discrete fuzzy sets and continuous fuzzy sets according to whether their universes are continuous or not. The fuzziness measures of discrete fuzzy sets are proposed. We extend the formulas and usability of these fuzziness measures in continuous fuzzy sets. Based on measures uncertainty of fuzzy sets, this paper proposes a fuzziness measure of a Z-number: simple addition of fuzziness measure of A and fuzziness measure of B. Then we use this method to find a fused Z-number with the best information quality from multiple sensors data.

The paper is organized as follows. The preliminaries of fuzzy sets, fuzzy numbers, some existing measures of uncertainty of discrete fuzzy sets and Z-numbers are briefly introduced in Section 2. In Section 3, the formulas and usability of different fuzziness measures are considered. The definition of a uncertainty measure of a Z-number is proposed in Section 4. In Section 5, we used fuzziness of a Z-number to measure the fusing methods. Finally, this paper is concluded in Section 6.

2. Preliminaries

In this section, some preliminaries including fuzzy sets, fuzzy numbers, some existing measures of uncertainty of discrete fuzzy sets and Z-numbers are briefly introduced.

2.1. Fuzzy Sets And Fuzzy Number

Some basic definitions of fuzzy sets and fuzzy number are briefly introduced.

Definition 1. Suppose X be a classical set of objects, whose generic elements are denoted x. The degree of x in a classical subset A of X is often viewed as a characteristic function μ_A from X to the real interval $[0, 1]$. Then the A is called a fuzzy set (Zadeh 1996),

$\mu_A(x)$ is the degree of membership of x in A $\mu_A : X \to [0, 1]$. The closer the value of $\mu_A(x)$ is to 1, the greater the degree of x belongs to A. $\mu : X \to [0, 1]$ is referred to as a membership function (Zadeh 1996).

Definition 2. The support of a fuzzy set A is the subset of X, it has nonzero membership function in A (Zadeh 1996):

$$supp(A) = A^{+0} = \{c \in X, \mu_A(x) > 0\}. \tag{1}$$

Definition 3. The (crisp) set of elements that belongs to the fuzzy set A at least to the degree α is called the α-level set (Zadeh 1996):

$$A^{\alpha} = \{c \in X, \mu_A(X) \geq \alpha\}. \tag{2}$$

Definition 4. A fuzzy set A is convex if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) > min(\mu_A(x_1), \mu_A(x_2)) \tag{3}$$

for all $x_1, x_2 \in R, \lambda \in [0,1]$, min denotes the minimum operator (Zadeh 1996).

Definition 5. A fuzzy number is a fuzzy set A on R which possesses the following properties: a) A is a normal fuzzy set; b) A is a convex fuzzy set; c) α-cut of A, A^α is a closed interval for every $\alpha \in (0,1]$; d) the support of A, $supp(A)$ is bounded (Zadeh 1996; Dubois and Prade 1978).

2.2. Measures of Fuzziness For Discrete Fuzzy Sets

Let x be a discrete random variable that takes value in $X = \{x_1, x_2, \cdots, x_n\}$. The set of all fuzzy subsets of $X = \{x_1, x_2, \cdots, x_n\}$ is denoted by $P(X)$.

Definition 6. The fuzziness measure of a discrete fuzzy set is a mapping $H : P(X) \to R^+$. Ebanks (Ebanks 1983) lists the properties of a measure of fuzziness to be satisfied.

$$\begin{aligned}
Sharpness\ P1: &\ H(A) = 0 \Leftrightarrow \mu_A(x) = 0\ or\ 1\ \forall x \in X; \\
Maximality\ P2: &\ H(A)\ is\ maximum \Leftrightarrow \mu_A(x) = 0.5\ \forall x \in X; \\
Resolution\ P3: &\ H(A) \geq H(A^*),\ where\ A^*\ is\ a\ sharpented\ version\ of\ A. \\
Symmetry\ P4: &\ H(A) = H(1-A),\ where\ \mu_{1-A}(x) = 1 - \mu_A(x)\ \forall x \in X.
\end{aligned}$$

Ebanks also presented the fifth and sixth requirement: $Valuation$ and $generalizedadditivity$, but the above four requires have been widely accepted and recognized. Some common measures of fuzziness for a discrete fuzzy set are introduced as follows.

For $A \in P(X)$ denotes A^{near} is the crisp set nearest to A, A^{far} is the crisp set farthest to A:

$$A^{near}(x) = \begin{cases} 1\ if\ \mu_A(x) \geq 0.5 \\ 0\ otherwise. \end{cases} \tag{4}$$

$$A^{far}(x) = \begin{cases} 0 \; if \; \mu_A(x) \geq 0.5 \\ 1 \; otherwise. \end{cases} \tag{5}$$

An index of fuzziness for $A \in P(X)$ was presented by Kaufmann (Kaufmann 1975),

$$H_{Ka}(A) = \frac{2 \times d(A, A^{near})}{n^{\frac{1}{q}}}, \tag{6}$$

where $q \in [1, \infty)$, d is a distance on $P(X) \times P(X)$:

$$d(A, A^{near}) = [\sum_{i=1}^{n} |u(x_i) - u_{A^{near}(x_i)}|^q]^{\frac{1}{q}}. \tag{7}$$

H_{Ka} is called the linear index of fuzziness when $q = 1$, and quadratic index of fuzziness when $q = 2$.

Yager (YAGER 1979; Yager 1980) also used the definition of distance d to define a new measure of fuzziness:

$$H_Y(q, A) = (d^q(Y, Y^c) - d^q(A, A^c))/d^q(Y, Y^c), \tag{8}$$

where Y is an arbitrary crisp subset of X, Y^c is the complement of Y defined by Zadeh, $\mu_{Y^c}(x) = 1 - \mu_Y(x)$. $d^q(Y, Y^c)$ is the maximum distance between any pair of sets in $P(X) \times P(X)$.

Kosko (Kosko 1986) defined a fuzziness measure as the ratio of the distance between the fuzzy set A and A^{near} to the distance between A and A^{far}, obviously $A^{far} = (A^{near})^c$.

$$H_{KoE}(q, A) = d^q(A, A^{near})/d^q(A, A^{far}), \tag{9}$$

where d^q is specified in Eq. (7).

2.3. *Z-number*

The concept of Z-number is related to the reliability of information.

Definition 7. A Z-number, Z, is an ordered pair of fuzzy numbers, $Z = (A, B)$. The first component, A, is a restriction (constraint) on the values of the real-world uncertain variable, X, is allowed to take. The second component, B, is a measure of reliability (certainty) of the first component (Zadeh 2011).

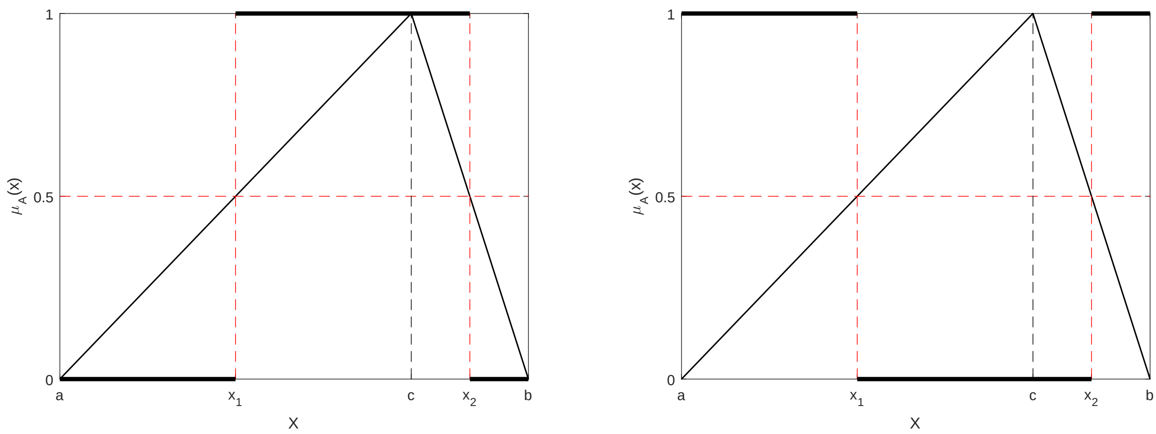

Figure 1: A^{near}(left) and A^{far}(right) of triangular Continuous fuzzy set A

3. Measures of Fuzziness For Continuous Fuzzy Sets

If X is continuous, thus all elements of $P(X)$ and $A \in P(X)$ is continuous. Take a triangular continuous fuzzy set A as an example. The crisp set nearest A^{near} and the crisp set farthest A^{far} of A are expressed the thick lines in Fig. 1. It can be seen the A^{near} and A^{far} are constant continuous functions:

$$A^{near}(x) = \begin{cases} 0, & x < x_1 \; and \; x > x_2 \\ 1, & x_1 \leq x \leq x_2 \end{cases} \quad A^{far}(x) = \begin{cases} 1, & x < x_1 \; and \; x > x_2 \\ 0, & x_1 \leq x \leq x_2 \end{cases}$$

Then the distance between A and A^c, $d^q(A, A^c)$ can given as follows. For convenience, let $q = 1$ in the following paragraphs.

$$\begin{aligned} d(A, A^c) &= \sum_{i=1}^{n} |\mu_A(x_i) - \mu_{A^c}(x_i)| \\ &= \sum_{i=1}^{n} |\mu_A(x_i) - (1 - \mu_A(x_i))| \\ &= \sum_{i=1}^{n} |2\mu_A(x_i) - 1|. \end{aligned}$$

If X is continuous and $X = [a, b], b \geq a$, then

$$d(A, A^c) = \int_a^b |2\mu(x) - 1| dx. \tag{10}$$

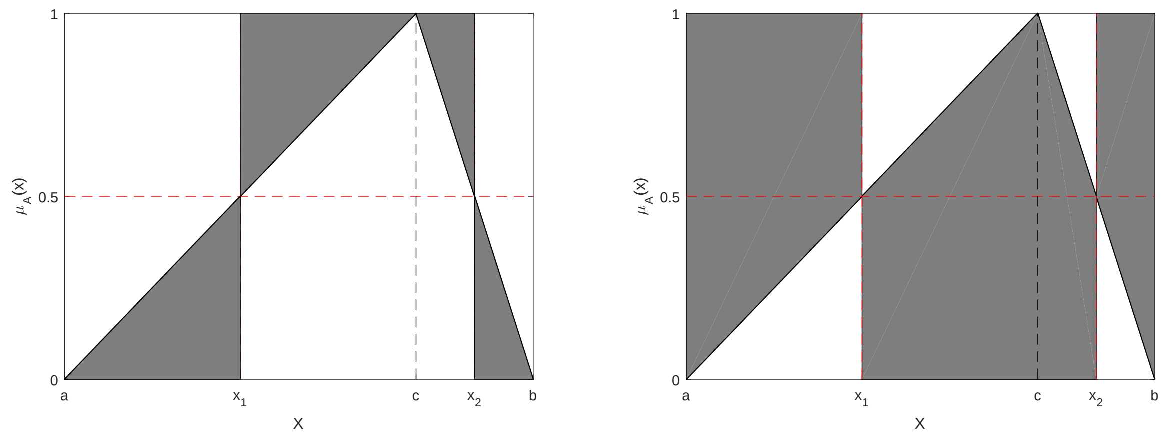

Figure 2: $d(A, A^{near})$(left) and $d(A, A^{far})$(right) of triangular Continuous fuzzy set A

The number of variables, n, is infinitely-great, so the fuzziness measure of Kaufmann is not suited to continuous fuzzy sets. In Yager's fuzziness measure (YAGER 1979; Yager 1980), $d(Y, Y^c) = \int_a^b 1dx = b - a$ is the maximum distance between any pair of sets in $P(X) \times P(X)$. So the fuzziness measure of Yager can be expressed as follows in continuous fuzzy sets.

$$\begin{aligned} H_Y(A) &= (d(Y, Y^c) - d(A, A^c))/d(Y, Y^c) \\ &= 1 - \frac{d(A, A^c)}{d(Y, Y^c)} \\ &= 1 - \frac{\int_a^b |2\mu_A(x) - 1|}{b - a} dx \end{aligned} \tag{11}$$

Using the triangular continuous fuzzy set A of Fig. 1, the distance between A and A^{near}, A and a^{far} are expressed as dashed area of Fig. 2. With the knowledge of integral, the formulas are

$$d(A, A^{near}) = \int_a^{x_1} \mu_A(x)dx + \int_{x_2}^{b} \mu_A(x)dx + \int_{x_1}^{x_2} (1 - \mu_A(x))dx \tag{12}$$

$$d(A, A^{far}) = \int_a^{x_1} (1 - \mu_A(x))dx + \int_{x_2}^{b} (1 - \mu_A(x))dx + \int_{x_1}^{x_2} \mu_A(x)dx \tag{13}$$

Obviously, $d(A, A^{far}) = b - a - d(A, A^{near})$.

The fuzziness measure of Kosko of continuous fuzzy sets can be given using Eq. (12)

and Eq. (13).

$$H_{KoE}(A) = \frac{d(A, A^{near})}{d(A, A^{far})} = \frac{\int_a^{x_1} \mu_A(x)dx + \int_{x_2}^{b} \mu_A(x)dx + \int_{x_1}^{x_2}(1-\mu_A(x))dx}{\int_a^{x_1}(1-\mu_A(x))dx + \int_{x_2}^{b}(1-\mu_A(x))dx + \int_{x_1}^{x_2} \mu_A(x)dx} \tag{14}$$

Example 3.1. *If A is the triangular continuous fuzzy set defined by*

$$\mu_A(x) = \begin{cases} \frac{x+2}{2}, & -2 \le x \le 0 \\ \frac{3-X}{3}, & 0 \le x \le 3 \\ 0, & otherwise. \end{cases}$$

Graphical description of A on the universe X is shown in Fig. 3. First step, calculate the points $P_1(x_1, 0.5)$ *and* $P_2(x_2.0.5)$ *where* $\mu_A(x)$ *cuts* $\mu_A(x) = 0.5$ *to get* $x_1 = -1, x_2 = 1.5$. *Then compute the fuzziness measure of A.*

$$\begin{aligned} H_Y(A) &= 1 - \frac{\int_a^b |2\mu_A(x) - 1|dx}{b-a} \\ &= 1 - \frac{\int_{-2}^{-1}(1 - 2\frac{x+2}{2})dx + \int_{1.5}^{3}(1 - 2\frac{3-x}{3}))dx + \int_{-1}^{0}(2\frac{x+2}{2} - 1)dx + \int_0^{1.5}(2\frac{3-x}{3} - 1)dx}{3-(-2)} \\ &= 1 - \frac{3.25}{5} = 0.35 \end{aligned}$$

$$\begin{aligned} H_{KoE}(A) &= \frac{d(A, A^{near})}{d(A, A^{far})} \\ &= \frac{\int_a^{x_1} \mu_A(x)dx + \int_{x_2}^{b} \mu_A(x)dx + \int_{x_1}^{x_2}(1-\mu_A(x))dx}{\int_a^{x_1}(1-\mu_A(x))dx + \int_{x_2}^{b}(1-\mu_A(x))dx + \int_{x_1}^{x_2} \mu_A(x)dx} \\ &= \frac{1.25}{5-1.25} = 0.33 \end{aligned}$$

4. Fuzziness Measure for Z-numbers

In this section, the definition of fuzziness measure of a Z-number is presented, and fuzziness measures for Z-numbers are proposed based on the fuzziness measures for fuzzy sets in Section 2 and Section 3. Some examples including discrete Z-numbers and continuous Z-numbers are used to show the efficiency of proposed method.

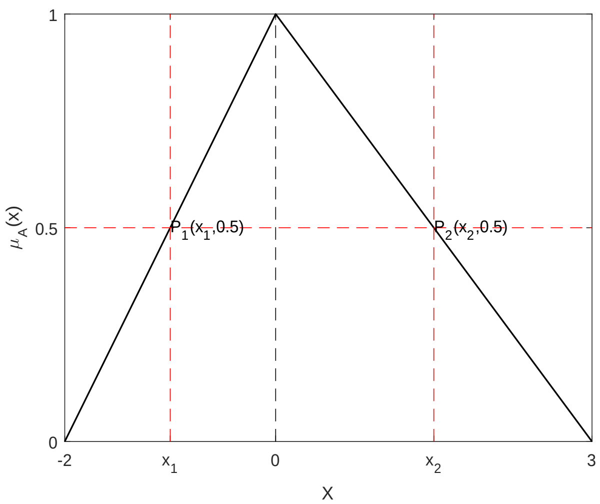

Figure 3: Graphical description of A in example 3.1

4.1. Definition

Denoted by $\mathcal{D}$ the space of fuzzy sets of $\mathcal{R}$. Denoted by $\mathcal{D}_{[a,b]}$ the space of fuzzy set of $[a,b] \subset \mathcal{R}$. Denoted $\mathcal{Z}$ the space of Z-number:

$$\mathcal{Z} = \{Z = (A,B) | A \in \mathcal{D}, B \in \mathcal{D}_{[0,1]}\}.$$

For a Z-number $Z = (A,B)$, denoted X_A is the universe of A, X_B is the universe of B.

Definition 8. A measure of fuzziness for a Z-number $Z = (A,B) \in \mathcal{Z}$ is a mapping $H_Z : \mathcal{Z} \to R^+$. The properties of a measure of fuzziness to be satisfied as follows.

$$Sharpness\ G1 : H_Z(Z) = 0 \Leftrightarrow \mu_A(x_A) = 0\ or\ 1\ and\ \mu_B(x_B) = 0\ or\ 1\ \forall x_A \in X_A,\ \forall x_B \in X_B;$$

$$Maximality\ G2 : H_Z(Z)\ is\ maximum \Leftrightarrow \mu_A(x_A) = 0.5\ and\ \mu_B(x_B) = 0.5\ \forall x_A \in X_A,\ \forall x_B \in X_B;$$

$$Resolution\ G3 : H_Z(Z) \geq H(Z^*),\ where\ Z^*\ is\ a\ sharpened\ version\ of\ Z.$$

$$Symmetry\ G4 : H_Z(Z) = H_Z(Z(1-A, 1-B)),\ where\ \mu_{1-A}(x_A) = 1 - \mu_A(x_A)\ and$$
$$\mu_{1-B}(x_B) = 1 - \mu_B(x_B)\ \forall x_A \in X_A,\ \forall x_B \in X_B;$$

A method of uncertainty measure for Z-numbers is $H_Z(Z) = H(A) + H(B)$, which $H(A)$ is the degree of fuzziness of A. Obviously, this method satisfy the above requires.

Proof. Assume the fuzziness measure, H, satisfy $P1 - P4$. For G1,

$$\mu_A(x_A) = 0 \; or \; 1 \; and \; \mu_B(x_B) = 0 \; or \; 1 \; \forall x_A \in X_A, \; \forall x_B \in X_B,$$

so, $H(A) = 0$ and $H(B) = 0$, therefore $H_Z(Z) = H(A) + H(B) = 0$ and vice versa.

For G2, $\mu_A(x_A) = 0.5 \; and \; \mu_B(x_B) = 0.5 \; \forall x_A \in X_A, \; \forall x_B \in X_B$, so, $H(A)$ and $H(B)$ are maximum, therefore $H_Z(Z) = H(A) + H(B)$ is maximum and vice versa.

For G3, denoted $A^* = (A^*, B^*)$, where A^*, B^* are sharpened version of A and B, respectively. So $H(A) \geq H(A^*)$ and $H(B) \geq H(B^*)$, therefore $H(A) + H(B) \geq H(A^*) + H(B^*) \Rightarrow H(Z) \geq H(Z^*)$.

For G4, $H(A) = H(1 - A)$ and $H(B) = H(1 - B)$, so $H(A) + H(B) = (H(1 - A)) + (H(1 - B)) \Rightarrow H_Z(Z) = H_Z(Z(1 - A, 1 - B))$. □

4.2. Fuzziness Measure for Discrete Z-numbers

There are three fuzziness measures of discrete Z-numbers, based on fuzziness measure for discrete fuzzy sets of Kaufmann, Yager and Kosko in Section 2.

$$H_{Z,Ka} = H_{Ka}(A) + H_{Ka}(B), \tag{15}$$

$$H_{Z,Y} = H_Y A + H_Y(B), \tag{16}$$

$$H_{Z,KoE} = H_{KoE}(A) + H_{KoE}(B). \tag{17}$$

Example 4.1. *A Z-number $Z = (A, B)$ given:*

$$\begin{aligned} A =& 0/1 + 0.3/2 + 0.5/3 + 0.6/4 + 0.7/5 + 0.8/6 + 0.9/7 + 1/8 \\ &+ 0.8/9 + 0.6/10 + 0/11, \end{aligned}$$

$$\begin{aligned} B =& 0/0 + 0.5/0.1 + 0.8/0.2 + 1/0.3 + 0.8/0.4 + 0.7/0.5 \\ &+ 0.6/0.6 + 0.4/0.7 + 0.2/0.8 + 0.1/0.6 + 0/1. \end{aligned}$$

The results are shown in Table 1.

It can be seen from these results, the proposed method can correctly describe the degree of uncertainty of Z-numbers and it's computational process is simple.

method	$H(A)$	$H(B)$	$H_Z(Z)$
KaufmannKaufmann (1975)	0.4364	0.3818	0.8182
YagerYAGER (1979); Yager (1980)	0.4364	0.3818	0.8182
KoskoKosko (1986)	0.2857	0.2692	0.5549

Table 1: The results of fuzziness measures for a Z-number in Example 4.1

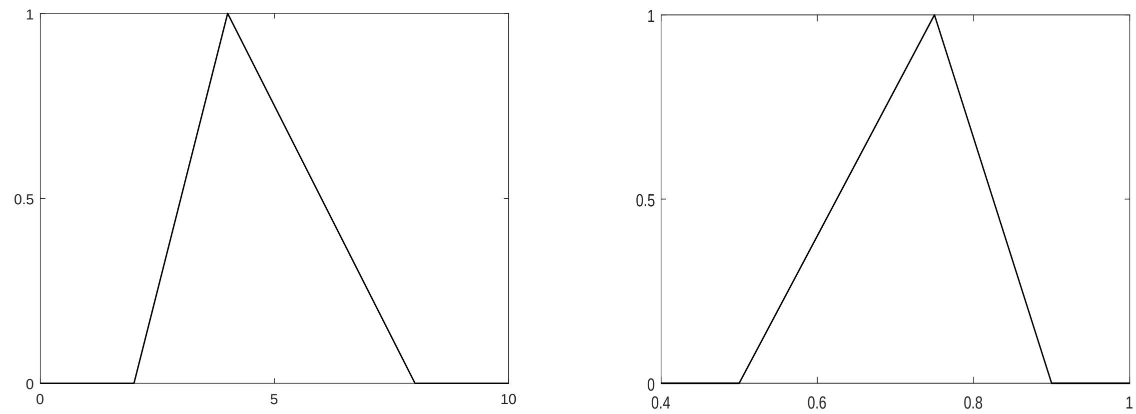

Figure 4: A Z-number $Z = Z(A(left), B(right))$

4.3. Fuzziness Measure for Continuous Z-numbers

There are two fuzziness measures of discrete Z-numbers, based on fuzziness measure for discrete fuzzy sets of Yager and Kosko in Section 2.

$$H_{Z,Y} = H_Y A + H_Y(B), \tag{18}$$

$$H_{Z,KoE} = H_{KoE}(A) + H_{KoE}(B). \tag{19}$$

Example 4.2. *A Z-number is shown in Fig. 4, the results are given in Table 2.*

5. Fuzziness measure in Z-number Information Fusion

What is clear that the smaller fuzziness measure the more information passed by a Z-number. It should be clear that for the purposes of decision-making we prefer Z-numbers

method	$H(A)$	$H(B)$	$H_Z(Z)$
YagerYAGER (1979); Yager (1980)	0.1667	0.4620	0.6287
KoskoKosko (1986)	0.3333	0.5	0.8333

Table 2: The results of fuzziness measures for a Z-number in Example 4.2

with smaller fuzziness measure as we have less uncertainty, more information. In the sensor data fusion problem (Baymuratov and Zhukova 2017), The fusion value with the highest quality can be found from the weighted aggregation of the Z-numbers sources using fuzziness measures of Z-numbers. Assume x is a variable that takes its value in the space $X = \{x_1, \cdots, x_n\}$, v is a reliability variable takeing its value in the reliability space $V = \{v_1, \cdots, v_m\}$ and a collection $\mathcal{Z} = \{Z_1, \cdots, Z_t\}$ of Z-numbers information about the value of X.

On weighted average fusion, there are t Z-numbers $Z_1, \cdots, Z_t$, where $Z_i = (A_i, B_i)$, $A_i = [\mu_{A_i}(x_1), \cdots, \mu_{A_i}(x_n)]$, $B_i = [\mu_{B_i}(v_1), \cdots, \mu_{B_i}(v_m)]$. Assume the fusion of t Z-numbers is $Z(A, B) = \sum_{i=1}^{t} w_i Z_i$. then

$$A = [\sum_{i=1}^{t} \frac{1}{t} \mu_{A_i}(x_1), \cdots, \sum_{i=1}^{t} \frac{1}{t} \mu_{A_i}(x_n)],$$

$$B = [\sum_{i=1}^{t} \frac{1}{t} \mu_{B_i}(v_1), \cdots, \sum_{i=1}^{t} \frac{1}{t} \mu_{B_i}(v_m)].$$

In this case, we use fuzziness measure of Yager as an example.

$$
\begin{aligned}
H_Z(Z) &= H_Y(A) + H_Y(B) \\
&= 1 - \frac{\sum_{j=1}^{n})|2\mu_A(x_j) - 1|}{n} + 1 - \frac{\sum_(j = 1)^{m})|2\mu_B(v_j) - 1|}{m} \\
&= 1 - \frac{\sum_{j=1}^{n} |2\sum_{i=1}^{t} \frac{1}{t}\mu_{A_i}(x_j) - 1|}{n} + 1 - \frac{\sum_{j=1}^{m} |2\sum_{i=1}^{t} \frac{1}{t}\mu_{B_i}(v_j) - 1|}{m} \\
&= 1 - \frac{\frac{1}{t}\sum_{j=1}^{n} |\sum_{i=1}^{t} 2\mu_{A_i}(x_j) - \sum_{i=1}^{t} 1|}{n} + 1 - \frac{\frac{1}{t}\sum_{j=1}^{m} |\sum_{i=1}^{t} 2\mu_{B_i}(v_j) - \sum_{i=1}^{t} 1|}{m} \\
&= 1 - \frac{1}{t}\sum_{i=1}^{t} \frac{\sum_{j=1}^{n} |2\mu_{A_i}(x_i) - 1|}{n} + 1 - \frac{1}{t}\sum_{i=1}^{t} \frac{\sum_{j=1}^{m} |2\mu_{B_i}(v_i) - 1|}{m} \\
&= \frac{1}{t}\sum_{i=1}^{t}(1 - \frac{\sum_{j=1}^{n} |2\mu_{A_i}(x_i) - 1|}{n}) + \frac{1}{t}\sum_{i=1}^{t}(1 - \frac{\sum_{j=1}^{m} |2\mu_{B_i}(v_i) - 1|}{m}) \\
&= \frac{1}{t}\sum_{i=1}^{t} H_Y(A_i) + \frac{1}{t}\sum_{i=1}^{t} H_Y(B_i) \\
&= \frac{1}{t}\sum_{i=1}^{t}(H_Y(A_i) + H_Y(B_i)) \\
&= \frac{1}{t}\sum_{i=1}^{t} H_Z(Z_i) \\
&= \sum_{i=1}^{t} \frac{1}{t} H_Z(Z_i).
\end{aligned}
\tag{20}
$$

If all Z-numbers are the crisp Z-number, then $H(A) = 1$ and $H(B) = 1$, so the fuzziness measure is minimum $H_Z(Z) = 0$. If all Z-numbers are fuzzy completely, then $H(A) = 0$ and $H(B) = 0$, so the fuzziness measure is maximum $H_Z(Z) = 2$.

It is can be given by referencing to Eq. (20):

$$H_Z(Z) = \sum_{i-1}^{t} \frac{1}{t} H_Z(Z_i). \tag{21}$$

Example 5.1. *The collection of relevant Z-number is $\mathcal{Z} = \{Z_1, Z_2, Z_3\}$ from three sensors.*

A Z-number $Z_1 = (A_1, B_1)$:

$$
\begin{aligned}
A_1 &= 0/0 + 0.5/1 + 0.8/2 + 1.0/3 + 0.8/4 + 0.7/5 + 0.6/6 + 0.4/7 \\
&\quad + 0.2/8 + 0.1/9 + 0/10,
\end{aligned}
$$

$$B_1 = 0/0 + 0.3/0.1 + 0.5/0.2 + 0.6/0.3 + 0.7/0.4 + 0.8/0.5 + 0.9/0.6 + 1.0/0.7 + 0.9/0.8 + 0.8/0.9 + 0/1.$$

A Z-number $Z_2 = (A_2, B_2)$:

$$A_2 = 0/0 + 0.2/1 + 0.4/2 + 0.6/3 + 0.8/4 + 1.0/5 + 0.8/6 + 0.6/7 + 0.4/8 + 0.2/9 + 0/10,$$

$$B_2 = 0/0 + 0.2/0.1 + 0.4/0.2 + 0.6/0.3 + 0.8/0.4 + 1.0/0.5 + 0.8/0.6 + 0.6/0.7 + 0.4/0.8 + 0.2/0.9 + 0/1.$$

A Z-number $Z_3 = (A_3, B_3)$:

$$A_3 = 0/0 + 0.4/1 + 0.5/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7 + 0.5/8 + 0.3/9 + 0/10,$$

$$B_1 = 0/0 + 0.4/0.1 + 0.5/0.2 + 0.6/0.3 + 0.8/0.4 + 0.9/0.5 + 1.0/0.6 + 0.7/0.7 + 0.5/0.8 + 0.2/0.9 + 0/1.$$

Then

$$H_Y(A_1) = \frac{23}{55},\ H_Y(B_1) = \frac{21}{55}, H_Z(Z_1) = \frac{44}{55};$$
$$H_Y(A_2) = \frac{24}{55},\ H_Y(B_2) = \frac{24}{55}, H_Z(Z_2) = \frac{48}{55};$$
$$H_Y(A_3) = \frac{30}{55},\ H_Y(B_3) = \frac{26}{55}, H_Z(Z_1) = \frac{56}{55}.$$

Then results of weighted average fusion are:

$$H_Z(Z_{1,2}) = \frac{46}{55}$$
$$H_Z(Z_{1,3}) = \frac{50}{55}$$
$$H_Z(Z_{2,3}) = \frac{52}{55}$$
$$H_Z(Z_{1,2,3}) = \frac{49.3}{55}$$

As a result, the Z-number $Z_{1,2}$ fusion is the Z-numbers fusion with the best information quality.

6. Conclusions

Z-numbers is one of the most widely used math tools to addressing the uncertainty information in real world. The fuzziness measure of Z-numbers can describe the degree of quality of information. The bigger the value of fuzziness of a Z-number, the less information it contains. The fuzziness measure of a Z-number can be expressed as the simple addition of fuzziness measures of two fuzzy numbers of it. Based on some well-known measures of fuzziness of discrete fuzzy sets, the method to measure fuzziness of the continuous fuzzy sets is extended and to deal with the discrete Z-numbers and continuous Z-numbers. The fuzziness measure of Z numbers can be seen as an index of information quality. It is used to obtain the best information quality in sensor data fusion when the output of sensor report are provided by Z-numbers.

Worden and Dulieu-Barton (2004); Balageas, Fritzen, and Gemes (2001); Rytter (1993); Chang (2003); Staszewski, Boller, and Tomlinson (2004); Jiang, Fu, and Zhang (2011); Guo (2006); OBrien and Loughlin (2007); Boutros and Liang (2007); He, Yan, and Zhang (2012); Aziz, Akif, and Rafig (2015); Aliev and Salimov (2017b); Malik (2012),

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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