Gauge Coordinates and Electromagnetic Field

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Abstract

We show there is a gauge and coordinate choice such that the electromagnetic field is zero.

1 Gauge transformation

Let $A_\mu(x)$ and $g_{\mu\nu}(x)$ be the electromagnetic potential and metric tensor respectively and $x = (x^0, \mathbf{x})$ a point of $\mathbb{R}^4$. The electromagnetic field is

$$F_{\mu\nu}(x) = A_{\nu,\mu}(x) - A_{\mu,\nu}(x) \tag{1}$$

A gauge transformation

$$A_\mu(x) \to A_\mu(x) + \phi_{,\mu}(x) \tag{2}$$

where $\phi(x)$ is a function on $\mathbb{R}^4$ leaves $F_{\mu\nu}(x)$ unchanged. Define

$$\hat{A}^\mu(x) = A^\mu(x) + g^{\mu\alpha}(x)\phi_{,\alpha}(x) \tag{3}$$

We have by (1) and (3)

$$\begin{aligned} F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} + \phi_{,\nu\mu} - \phi_{,\mu\nu} = (A_\nu + \phi_{,\nu})_{,\mu} - (A_\mu + \phi_{,\mu})_{,\nu} \\ &= (g_{\nu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\mu} - (g_{\mu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\nu} \\ &= (g_{\nu\alpha}\hat{A}^\alpha)_{,\mu} - (g_{\mu\alpha}\hat{A}^\alpha)_{,\nu} \end{aligned} \tag{4}$$

2 Coordinate transformation

Require that $g_{\mu\nu}(x)$ satisfies the Einstein field equations

$$G_{\mu\nu} = 8\pi\Big(g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau} - \frac{1}{4}g_{\mu\nu}g^{\sigma\alpha}g^{\tau\beta}F_{\sigma\tau}F_{\alpha\beta}\Big) + 8\pi T_{\mu\nu} \tag{5}$$

where $T^{\mu\nu}(x)$ is the energy-momentum tensor of a perfect fluid. $T^{\mu\nu}(x)$ will depend on $g_{\mu\nu}(x)$. Require that $A_\mu(x), g_{\mu\nu}(x)$, and $g^{\mu\nu}(x)$ are smooth and there is an a so that

$$|A^\mu(x)| < a \qquad |g^{\mu\nu}(x) - \eta^{\mu\nu}| < \frac{1}{2} \tag{6}$$

for all x. Let

$$\phi(x) = -2(1+a)x^0 \tag{7}$$

We then have that $\hat{A}^\mu(x)$ is smooth and by (3), (6), and (7) that

$$\hat{A}^0(x) > 1 \qquad |\hat{A}^\mu(x)| < 3 + 4a \tag{8}$$

for all x. Consequently there are coordinates x' such that the coordinate transformation $x'(x)$ is smooth, hence $A'_\mu(x), g'_{\mu\nu}(x')$, and $g'^{\mu\nu}(x')$ are smooth, and

$$\hat{A}'^0(x') = 1 \qquad \hat{A}'^1(x') = \hat{A}'^2(x') = \hat{A}'^3(x') = 0 \tag{9}$$

for all x' (see appendix).

3 Electromagnetic field is zero in synchronous coordinates

After making a gauge transformation (2) we have $g_{\mu\nu}(x)$ will still be a solution of (5). Make a gauge tranformation and change to coordinates x' so that everywhere (9) holds. By (4) and (9) we have in x' coordinates

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (g'_{\nu\alpha}\hat{A}'^{\alpha})_{,\mu} - (g'_{\mu\alpha}\hat{A}'^{\alpha})_{,\nu} = g'_{\nu 0},_{\mu} - g'_{\mu 0},_{\nu} \tag{10}$$

so we can write (5) transformed to x' coordinates as

$$G'_{\mu\nu} = 8\pi g'^{\sigma\tau}[g'_{\sigma 0,\mu} - g'_{\mu 0,\sigma}][g'_{\tau 0,\nu} - g'_{\nu 0,\tau}] - 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau}[g'_{\tau 0,\sigma} - g'_{\sigma 0,\tau}][g'_{\beta 0,\alpha} - g'_{\alpha 0,\beta}] + 8\pi T'_{\mu\nu} \tag{11}$$

Now if $g'_{\mu\nu}(x')$ is a solution of (11) then we expect that $g''_{\mu\nu}(x')$ is also a solution of (11) where $g''_{\mu\nu}(x')$ is related to $g'_{\mu\nu}(x')$ by a smooth general coordinate transformation $x''(x')$ [1]. Choose the coordinate transformation $x''(x')$ so that $g''_{\mu\nu}(x')$ also satisfies the synchronous coordinate conditions

$$g''_{00}(x') = -1 \qquad g''_{10}(x') = g''_{20}(x') = g''_{30}(x') = 0 \tag{12}$$

In (11) replacing $g'_{\mu\nu}(x')$ by $g''_{\mu\nu}(x')$ and using (12) gives

$$G''_{\mu\nu}(x') = 8\pi T''_{\mu\nu}(x') \tag{13}$$

Here $G''_{\mu\nu}(x')$ is determined by $g''_{\mu\nu}(x')$ and $T''^{\mu\nu}(x')$ is dependent on $g''_{\mu\nu}(x')$. Now from (13) we have the energy-momentum tensor of the electromagnetic field is zero hence the electromagnetic field is zero. We expect whether an electromagnetic field is zero or not to be independent of coordinate conditions such as (12).

Appendix

Let $B^{\mu}(x)$ be a smooth vector field on $\mathbb{R}^4$ and there is a b such that

$$B^0(x) > 1 \qquad |B^{\mu}(x)| < b \tag{14}$$

or all x. We will show there are coordinates $x' = (x'^0, \mathbf{x}')$ such that

$$B'^0(x') = 1 \qquad B'^1(x) = B'^2(x') = B'^3(x') = 0 \tag{15}$$

for all x'. Also the coordinate transformation $x'(x)$ is smooth.

There is for each $\mathbf{x}' \in \mathbb{R}^3$ a curve $\gamma_{\mathbf{x}'}(\lambda)$ of $\mathbb{R}^4$ defined by

$$\frac{d\gamma^{\mu}_{\mathbf{x}'}}{d\lambda}(\lambda) = B^{\mu}(\gamma_{\mathbf{x}'}(\lambda)) \qquad \gamma^0_{\mathbf{x}'}(0) = 0, \gamma^1_{\mathbf{x}'}(0) = x'^1, \gamma^2_{\mathbf{x}'}(0) = x'^2, \gamma^3_{\mathbf{x}'}(0) = x'^3 \tag{16}$$

Smoothness of $B^{\mu}(x)$ will quarantee (16) has a solution [2]. The curves traced out by the $\gamma_{\mathbf{x}'}(\lambda)$ never intersect except possibly at points where $B^{\mu}(x)$ is zero but by (14) this never happens. There is then a one-to-one map

$$\Theta : \mathbb{R}^4 \to \mathbb{R}^4 \qquad (\lambda, \mathbf{x}') \mapsto \gamma_{\mathbf{x}'}(\lambda) \tag{17}$$

We now show Θ is onto. Take a point $(x^0, \mathbf{x}) \in \mathbb{R}^4$ with $x^0 \leq 0$. The argument for the case when $x^0 > 0$ is similar. There is a curve $\sigma(\xi)$ of $\mathbb{R}^4$ such that

$$\frac{d\sigma^{\mu}}{d\xi}(\xi) = B^{\mu}(\sigma(\xi)) \qquad \sigma^0(0) = x^0, \sigma^1(0) = x^1, \sigma^2(0) = x^2, \sigma^3(0) = x^3 \tag{18}$$

Begin at $(x^0, \mathbf{x})$ and follow the curve traced out by $\sigma(\xi)$ in the direction of increasing ξ. Using (14) it will eventually intersect the plane $x^0 = 0$ at say the point $(0, \mathbf{y})$. We have that $\sigma(\xi)$ will trace out the same curve as $\gamma_{\mathbf{y}}(\lambda)$. Consequently there is a λ_0 such that

$$\gamma_{\mathbf{y}}(\lambda_0) = (x^0, \mathbf{x}) \tag{19}$$

so Θ is then onto and hence is a bijection. The map Θ defines a coordinate change such that (15). Also since $B^\mu(x)$ is smooth we have the coordinate transformation $x'(x)$ is smooth.

References

[1] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, 1972) p. 161

[2] B.F. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1980) p. 42

[3] K. De Paepe, *Physics Essays*, September 2007