Sum of the first n factorial numbers

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Abstract

In this article, we proceed to study and apply the properties of the Gamma function to obtain a formula that allows us to calculate the sum of the first n factorial numbers.

Keywords: Gamma function, Factorial numbers.

1. Introduction

The Gamma function is a function that extends the concept of factorial to complex numbers. If the real part of the complex number z satisfies Re (z)> 0, then the Gamma function is defined as:

$$\Gamma(\mathrm{z}) = \int_0^\infty \mathrm{t}^{\mathrm{z}-1}\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$$

The integral converges absolutely and can extend to the entire positive complex plane. If n is a positive integer, then.

$$\Gamma(n) = (n-1)!$$

With what is shown the relationship of the Gamma function with the factorial function.

2. Sum of the first n factorial numbers.

We have the following sum:

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = 1! + 2! + 3! + \cdots + (\mathrm{n}-2)! + (\mathrm{n}-1)! + \mathrm{n}!$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = \Gamma(2) + \Gamma(3) + \Gamma(4) + \cdots + \Gamma(\mathrm{n}-1)! + \Gamma(\mathrm{n})! + \Gamma(\mathrm{n}+1)!$$

Then, by defining the Gamma function, we have:

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = \sum_{\mathrm{i}=2}^{\mathrm{n}+1} \Gamma(\mathrm{i}) = \int_0^\infty \mathrm{te}^{-\mathrm{t}}\mathrm{dt} + \int_0^\infty \mathrm{t}^2\mathrm{e}^{-\mathrm{t}}\mathrm{dt} + \cdots + \int_0^\infty \mathrm{t}^{\mathrm{n}-1}\mathrm{e}^{-\mathrm{t}}\mathrm{dt} + \int_0^\infty \mathrm{t}^{\mathrm{n}}\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$$

For integration properties.

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = \sum_{\mathrm{i}=2}^{\mathrm{n}+1} \Gamma(\mathrm{i}) = \int_0^\infty (\mathrm{t} + \mathrm{t}^2 + \cdots \mathrm{t}^{\mathrm{n}-1} + \mathrm{t}^{\mathrm{n}})\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$$

Performing the sum of the geometric progression.

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = \sum_{\mathrm{i}=2}^{\mathrm{n}+1} \Gamma(\mathrm{i}) = \int_0^\infty \left(\mathrm{t}\frac{\mathrm{t}^{\mathrm{n}}-1}{\mathrm{t}-1}\right)\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{n_i})! = \sum_{\mathrm{i}=2}^{\mathrm{n}+1} \Gamma(\mathrm{i}) = \int_0^\infty \left(\frac{t^{n+1}-t}{t-1}\right)\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$$

$$\sum_{i=1}^{n}(n_i)! = \int_{0}^{\infty}\left(\frac{t^{n+1}-t}{t-1}\right)e^{-t}dt$$

This is the expression that allows to obtain the sum of the first n factorial numbers.

3. Conclusions.

In this article we obtained an expression that allows us to obtain the sum of the first n factorial numbers. The expression obtained is expressed as an indefinite integral and emerges as a variation of the Gamma function.

References

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