Gravitational Effect on the Final Stellar to Planetary Mass Ratio

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The discovery of exoplanets and possible future detection of exomoons led to the question if any relationship holds between the mass of its host star and mass of its planets. That is, if the mass of planets is constrained by the final mass of the hosting star. If relationship such as this exists, then several key questions can be answered. First of all, based on the mass of the star, what is the upper limit mass budget available for the creation of terrestrial planets. Secondly, if we know the mass budget availability, then what is the upper limit of the mass budget for the water in the system. Thirdly, finding the probability of the formation of exomoons with sizes comparable to earth around Jovian planets. Fourthly, explaining the rare occurrence of Jupiter sized planets around red dwarf systems.

1 Overview

Over thousands of exoplanets have been discovered in the last decade, yet their mass relationship with their parent star is poorly investigated. We shall start with the solar system. We assume that the solar system is typical, that is, its formation via the accretion process from the collapse of molecular cloud possibly caused by a nearby supernovae explosion is the standard star creation process. Out of the stellar accretion process, local accretion started around ice gas giants Jupiter, Saturn, Uranus, and Neptune. Since the accretion material is primarily composed of hydrogen and helium and follows the same laws of physics, one expects any relationship derivable naturally extends to all.[5] We omit terrestrial planets and dwarf planets. Terrestrial planets formed through the bombardment of protoplanetary embryos. Unlike gas giants which reaches a critical mass (10 earth mass or greater) and results in a runaway mass accretion and form their own accretion disk, terrestrial and dwarf planet's mass is tiny and mass increase occurs only through collision with bodies of similar size.[5] As a result, Accretion disk never forms around a terrestrial-sized planet or smaller. There is no significant difference between the initial formation of gas giants and terrestrial planets. Both undergo a period of body collisions, but gas giants are able to gain mass greater than 10 earth mass and transitions into the next stage of planetary evolution. In a sense, terrestrial planets can be called failed gas giants. Gas giants can be called failed stars.

2 Empirical Law Derivation and Proof

We investigate the mass of the sun relative to the total mass of all planets, and the mass ratio is 745.29 to 1. The mass of Jupiter, relative to the total mass of all its moons (99% of its mass are from Io, Callisto, Ganymede, and Europa), is 4,829 to 1. The mass of Saturn, relative to the total mass of all its moons (99% of its mass comes from Titan, Enceladus, and Mimas) is 4,137 to 1. The mass of Uranus, relative to the total mass of all its moons (99% of its mass comes from Miranda, Ariel, Oberon, Umbriel, and Titania) is 9,430 to 1. Neptune, however, can not be used for data collection since its moon Triton is in a retrograde motion. Most theories support the idea that Triton is captured by Neptune from the Kuiper belt. According to the Nice Model, Neptune and Uranus were in a much tighter orbits around the Sun within 20 AU at its formation, and Neptune was also in the lower orbit. It later switched position with Uranus and ventured into the Kuiper belt and captured Triton. It remains a mystery regarding the original moons that formed through the accretion process. Those moons either fall into Neptune or get ejected by unstable N-body orbits of the additional captured moons. As a result, we have four data points can be plotted and we find the best power-law fit for the empirical data, and we found that the following empirical law is describing the mass ratio relationship for the accretion process of different mass.

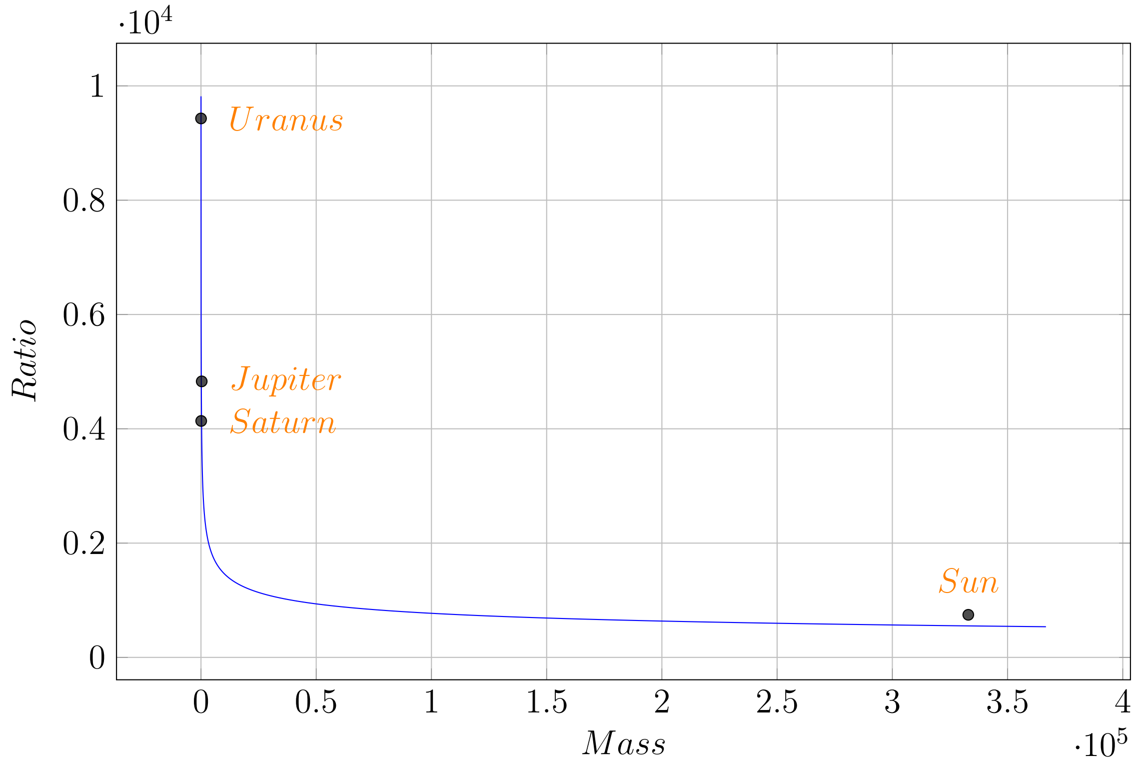

Figure 2.1: Primary to satellite mass ratio

$$y = 19,250x^{-0.2794} \tag{2.1}$$

If we plug in the solar mass value of 333,000 earth mass, the predicted total planet budget is 603.927 earth mass for the solar system. This is higher than our solar system budget at 446.719 earth mass. With this budget estimate and sun's metallicity, the total water budget's upper bound is 6.107 earth mass. It is only 1.897 earth mass if oxygen is counted toward the composition of terrestrial planet creation. The total mass available to create celestial bodies with metals is 11.2934 earth mass (excluding H, He, Ne, and other trace non-interacting gas). The budget for terrestrial planet creation is 4.747 earth mass excluding oxygen (Those elements with a boiling point higher than 500 K, simulating the accretion disc temperature of the inner planets composed of elements such as C, Fe, Si, and Mg). It is 8.957 earth masses including oxygen assuming all terrestrial planet contains 47% oxygen as observed on earth.

Now, we can interpret the empirical derivation to that of the physical reality. We establish the following hypothesis:

The downward slope is simply a comparison of the gravitational limit/strength of the planets, arising from host stars of varying stellar mass, on their satellites and their effective strength in terms of a fraction out of the radius of the original planetary accretion that formed them.

Because stars with smaller mass also hosts planets with smaller mass, the planetary accretion Keplerian discs radius is accordingly smaller. The force between the planet and the edge of the planetary disc supposed to get stronger, but the increase in strength due distance shrink-

age could not compensate the decrease in force strength due to decrease in planetary and its satellites mass at the edge of the planetary disc.
On the other hand, for more massive stars, the expected distance between their planets and planet's satellites grows due to greater planetary accretion disc mass, so the force of attraction suppose to decrease. However, the decrease in strength at the edge of the planetary disc due to distance expansion can not offset the gravitational attraction strength increase due to planetary and satellite mass increase.
For a point of reference, we introduce the graph for the attraction between two masses with a unit mass of 1 vs. r_0, the accretion distances.

$$\frac{1}{r_0^2} \tag{2.2}$$

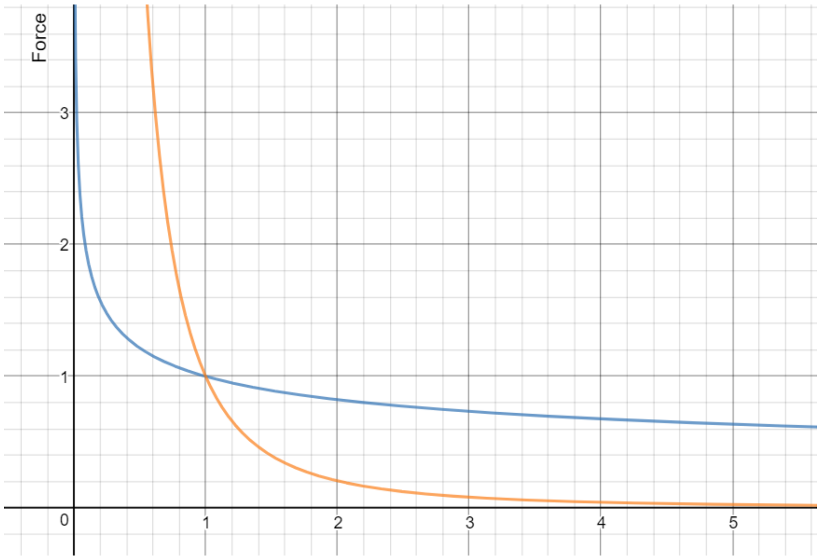

Figure 2.2: Gravitational force grows faster than primary to satellite ratio with separation<1

It is self-evident that for two objects with a fixed unit mass of 1, with distance separation below 1, the force of attraction grows faster than our empirical equation for two objects with variable mass (suggesting the empirical law holds smaller mass, slower decrease in separation distance, or both) With separation above 1, the force of attraction drops faster than our empirical equation for two objects with variable mass. (suggesting the empirical law holds larger mass, slower increase in separation distance, or both)
Once we derived the equation and formulated our hypothesis, we want to show how such relationship can be truly derived mathematically and proved using universal law of gravitation. One has to be very careful at deriving and proving this empirical law starting with N body interacting with each other through gravitational force and possibly viscosity in the accretion disk. Only 2 body simulation can be computed to precision, and 3 or N body problem is hard if not utterly impossible to solve. Making the matter worse, one needs to find the final accretion

state which takes millions of years and millions of cycles of interaction and revolution, rendering the problem intractable.
We can, however, simplify this problem, by circumventing around the complexity of N-body simulation by simply stating the initial condition and the final condition of the system. The intermediate steps can be omitted to make the process tractable. The initial condition of the system is utterly a flat circular disk of a certain diameter, and the final condition of the system is utterly a sphere with a certain radius.

Once we have specified these conditions, we can start our derivation:
First, the radius of the accretion disk grows in size proportionally to the final mass of the star:

$$\left(\frac{4}{3}\pi r_0^3\right)\rho_0 = \left(\pi r_1^2 h\right)\rho_1 \tag{2.3}$$

where the left-hand side is the stellar mass with a volume based on its radius $r_0$assuming a density of $\rho_0 = 1$ and the right-hand side is the volume of the disk which forms the original accretion disc with spread out density of ρ_1, where $\rho_1 < \rho_0$.
Then the equation simplifies to:

$$\frac{4}{3}\pi r_0^3 = \left(\pi r_1^2 h\right)\rho_1 \tag{2.4}$$

alternative, it can expressed as:

$$M_{sol} = \left(\pi r_1^2 h\right)\rho_1 \tag{2.5}$$

We rearrange the equation and solve for r_1, the radius of the disc:

$$r_1 = \left(\frac{M_{sol}}{\pi h \rho_1}\right)^{\frac{1}{2}} \tag{2.6}$$

However, we do not have any specific information regarding the value of h, the height of the accretion disc. Fortunately, the height of the accretion disc can be expressed as a fraction of the final stellar mass M. First of all, sun's accretion disc's height is bounded by $a_{earth} \cdot i_{earth} < h_{sun} < 3H_{earth}$. [7] That is, the height is bounded by the semi-major axis of earth times its angle of inclination and 3 times the Hill radius of earth. The disc height is correlated with the disc radius. This is self-evident from the lower bound $a_{earth} \cdot i_{earth}$. Since earth's inclination to the invariable plane is 1.57 degrees, we have $0.0174a < h_{sun}$.
The upper bound Hill radius is also correlated with the semi-major axis. It is known that the Hill sphere can be calculated from the equation and the final height is:

$$r_{\mathrm{H}} \approx a(1-e)\sqrt[3]{\frac{m}{3M}} \tag{2.7}$$

When eccentricity is negligible, it simplifies to:

$$r_{\mathrm{H}} \approx a\sqrt[3]{\frac{m}{3M}} \tag{2.8}$$

and the relation in terms of the volume of the Hill sphere compared with the volume of the second body's orbit around the first, whereas m = mass of earth and M = mass of the Sun:

$$3\frac{r_{\mathrm{H}}^3}{a^3} \approx \frac{m}{M} \tag{2.9}$$

As a result, we know that the height of accretion disc for creating a sun like star is:

$$3\frac{r_{\mathrm{H}}^3}{a^3} = \frac{M_{earth}}{M_{sol}} \tag{2.10}$$

$$\frac{r_H}{a} = \left(\frac{1}{3}\cdot\frac{M_{earth}}{M_{sol}}\right)^{\frac{1}{3}} \tag{2.11}$$

$$r_H = 0.010a \tag{2.12}$$

We constraint the height of the disc to be $0.0174a < h < 0.030a$, and choose our height as $h = 0.0237a$. Since the semi-major axis of earth is 1 AU, $h = 3,545,470$ km.
The only remaining term we needs to derive is ρ_1. Unfortunately, ρ_1 is much more tricky. We generally do not know the average size and density of the accretion disc, which changes throughout the stellar's formation history.
We can fix ρ_1 by set the typical size of the accretion disc to 100 AU for a sun like star. By fixing the size of the accretion disc, one can determine the density ρ_1. We will later adjust our values to fit the empirical observation.
Furthermore, we need rethink the final shape of the sun not as a sphere, rather a condensed, vertical cylinder pipe whereas the height of the pipe is the height of the original accretion disc assumed to be 3.54 million km, or 2.37 times the Hill radius of earth.

$$r_{sun} = \left(\frac{\left(\frac{4}{3}\pi\,(696,000)^3\right)}{\pi\cdot 3,545,470\cdot\rho_1}\right)^{\frac{1}{2}} \tag{2.13}$$

$$r_{sun} = 356,078.96 \text{ km} \tag{2.14}$$

Then, the final stellar radius with density of $\rho_1 = \rho_0 = 1$ is 356,079 km. However, we know that initially the disc has to be much larger. One can parameterize to find values which closely matches 100 AU, without even knowing the density of ρ_1.

$$\frac{\left(\frac{\pi r_{sun}^2}{\pi}\right)^{\frac{1}{1.091}}}{149,597,870} \tag{2.15}$$

$$\approx 100 \text{ AU}$$

That is, if one were unaware of the density ρ_1 and $\rho_1 < \rho_0$ holds, then one can take the exponent with $\frac{1}{c}$ where $C < 2$ so that the final resulting radius of the disk is always larger than by taking the square root ($C = 2$).
Then, the radius of the disc can be expressed as:

$$r_{disc} = \left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}} \tag{2.16}$$

Assuming that the height of the disc is proportional not just to earth's semi-major axis but to the entire disc radius, the height of the disc in terms of the mass of the star itself is then :

$$h = H\left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}} \tag{2.17}$$

$$H = \frac{0.0237}{10^2} \tag{2.18}$$

The ratio is reduced by a factor of $\left(\frac{1}{100}\right)^2$ because we formerly assumed that the height of the accretion disc is in proportion to the semi-major axis of earth at 1 AU and we substituted by 100 AU.
A caveat must be raised. The strength of the correlation between the accretion disc radius and the height remains unkown. If the a strong correlation exists, the height changes as the stellar accretion disc size and stellar mass changes. If a weak correlation exists, the height remains largely independent as the stellar accretion disc size and stellar mass changes. We can later adjust our values to fit the empirical observation and reach a conclusion regarding the strength of the correlation.
The final disk size is then given by the equation:

$$r_1 = \left(\frac{M_{sol}}{\pi \cdot H \cdot \left(\frac{M_{sol}}{\pi}\right)}\right)^{\frac{1}{1.091}} \tag{2.19}$$

Then, we use the law of universal gravitation, to define the attraction between two masses with gravitational constant G, G is assumed to be unit 1, because its real value is irrelevant for our discussion.

$$F = G \cdot \frac{x \cdot x}{r_1^2} \tag{2.20}$$

Where r_1 is the radius of previously derived results.

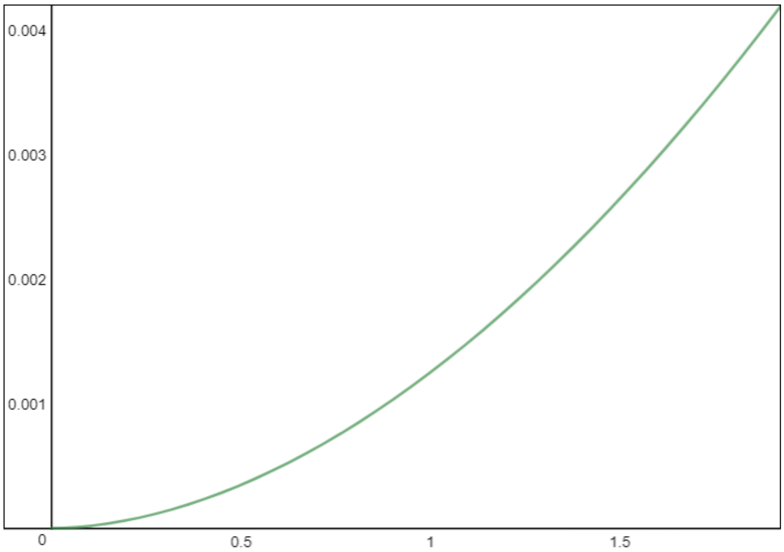

Figure 2.3: Unit mass pairs' attraction at their expected formation distance

Then, to show that planets and smaller stars ought to have a higher ratio of planet to satellites mass, we use the following equation:

$$r_{small} = \left(\frac{k \cdot M_{sol}}{\pi \cdot H \cdot \left(\frac{k \cdot M_{sol}}{\pi} \right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.21}$$

$$r_{small} \approx \left(\frac{k \cdot M_{sol}}{k \cdot (M_{sol})^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.22}$$

$$F = \frac{(kx)\,(kx)}{(cr_{small})^2} \tag{2.23}$$

Where k stands for the fraction of the original solar mass of the smaller planet embedded inside the existing star, and c stands for the coefficient that re-adjusted, finding at which fraction of the accretion disc of the planet the force of planet's gravitation dominates over the host star. We used k=0.5 and c =1. The plots are shown below:

$$F = \frac{0.5x \cdot 0.5x}{(1 \cdot r_{small})^2} \tag{2.24}$$

$$r_{small} = \left(\frac{0.5 \cdot M_{sol}}{\pi \cdot H \cdot \left(\frac{0.5 \cdot M_{sol}}{\pi}\right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.25}$$

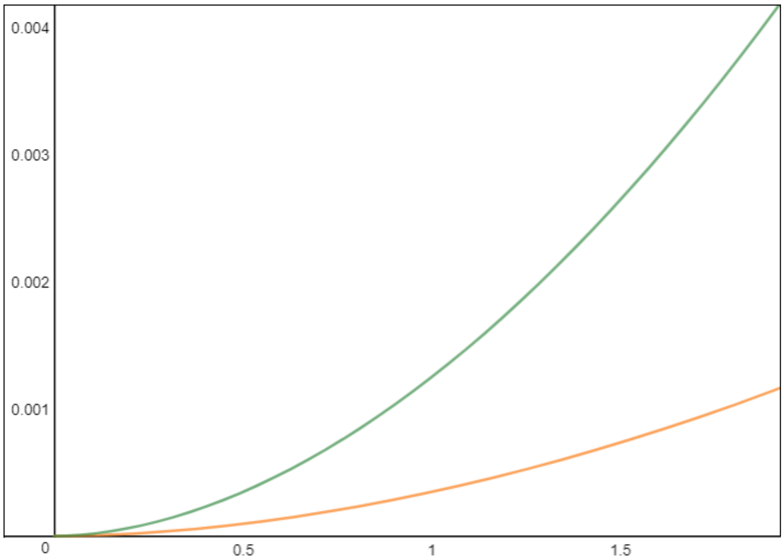

Figure 2.4: Two 1/2 mass pairs' attraction at their expected formation dist falls below 2 unit mass pairs' attraction at their expected formation dist

The 1/2th masses curve sits below the original unit mass curve for all range of masses, indicating that the attraction at the edge of smaller accretion disc between two smaller mass is always less than the attraction between two larger mass at the edge of its own accretion disc. Re-adjusting c=0.527, we have

$$F = \frac{0.5x \cdot 0.5x}{(0.527 \cdot r_{small})^2} \tag{2.26}$$

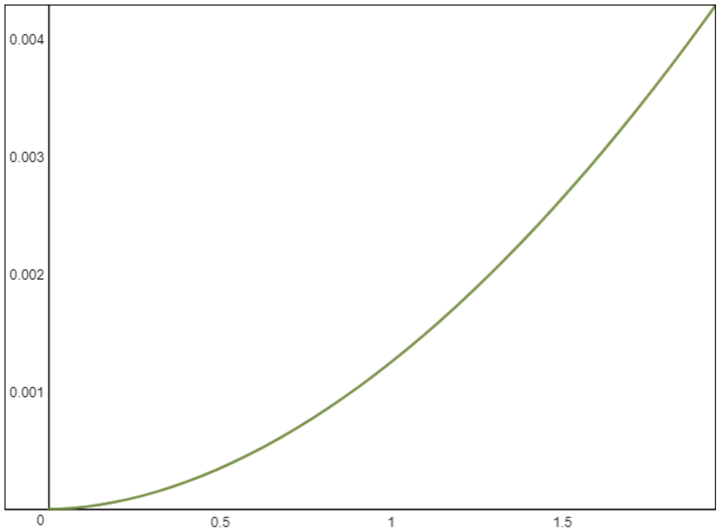

Figure 2.5: Two 1/2 mass pairs' attraction matched 2 unit mass pairs' attraction by narrowing their separation to 52.7 percent of their expected formation dist

where the 1/2th masses curve shifts upward vertically and coincides with the unit mass curve. This indicates that only when satellites were extending at or below 52.7 percent of the smaller mass's accretion disc, and it is maintained by the gravitational attraction of its planet, the rest is lost due to greater attraction by the host star.
We then used k=0.1 and c =1. The plots are shown below:

$$F = \frac{0.1x \cdot 0.1x}{(1 \cdot r_{small2})^2} \tag{2.27}$$

$$r_{small2} = \left(\frac{0.1 \cdot M_{sol}}{\pi \cdot H \cdot \left(\frac{0.1 \cdot M_{sol}}{\pi} \right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.28}$$

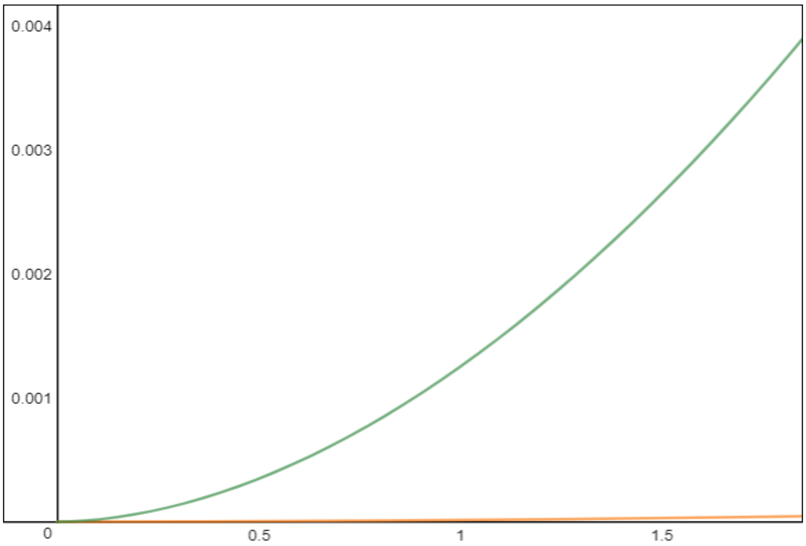

Figure 2.6: Two 1/10th mass pairs' attraction at their expected formation dist falls below 2 unit mass pairs' attraction at their expected formation dist

The 1/10th mass curve sits below the original unit mass curve for all range of masses, indicating that the attraction at the edge of smaller accretion disc between two smaller mass is always less than the attraction between two larger mass at the edge of its own accretion disc. Its force is also much weaker than a 0.5 solar mass case. Re-adjusting c = 0.119, we have

$$F = \frac{0.1x \cdot 0.1x}{(0.119 \cdot r_{small2})^2} \tag{2.29}$$

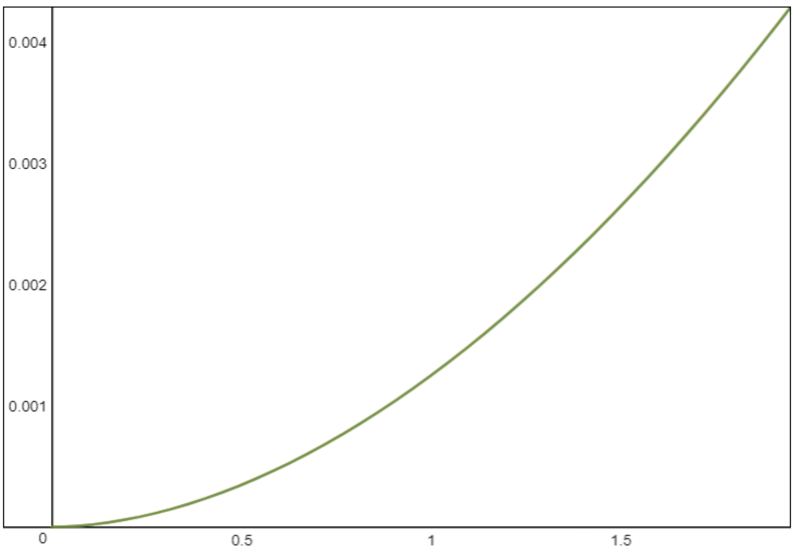

Figure 2.7: Two 1/10th mass pairs' attraction matched 2 unit mass pairs' attraction by narrowing their separation to 11.9 percent of their expected formation dist

Where the 1/10th mass curve shifts upward vertically and coincides with the existing unit mass curve. This indicates that only when satellites were extending at or below 10.55 percent of the

smaller mass's accretion disc, and it is maintained by the gravitational attraction of its planet, the rest is lost due to greater attraction by the host star.
The percentage threshold radius for the substellar object of different mass' gravitational limit can be derived and its form:

$$F = \frac{x \cdot x}{(r_1)^2} = \frac{kx \cdot kx}{(cr_{small})^2} \tag{2.30}$$

$$\frac{x^2}{r_1^2} = \frac{k^2 \cdot x^2}{(cr_{small})^2} \tag{2.31}$$

divide both sides by x^2:

$$\frac{1}{r_1^2} = \frac{k^2}{c^2 \cdot r_{small}^2} \tag{2.32}$$

$$c^2 r_{small}^2 = k^2 r_1^2 \tag{2.33}$$

$$c^2 = \frac{k^2 r_1^2}{r_{small}^2} \tag{2.34}$$

$$c^2 = \left(\frac{kr_1}{r_{small}}\right)^2 \tag{2.35}$$

taking square root on both sides:

$$c = \frac{k \cdot r_1}{r_{small}} \tag{2.36}$$

Where k stands for the mass ratio relative to the host star, r_1 is the host star accretion disc radius, and r_{small} is the planet accretion disc radius. The plot is given below:

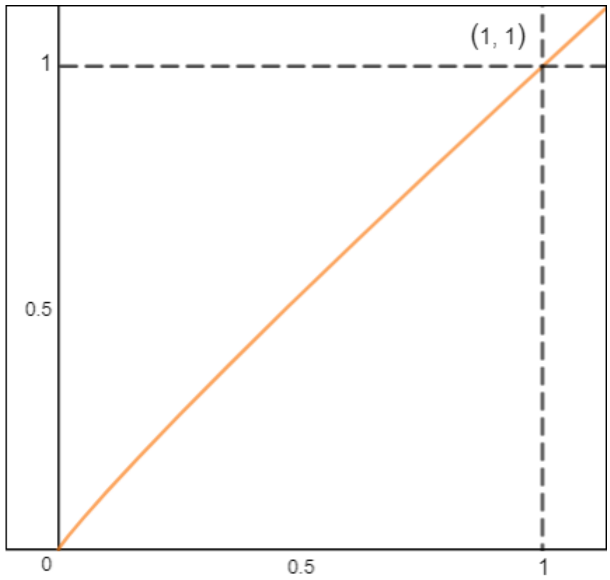

Figure 2.8: The amount of percentage of radius narrowing required based on their expected disc/stellar mass

The interpretation of the graph is the following:
In general, the smaller the planet, more of its accretion mass will be lost in the gravitational tug of war with its host star. It is ultimately a consequence of mathematics. Although the mass decreases linearly, and the gravitational attraction decreases to the 2nd power, but the accretion disc's radius for smaller mass only shrinks by the factor $\left(\left(\frac{1}{x}\right)^{\frac{-0.098}{1.098}}\right)^{\frac{1}{1.098}}$.

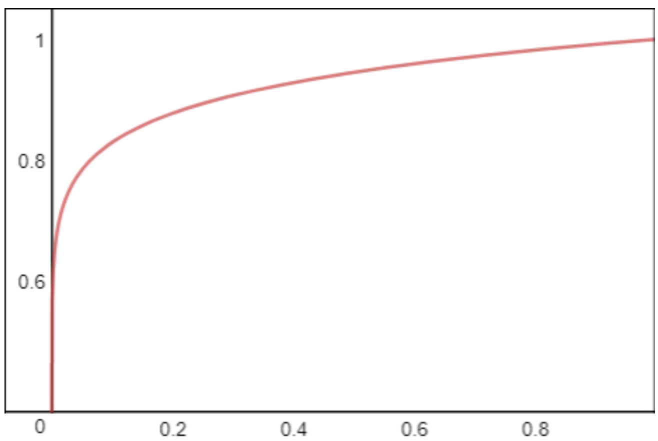

Figure 2.9: Disc radius shrinking sublinearly

The combined effect on the gravitational strength at the edge of the disc radius is:

$$f = \frac{r_{small}}{r_1} \tag{2.37}$$

$$f = \frac{\left(\frac{xM}{H\cdot\left(\frac{xM}{\pi}\right)^{\frac{1}{1.098}}\pi}\right)^{\frac{1}{1.091}}}{\left(\frac{M}{H\cdot\left(\frac{M}{\pi}\right)^{\frac{1}{1.098}}\pi}\right)^{\frac{1}{1.091}}} \tag{2.38}$$

$$f = \frac{\left(\frac{xM}{(xM)^{\frac{1}{1.091}}}\right)^{\frac{1}{1.091}}}{\left(\frac{M}{M^{\frac{1}{1.091}}}\right)^{\frac{1}{1.091}}} \tag{2.39}$$

$$f = \left(\frac{xM}{(xM)^{\frac{1}{1.091}}} \cdot \frac{M^{\frac{1}{1.091}}}{M}\right)^{\frac{1}{1.091}} \tag{2.40}$$

$$f = \left(x \cdot \left(\frac{M}{xM}\right)^{\frac{1}{1.091}}\right)^{\frac{1}{1.091}} \tag{2.41}$$

$$f = \left(x \cdot \left(\frac{1}{x}\right)^{\frac{1}{1.091}}\right)^{\frac{1}{1.091}} \tag{2.42}$$

$$f = \left(\left(\frac{1}{x}\right)^{\frac{-0.091}{1.091}}\right)^{\frac{1}{1.091}} \tag{2.43}$$

So the force of gravitation decreases relative to the unit mass by the curve $\frac{x^2}{f^2}$, which is just placed slightly higher than the force decreasing curve x^2 thanks to the sublinear decreases in disc radius.

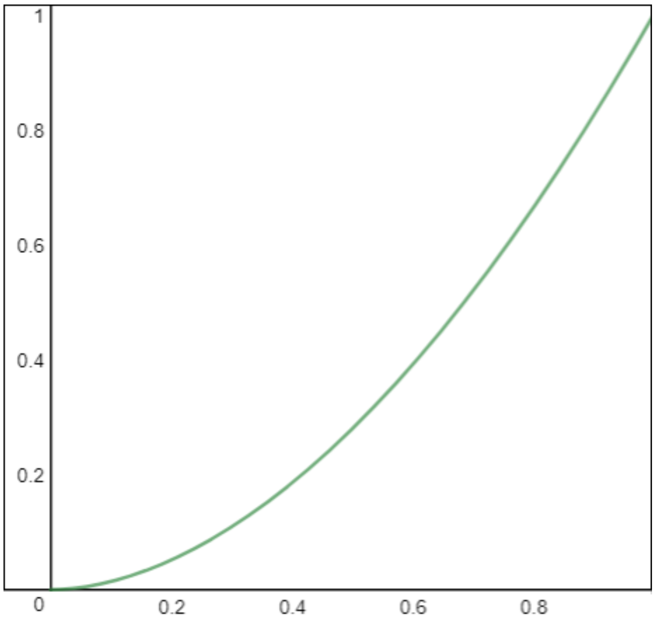

Figure 2.10: curve $\frac{x^2}{f^2}$

As a result, one can think the curve $\frac{x^2}{f^2}$ as the transformation factor for the percentage of the accretion disc under the gravitational pull of accreting object of varying mass.
For a lighter mass relative to a reference object (sun or earth) comes with a smaller percentage of the accretion disc under gravitational attraction by the factor

$$\frac{x^2}{f^2} < 1 \tag{2.44}$$

For a greater mass relative to a reference object (sun or earth) comes with a greater percentage of the accretion disc under gravitational attraction by the same factor

$$\frac{x^2}{f^2} > 1 \tag{2.45}$$

and for any values of x, the percentage threshold radius for the substellar object of different mass' gravitational limit we have $c = \frac{x}{f}$ so that:

$$\frac{x^2}{\left(\left(\frac{x}{f}\right) f\right)^2} = 1 \tag{2.46}$$

whereas$\frac{x}{f} = x\left(\frac{1}{x^{0.091}}\right)^{\frac{1}{1.091}} = c = \frac{k \cdot r_1}{r_{small}}$[1]
$x\left(\frac{1}{x^{0.091}}\right)^{\frac{1}{1.091}}$ is derived based on the simplification of

[1]The literal interpretation of $\frac{x}{f}$ is that, in order for $\frac{x^2}{f^2}$ to stay at the strength at the parity of unit mass at the unit distance for any arbitrary accretion radius, a factor of $\frac{f^2}{x^2}$, the inverse of $\frac{x^2}{f^2}$ is required. Moreover, in order to reach parity, we can only set limits on the radius not the mass, so the factor $\frac{f^2}{x^2}$ can only appear

$$c = \frac{k \cdot r_1}{r_{small}} \tag{2.47}$$

$$k = x \tag{2.48}$$

$$r_1 = \left(\frac{M_{sol}}{\pi \cdot H \cdot \left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.49}$$

$$r_{small} = \left(\frac{k \cdot M_{sol}}{\pi \cdot H \cdot \left(\frac{k \cdot M_{sol}}{\pi}\right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}} \tag{2.50}$$

then substituting becomes:

$$\frac{x \left(\frac{M_{sol}}{\pi \cdot H \cdot \left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}}}{\left(\frac{x \cdot M_{sol}}{\pi \cdot H \cdot \left(\frac{x \cdot M_{sol}}{\pi}\right)^{\frac{1}{1.091}}} \right)^{\frac{1}{1.091}}} \tag{2.51}$$

simplifies to:

in the denominator, then, it becomes $\frac{x^2}{f^2\left(\frac{f^2}{x^2}\right)^{-1}} \Rightarrow \frac{x^2}{f^2\left(\frac{x^2}{f^2}\right)^{1}} \Rightarrow \frac{x^2}{f^2\left(\frac{x}{f}\right)^{2}}$. Furthermore, The factor has to be inside the sublinear decreasing radius expressed as f (as a fraction of f), so we have $\frac{x^2}{\left(\left(\frac{x}{f}\right)f\right)^2}$

$$\Rightarrow x\left(\frac{\frac{M_{sol}}{\pi\cdot H\cdot\left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}}}}{\frac{x\cdot M_{sol}}{\pi\cdot H\cdot\left(\frac{x\cdot M_{sol}}{\pi}\right)^{\frac{1}{1.091}}}}\right)^{\frac{1}{1.091}} \tag{2.52}$$

$$\Rightarrow x\left(\frac{M_{sol}}{\pi\cdot H\cdot\left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}}}\cdot\frac{\pi\cdot H\cdot\left(\frac{x\cdot M_{sol}}{\pi}\right)^{\frac{1}{1.091}}}{x\cdot M_{sol}}\right)^{\frac{1}{1.091}} \tag{2.53}$$

$$\Rightarrow x\left(\left(\frac{\frac{x\cdot M_{sol}}{\pi}}{\frac{M_{sol}}{\pi}}\right)^{\frac{1}{1.091}}\cdot\frac{1}{x}\right)^{\frac{1}{1.091}} \tag{2.54}$$

$$\Rightarrow x\left(\left(\frac{x\cdot M_{sol}}{\pi}\cdot\frac{\pi}{M_{sol}}\right)^{\frac{1}{1.091}}\cdot\frac{1}{x}\right)^{\frac{1}{1.091}} \tag{2.55}$$

$$\Rightarrow x\left(\left(\frac{x}{1}\cdot\frac{1}{1}\right)^{\frac{1}{1.091}}\cdot\frac{1}{x}\right)^{\frac{1}{1.091}} \tag{2.56}$$

$$\Rightarrow x\left(x^{\frac{1}{1.091}}\cdot\frac{1}{x}\right)^{\frac{1}{1.091}} \tag{2.57}$$

$$\Rightarrow x\left(\frac{1}{x^{0.091}}\right)^{\frac{1}{1.091}} \tag{2.58}$$

Hence, we have shown that this empirical law holds as a consequence of mathematics and physics.
Since the empirical law is represented in terms of mass ratio (not as disc area ratio in terms of its radius), one has to take the inverse of equation $c = \frac{k\cdot r_1}{r_{small}}$ to shows how much mass is lost during the accretion process. Furthermore, the mass ratio is a consequence of the tug of war of the gravitational force between the planet and the host star and occurred in a 2 dimensional plane, so we have:

$$\left(\frac{k\cdot r_1}{r_{small}}\right)^{-2} \tag{2.59}$$

The tug of gravitation also occurs immediately above or below the plane. We assume that a tiny bit of interaction occurs in a three dimensional space. Therefore, we added the disc height in terms of the disc radius. As a result, one has to check the diagonal distance from the accreting planet to the furthermost point above or below the edge of the accreting planet's effective gravitational perimeter.

$$r_{1diagonal} = \sqrt{r_1^2 + H \cdot r_1^2} \tag{2.60}$$

$$r_{smalldiagonal} = \sqrt{r_{small}^2 + H \cdot r_{small}^2} \tag{2.61}$$

$$\left(\frac{k \cdot r_1}{r_{small}}\right)^{-2} = \left(\frac{k \cdot r_{1diagonal}}{r_{smalldiagonal}}\right)^{-2} \tag{2.62}$$

$$\left(\frac{k \cdot r_1}{r_{small}}\right)^{-2} = \left(\frac{\sqrt{\left(\frac{kr_1}{r_{small}}\right)^2 + H \cdot \left(\frac{kr_1}{r_{small}}\right)^2}}{(1^2 + H \cdot 1^2)^{\frac{1}{2}}}\right)^{-2} \tag{2.63}$$

Fortunately, the ratio remains the same regardless of whether taking consideration of diagonal conditions or not.

Ideally, we could re-adjusted it to 2.0237 or $2 + H$, so we have:

$$\left(\frac{k \cdot r_1}{r_{small}}\right)^{-(2+H)} \tag{2.64}$$

However, the curve does not match our empirical derived equation.

$$\left(\frac{k \cdot r_1}{r_{small}}\right)^{-2} \neq y = x^{-0.2794} \tag{2.65}$$

where the coefficient is not relevant and is reduced to 1. The graph shows inverse relationship; however, the curvature is different.

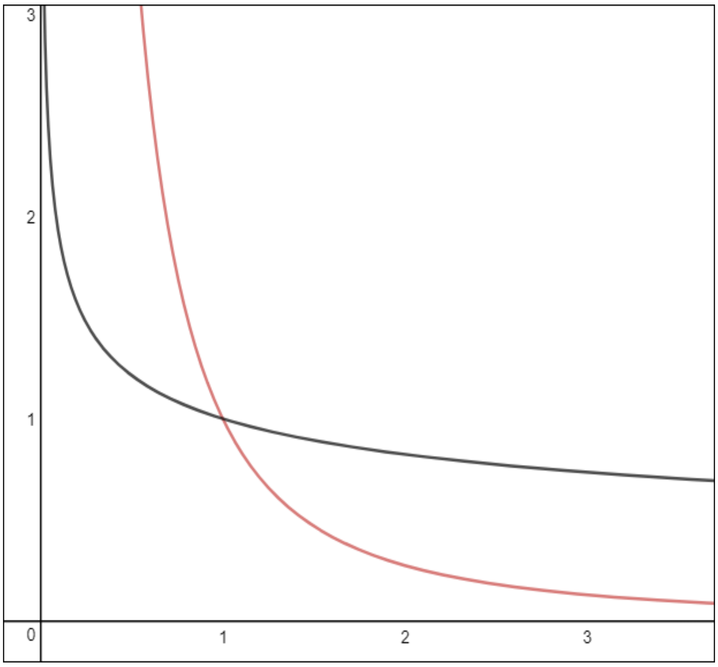

Figure 2.11: Our derivation does not match empirical derivation

In order to fit our equation into the empirical observation:
First of all, the original ratio can be simplified:

$$\left(\frac{k\cdot r_1}{r_{small}}\right)^{-2} = \left(\frac{k\left(\frac{M_{sol}}{H\cdot\left(\frac{M_{sol}}{\pi}\right)^{\frac{1}{1.091}}\pi}\right)^{\frac{1}{1.091}}}{\left(\frac{xM_{sol}}{H\cdot\left(\frac{xM_{sol}}{\pi}\right)^{\frac{1}{1.091}}\pi}\right)^{\frac{1}{1.091}}}\right)^{-2} \tag{2.66}$$

$$= \left(\frac{k\left(\frac{M_{sol}}{M_{sol}^{\frac{1}{1.091}}}\right)^{\frac{1}{1.091}}}{\left(\frac{xM_{sol}}{(xM_{sol})^{\frac{1}{1.091}}}\right)^{\frac{1}{1.091}}}\right)^{-2} \tag{2.67}$$

We then substitute both exponents $\frac{1}{1.091}$ with $\frac{1}{v}$ and $\frac{1}{j}$

$$\left(\frac{k \left(\frac{M_{sol}}{M_{sol}^{\frac{1}{v}}} \right)^{\frac{1}{j}}}{\left(\frac{xM_{sol}}{(xM_{sol})^{\frac{1}{v}}} \right)^{\frac{1}{j}}} \right)^{-2} = \left(\frac{k}{\left(\frac{x}{x^{\frac{1}{v}}} \right)^{\frac{1}{j}}} \right)^{-2} \tag{2.68}$$

[2]

Careful parameterization reveals that, in order for (2.67) to match the empirical result, one can substitute a range of value pairs for v and j to match the empirical observation.

The list of pair values are listed below, whereas the accretion radius size is defined by: $\frac{\left(r_{sun}^2\right)^{\frac{1}{j}}}{149{,}597{,}870 \text{ km}}$

$\frac{1}{v}$ **(Disc Height Factor)**	$\frac{1}{j}$ **(Disc Radius Factor)**	**Accretion Radius**
$\frac{1}{1.25}$	$\frac{1}{0.231}$	$2.53{\cdot}10^{35}$ ly
$\frac{1}{1.5}$	$\frac{1}{0.387}$	$7.97{\cdot}10^{15}$ ly
$\frac{1}{2}$	$\frac{1}{0.58}$	1,960,781 ly
$\frac{1}{5}$	$\frac{1}{0.93}$	6,948 AU
$\frac{1}{10}$	$\frac{1}{1.05}*$	294 AU
$\frac{1}{20}$	$\frac{1}{1.1}*$	96.55 AU
$\frac{1}{40}$	$\frac{1}{1.13}*$	51.88 AU
0	$\frac{1}{1.16}*$	28.8 AU
$-\frac{1}{30}$	$\frac{1}{1.2}*$	13.74 AU
$-\frac{1}{1.4}$	$\frac{1}{2}$	0.0025 AU

Table 2.1: The possible pairs of $\left(\frac{1}{v}, \frac{1}{j}\right)$ which fits the empirical observation

and now:

$$\left(\frac{k{\cdot}r_1}{r_{small}} \right)^{-2} = y = x^{-0.2794} \tag{2.69}$$

The results shows a whole range of values permissible mathematically; however, only a very limited set fits astronomical observations. Only $\frac{1}{j}$ for $1.2 < j < 1.05$ is considered because we

[2]after substitution, formerly $f = \frac{r_{small}}{r_1} = \left(\left(\frac{1}{x}\right)^{\frac{-0.091}{1.091}} \right)^{\frac{1}{1.091}}$ becomes $f = \frac{r_{small}}{r_1} = \left(\left(\frac{1}{x}\right)^{\frac{1-v}{v}} \right)^{\frac{1}{j}}$ and $\frac{x}{f} = x \left(\frac{1}{x^{0.091}}\right)^{\frac{1}{1.091}} = c = \frac{k{\cdot}r_1}{r_{small}}$ becomes $\frac{x}{f} = x \left(\frac{1}{x^{v-1}}\right)^{\frac{1}{j}} = c = \frac{k{\cdot}r_1}{r_{small}}$

assumed that for typical accretion disc at the solar mass, its radius ranges between 13 AU to 300 AU. The pair (v=∞, j=1.16), equivalent to (0, $\frac{1}{1.16}$) and indicates that if the height of the disc is completely independent from the disc radius, the accretion disc size at solar mass should be 28.8 AU. In such a case, our equation simplifies to:

$$\left(\frac{kM_{sol}^{\frac{1}{j}}}{(xM_{sol})^{\frac{1}{j}}}\right)^{-2} = \left(\frac{k}{x^{\frac{1}{j}}}\right)^{-2} \tag{2.70}$$

in which the height variable is completely removed from the equation.

The interpretation of the parameterization indicates that the height of the accretion disc, in fact, shows a weak correlation with stellar mass. Within the astronomical permissible value ranges, The graph of $y = M^{0\pm^{0.10}_{0.03}}$ indicates that the height of the accretion disk remain almost exactly the same across disks of different mass. The constant height is at least partially justified by equations describing the accretion dynamics concerning the height of the disc. It is stated that the scaled height of a Keplerian disk is given by [6]:

$$H = \frac{C_s T}{\Omega} \tag{2.71}$$

Where Ω is the angular Keplerian velocity, T is the local temperature, and C_s is the local sound speed. Whereas the angular velocity approximately equals the orbital velocity and is bounded by the escape velocity:

$$\sqrt{\frac{GM}{r_1}} \approx \Omega < \sqrt{\frac{2GM}{r_1}} \tag{2.72}$$

When G=1, M=x, we have:

$$H \approx \frac{1}{\sqrt{\frac{M_{sol}}{\left(\frac{M_{sol}}{(M_{sol})^{\frac{1}{v}}}\right)^{\frac{1}{j}}}}} \approx \frac{1}{\Omega} < \frac{1}{\sqrt{\frac{2M_{sol}}{\left(\frac{M_{sol}}{(M_{sol})^{\frac{1}{v}}}\right)^{\frac{1}{j}}}}} \tag{2.73}$$

It is assumed that C_s is the local sound speed and T does not significantly changes at all, it is speculated T may increase a little due to increase in density of the disc as the mass of the disc increases due to self-gravity, (which enables the curve to turn slightly positive and matches well with our original prediction $y = M^{0\pm^{0.10}_{0.03}}$) but in general, it is shown that the scaled height of the disk does not change much as the mass of the disc increases. That is, the scaled height for the accretion disk is largely independent of the mass of the accretion disk.

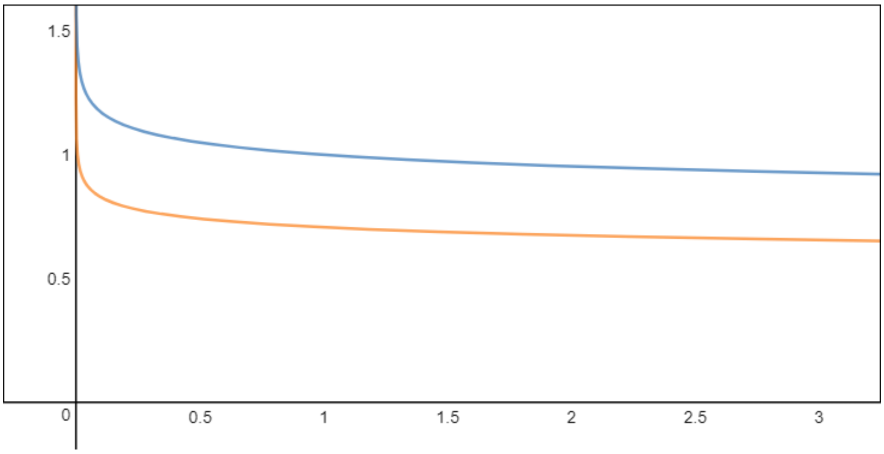

Figure 2.12: Scaled height for the disk remain pretty much constant as the mass of the disk increases

Not only the height of the accretion disc follows a weak correlation with stellar mass within the permissible range of astronomical observation, but the height of the disc also follows a weak negative correlation with accretion disc radius. The height of the accretion disk decreases as the accretion disc density increases, indicated as the stellar accretion radius decreases. This implies that as the accretion disc density increases, the gravitational pull and possibly viscosity on the disc increases the self-gravity of the disc. Self-gravity dominates when the initial accretion disc radius drops below ($\frac{1}{v} = 0$,$j = 1.16$) which is $\frac{\left(r_{sun}^2\right)^{\frac{1}{1.16}}}{149{,}597{,}870\,\mathrm{km}} = 28.8$ AU for solar mass.

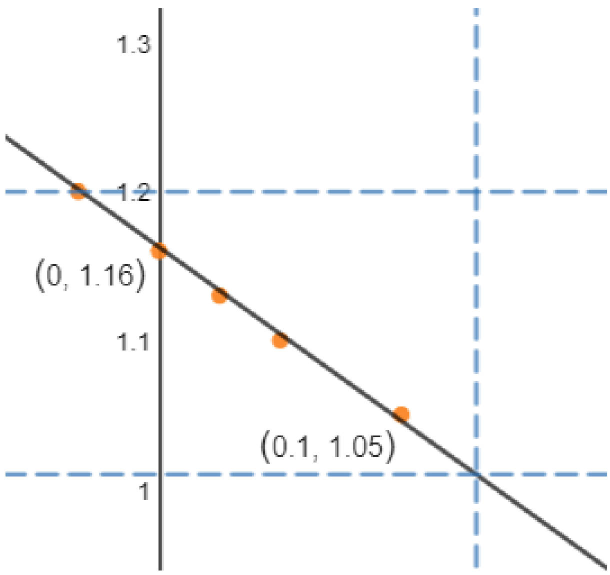

Figure 2.13: The greater the accretion disc density (the smaller the accretion disc radius), the greater self-gravity of the disc asserted on the disc height

In conclusion, the literal interpretation of this law can simply be stated as the follows. For any given celestial body, the size of the initial accretion disk and its radius grows as their mass increase. This also means that the orbits of the planets formed inside the disk also extends further out. However, with an increase in mass comes with an increase of gravity. Furthermore, in response to the increase in mass, the gravitational attraction from the host star and planet grows fast enough to compensate the decrease of gravitational attraction due to the increase in distance of the orbits of the forming planets.

If no other gravitational forces act on the moons of gas giants or planets around the star, then, regardless of the strength of the gravitational field, the planets stay. However, the formation of moons around gas giants is in a constant gravitational tug between the host star and the hosting giant gas planet. If the gas giant's mass is greater, then the moon's orbit extends further out. However, the planet's gravitational attraction on the moon is greater still, so, at the end of the day, the gas giant wins. The mass ratio of gas giant to moons is then low.

The formation of planets around stars are also in a constant gravitational tug between the stars and nearby stars. Nearly all stars formed in a star nursery, one of the most famous are the Orion nebulae. Each star forms within its own pocket roughly few thousand AU across, and its outer planets and gas giants can be gravitationally attracted to its neighbors. If the stellar mass is more massive than its neighbors, then its planets are more likely maintained. This is most dramatic in a scenario where a red dwarf's nursery surrounded by class O stars with mass 100 times greater than the sun, and many protoplanets of the red dwarves are seized by the class O stars. Therefore, the planet to satellite mass ratio also extends to the stellar to planet mass ratio.

3 Stellar Data and Derivation

We now can generalize our equation to all exoplanet data we have so far. We used the European exoplanet catalog and filtered out certain data (including binary brown dwarves) and maintained a list of 1,189 candidate sample points. For a system with multiple planets, we sum the total mass of the planets and label them as a single sample point because we are only interested in the mass ratio between the host star and the total mass of all of its planets. We sort the data points by their solar mass and place them into mass group brackets. From the table below, we can see that the number of planetary systems within each mass range bracket where the solar mass bracket is highlighted with an asterisk.

Mass Range mass	Samples N	Mass Range mass	Samples N
0.2	13	1.2	105
0.3	25	1.3	55
0.4	27	1.4	37
0.5	26	1.5	33
0.6	35	1.6	18
0.7	72	1.7	14
0.8	106	2	14
0.9	168	3	22
1*	205	–	–
1.1	135	–	–
Total			**1110**

Table 3.1: Stellar mass breakdown and their numbers

The results of each bracket are plotted int the graphs. The vertical axis represents the total number of planetary systems with a given mass. The horizontal axis represents the range of total mass of its planets. The mass is represented in the base of 340 Jupiter mass, which is the derived mass budget for a star with one solar mass from our earlier empirical law. We also run statistical distribution (generalized extreme value, normal distribution) on each plot, and recorded the inflection point on each plot. The result is reported below:

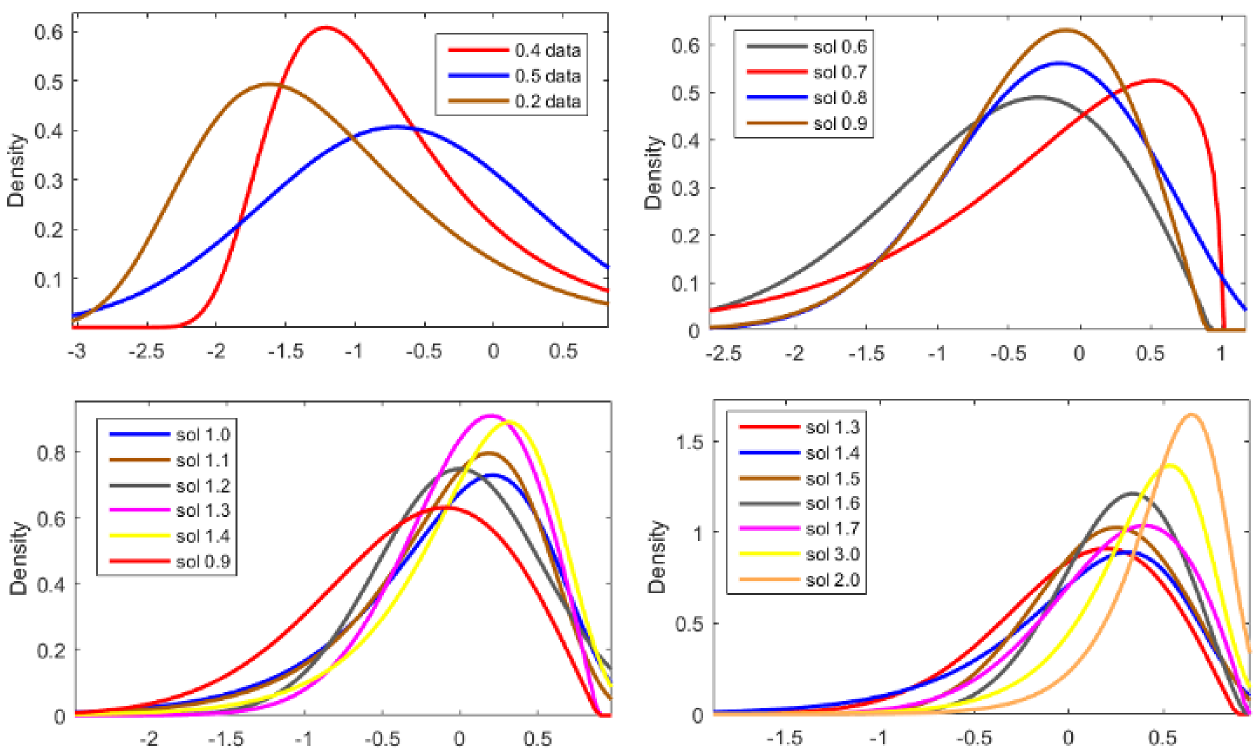

Figure 3.1: Plots for PDF of different stellar masses

Only in three cases, brackets with solar mass 0.7, solar mass 1.3, and solar mass 3.0 or greater does not conform to the trend. (though their right hand tail does fall on the prediction curve) The general trend is evident that the total planetary mass budget grows exponentially as the mass of the host star increases linearly.
Although the ratio of stellar mass to their planets grows exponentially, the variance within each bracket is large and skew toward the right. It can be stated that within one standard deviation, 66% of the ratio of stellar mass to their planets grows exponentially with a linear increase in stellar mass. Outliers in both extremes (about 10% shows a higher planetary total mass or lower ratio than prediction) and a tail (24%) shows a lower planetary total mass or higher ratio than the prediction The predictive power wanes at both extremes.
The causes for large variance remains a mystery and requires future investigation. It is also noted that variance grows in proportion to the ratio, in other words, grows as the stellar mass decreases. It seems to suggest that other factors (temperature, disc pressure, or stellar wind) may have a more effective role at minimizing the final planetary mass when the total planetary mass budget decreases and overwhelming the effects of gravity. Since the formation of planets and the stellar system takes on many different variables and are likely independent or only slightly correlated, extreme values permissible within each condition can contribute toward the left and right tails.

We recompute our empirical law for the entire data set, by finding the most likely value for each stellar mass brackets, along with solar system data points.

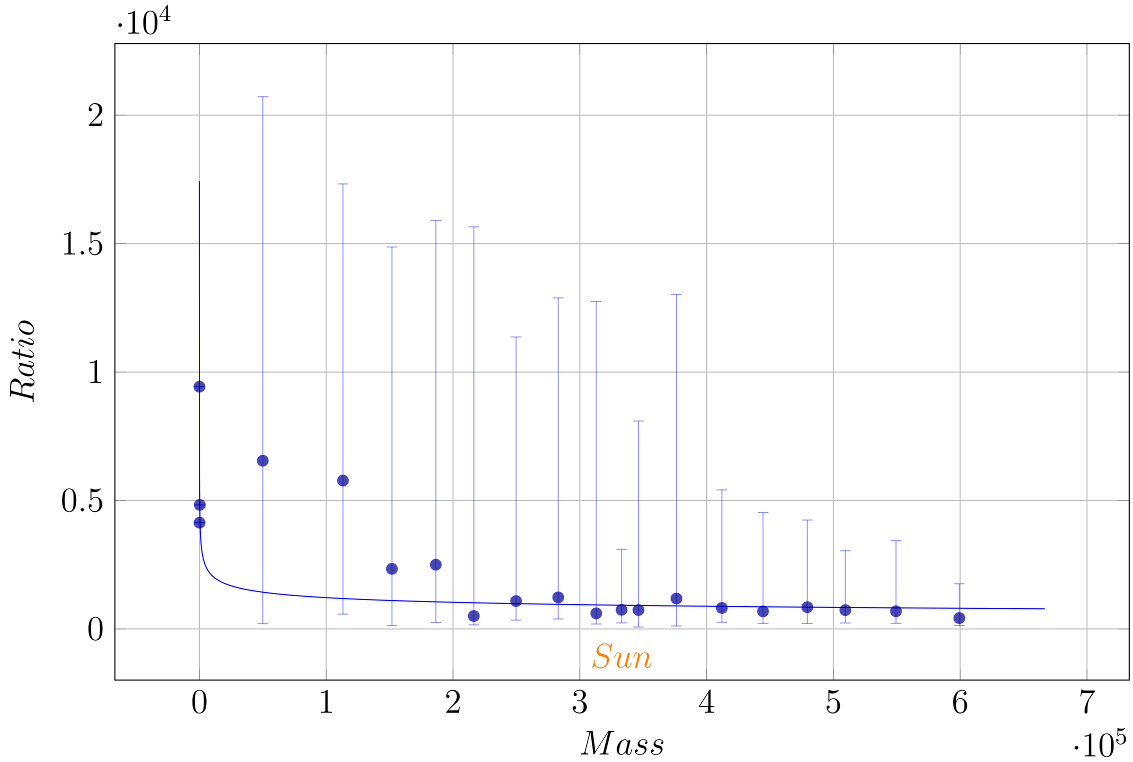

Figure 3.2: Stellar mass to planetary ratio across different range of stellar mass

$$y = 17520x^{-0.2315} \tag{3.1}$$

$$y = 10^{\left[\frac{Fe}{H}\right]} 17520x^{-0.2315} \tag{3.2}$$

Equation (11.3.2) is the generalized version of Equation (11.3.1), where the metallicity of a stellar system is taking into account to compute the mass budget for terrestrial planets.

If we plug in the solar mass value of 333,000 earth mass, the predicted mean total planet budget is $360.87\pm^{780.3}_{246.75}$ earth mass for the solar system. This is lower than our solar system budget at 446.719 earth mass. With this budget estimate and sun's metallicity, the total water budget is $3.65\pm^{7.890}_{2.495}$ earth mass. It is only $1.1342\pm^{2.4524}_{0.7750}$ earth mass if oxygen is counted toward the composition of terrestrial planet creation. The total mass available to create celestial bodies with metals is $6.748\pm^{14.59}_{4.614}$ earth mass, whereas the budget for terrestrial planet creation is $2.837\pm^{6.134}_{1.94}$ earth mass excluding oxygen. (Those elements with a boiling point higher than 500K, simulating the accretion disc temperature of the inner planets.) It is $5.3528\pm^{11.6}_{3.66}$ earth masses if including oxygen.

The value derived is less than the budget computed for the solar system, whereas the exoplanet's metallicity from the data is plotted below, which is generally comparable and exceeds that of the metallicity of the sun.

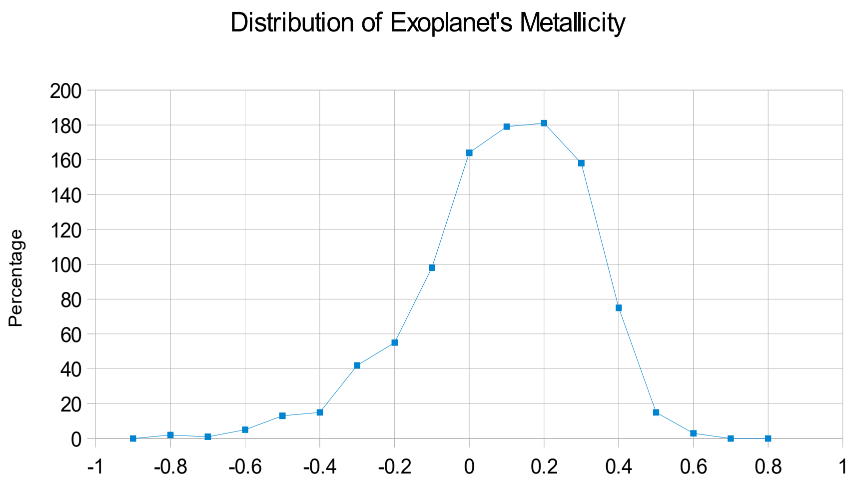

Figure 3.3: Distribution of exoplanet's metallicity

The plot indicates that sun's metallicity cannot account for the extra 85.85 earth masses we have observed. This discrepancy requires further analysis in the future when more data becomes available. One possible explanation is that some, if not all detection methods, such as the radial velocity method for detecting cold Jupiters and ice giants have too high noise to signal ratios. Therefore, the empirical equation serves as a lower bound on planetary mass budget.

Another possible explanation, as a clue offered by exponent re-adjusting and fit the new empirical curve.

V (Disc Height Factor)	J (Disc Radius Factor)	Accretion Radius
$\frac{1}{1.258}$	$\frac{1}{0.231}$	$2.53{\cdot}10^{35}$ ly
$\frac{1}{1.52}$	$\frac{1}{0.387}$	$7.97{\cdot}10^{15}$ ly
$\frac{1}{2.05}$	$\frac{1}{0.58}$	1,960,781 ly
$\frac{1}{5.7}$	$\frac{1}{0.93}$	6,948 AU
$\frac{1}{14}$	$\frac{1}{1.05}*$	294 AU
$\frac{1}{40}$	$\frac{1}{1.1}*$	96.55 AU
0	$\frac{1}{1.13}*$	51.88 AU
$-\frac{1}{40}$	$\frac{1}{1.16}*$	28.8 AU
$-\frac{1}{16}$	$\frac{1}{1.2}*$	13.74 AU
$-\frac{1}{1.3}$	$\frac{1}{2}$	0.0025 AU

Table 3.2: The possible pairs of $\left(\frac{1}{v}, \frac{1}{j}\right)$ which fits the empirical observation

with graph plots:

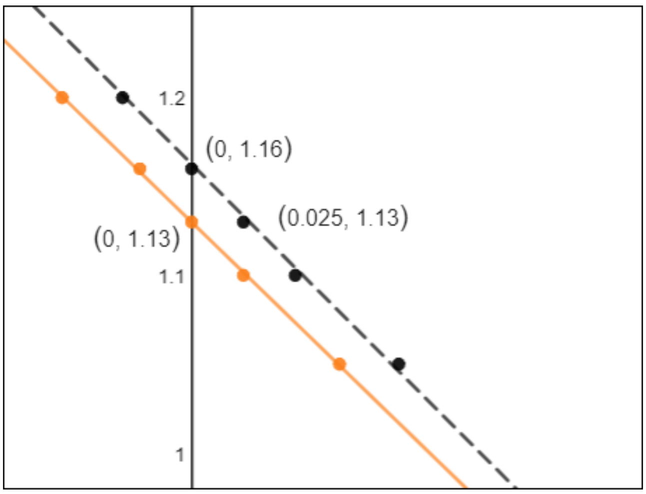

Figure 3.4: The new curve places in parallel in the previous one

Based on the graph, under the same disc radius and density (i.e. $j = 1.13$ with disc radius = 51.88 AU for solar mass disc), the new fit indicates a stronger negative correlation between the height and the disc radius with a lower disc height compares to the earlier fit. Under the same strength of correlation between the height and disc radius (i.e. $v = 0$), earlier fit requires a smaller radius ($j = 1.16$) or 28.79 AU for solar mass disc (greater density), and new fit requires

only a radius ($j = 1.13$) or 51.88 AU for solar mass disc . This implies that the accretion disc of the solar system was possibly less viscous, or having a higher temperature, increasing the pressure along the accretion plane that defied the self gravity of the disc.
A third possible explanation is that our earlier derived empirical law is non-accurate description of reality. The ealier formula predicts 603.927 earth mass for the solar system. This is higher than our solar system budget at 446.719 earth mass. The new formula's prediction of 360.87 earth mass is much closer to our solar system budget at 446.719 earth mass.

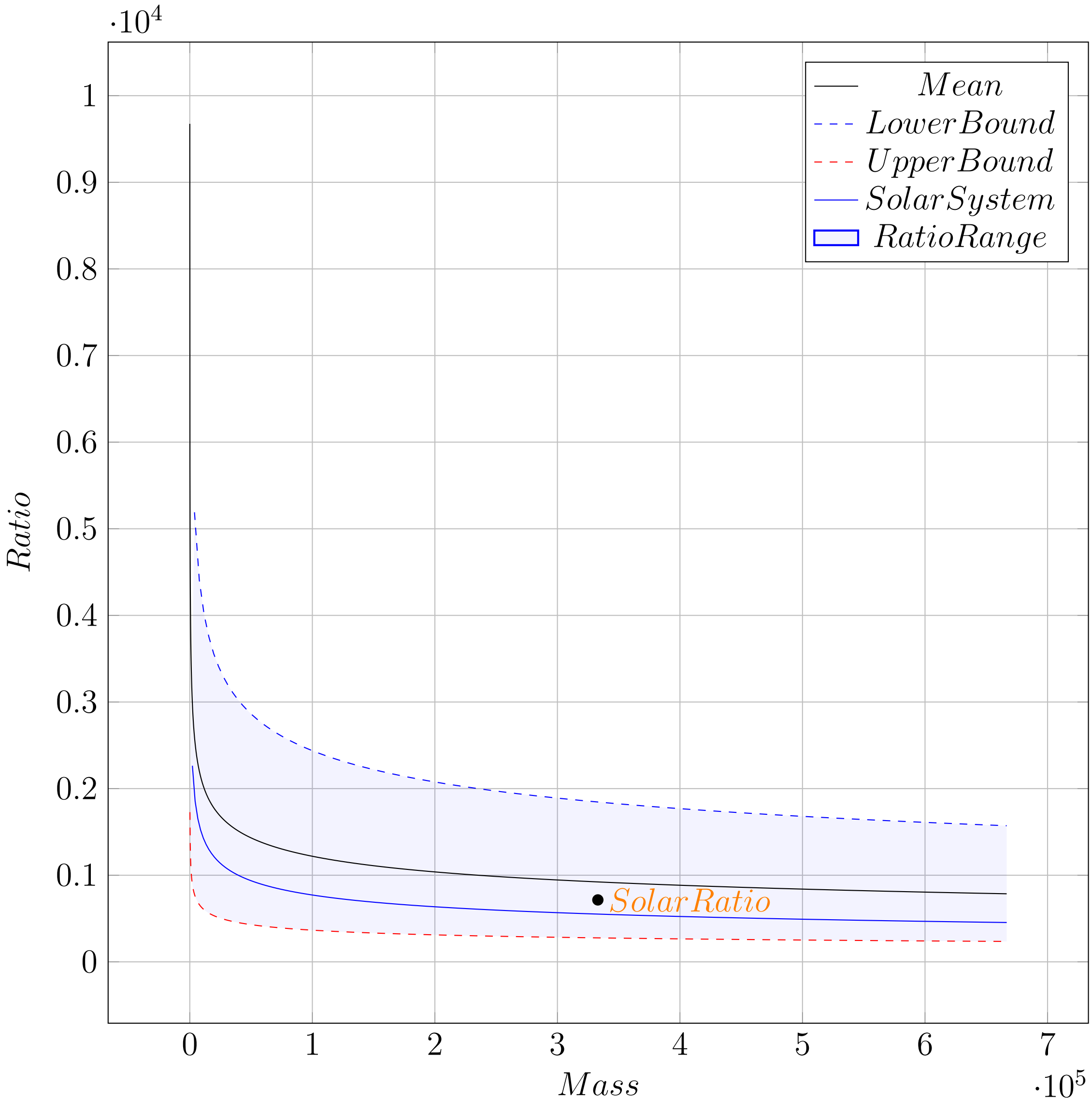

Figure 3.5: Lower and upper bound for primary to satellite ratio

One of the greatest challenge facing the host star and its planetary total mass budget theory is the formation of binary stars. Binary and multiple star system are also a product of accretion and solar nebulae disc consolidation. It turns out that binary stars formation is a consequence of stellar nebulae fragmentation. Since all stellar nebulae start as molecular cloud and gradually increases its density toward their gravitational center of mass, at some point along the transition, the density was high enough so that fragmentation creates two or more gravitational center point, which all evolved into stars on their own with different masses. A positive correlation exists for binary or multiple star systems with higher solar masses. It is speculated 44% of solar mass stars are binaries or multiples, while the majority (66%) of the red dwarfs are single stars.[1] From observational data, we also found that the average separation distance between binary pairs is 150 AU, which is well within the radius of forming the stellar disc, which usually extends hundreds of AU.

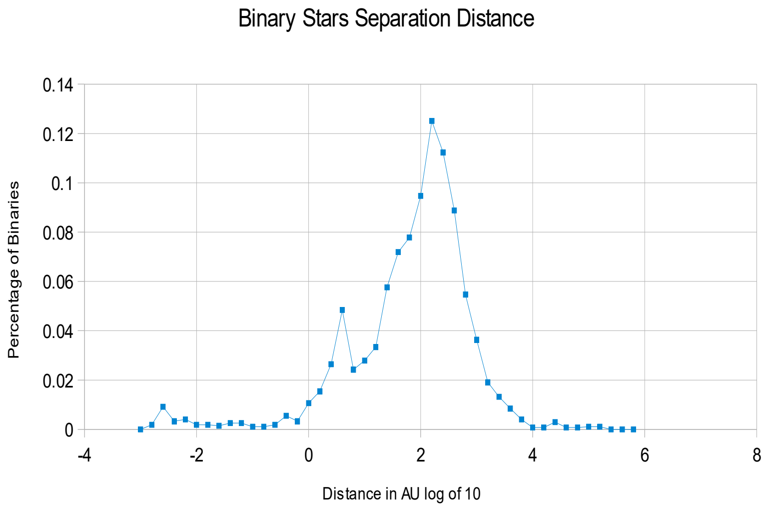

Figure 3.6: PDF of binary stars by separation distance

For some multiple star system with stars orbiting the primary star thousands to tens of thousands of AU in separation distance such as Proxima Centauri. Those configurations are captured stars during the star nursery period, in which the neighboring nurturing stars are just a few thousands AU units away. Furthermore, for data observed majority of the captured stars in wide orbits are red dwarves, which is no surprise as the most commonly formed stars and most frequently captured. Therefore, binary star formation with comparable separation and distance from their host star posed a challenge because their host star to their binary pairs' mass ratio dramatically falls below the threshold for the planetary mass budget. Upon close examination, however, the formation and mechanism of binary and multiple stars system are radically different from that of the planets. They occur at different stages of star formation. The formation of star pairs occurs in the very early stage of stellar formation, all within the first million years or earlier and driven by disc fragmentation. [4][3][2] While the formation of the planets, the remnants leftover from the stellar disc, occurs more slowly over the course of hundreds of millions of years throughout the T-Tauri Star phase until the star enters its main sequence.

4 Conclusion

It is surprising that after the discovery of the law of universal gravitation, it takes more than two hundred years to discover a relationship of the ratio regarding the primary mass and the total mass of its satellite. However, the derivation of this law was not a necessity since no exoplanets data were available and the concept of planets beyond the solar system was simply speculative. Furthermore, this law distinguishes from the classical law of gravitation and general theory of relativity by assessing the long-term trend from millions of years of gravitational interactions of n bodies during the accretion phase of the solar system instead of extreme precision and description of two body interaction at the present.

Based on this law, red dwarf rarely hosts Jupiter sized planets because its stellar mass less than 0.4 solar mass has a mean planetary mass budget merely 32.37% of solar mass stars. It is merely $116.8\pm^{252.55}_{79.86}$ earth masses. If one takes 0.2 solar mass as the mean stellar mass for red dwarves, then, the expected mean planetary mass budget is $49.72\pm^{107.50}_{33.997}$ earth masses, or only as the combined mass of Uranus and two Neptunes in the solar system. Because extreme values lie up to 3 standard deviations above the mean, gas giants revolve around red dwarves are still possible. However, it is no more likely in 17% of the systems, or only 1 in 6 red dwarf system hosts gas giants, and their gas giants size is only comparable to the mass of Saturn at the most.

Exomoons at the size of earth or greater require its hosting planet at the size of 8.78 Jovian mass (2,793.87 earth mass, 0.839% solar mass) or above.

$$\begin{aligned} y &= 0.00839 \cdot M_{sol} \left(17,520 \left(0.00839 \cdot M_{sol}\right)^{-0.2315}\right)^{-1} \\ &= 1.00108009577 \, \mathrm{M_{earth}} \end{aligned} \tag{4.1}$$

A planetary system with more than 8.7857 Jovian masses (2,793 $\mathrm{M_{earth}}$) implies a stellar mass at least 5.2694 solar masses or above.

$$\begin{aligned} y &= 5.2694 \cdot M_{sol} \left(17,520 \left(5.2694 \cdot M_{sol}\right)^{-0.2315}\right)^{-1} \\ &= 2,793.86134545 \, \mathrm{M_{earth}} \end{aligned} \tag{4.2}$$

Based on stellar evolution model, a star with 5.2694 solar mass or above devolve from the main sequence in just 156.89 million years or less.

$$T_{MS} \approx 10^{10} \left[\frac{M}{M_{sol}}\right] \left[\frac{L_{sol}}{L}\right] = 10^{10} \left[\frac{M}{M_{sol}}\right]^{-2.5} \tag{4.3}$$

The timeframe is not adequate for the development of complex multicellular life. (at least a few billions of years in preparation for cyanobacteria to create extra oxygen in the atmosphere and ocean for the emergence of eukaryotic cells and their multicellular descents.) Therefore, the science fiction movie Avatar's planet Pandora is not a realistic description of the physical

reality of the universe.
Furthermore, deriving values from the equation, the total budget for each planetary system for terrestrial planets is very limited. For solar mass systems with a metallicity of 0, there is enough budget to create at most $1.66\pm^{3.589}_{1.135}$ earth mass pure carbon planet, and $0.2346\pm^{0.507}_{0.1604}$ earth mass silicon planet. Finally, the water content for each planetary system is very limited. For a system with a metallicity of 0 and solar mass, the entire system has an upper bound of just $6.107\pm^{13.205}_{4.176}$ earth mass. The system has an upper bound of only $1.897\pm^{4.101}_{1.297}$ earth mass worth of water if oxygen is counted toward the composition of terrestrial planet creation as 47% of earth's mass composed of oxygen.
With lower metallicity, the water budget can be much lower, even with much higher metallicity, the upper bound of the water budget for the system is less than $24.13\pm^{52.18}_{16.499}$ earth masses. However, studies done on the process of terrestrial planet creation indicates that stellar systems with high metallicities are destroyed by migrating hot Jupiters early in its formational period. As a result, the upper bound of the total water budget in any extraterrestrial system with surviving terrestrial planets is limited to a mean of $10\pm^{21.622}_{6.838}$ earth mass or below.
Tau Ceti, one of the closest star to the Sun with just 12 light years in distance has confirmed 4 planets g, h, e, and f with mass 1.75, 1.83, 3.93, 3.93 earth mass respectively, their combined mass of 11.44 earth mass. Currently, no one is able to detect the composition of the planets, by using our planetary mass budget equation, one can make certain conclusion despite unobservability at the current time. Tau Ceti has 0.783 solar mass and 28% solar metallicity, and then one arrives at the following value:

$$\begin{aligned} y_{ceti} &= 0.783 \cdot M_{sol} \left(17,520\,(0.783 \cdot M_{sol})^{-0.2315}\right)^{-1} \\ &= 267.007574532\ \mathrm{M_{earth}} \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\left(\frac{5.3528}{360.87}\right) \cdot y_{ceti} \cdot 0.28 \\ &= 1.10895596076\ \mathrm{M_{earth}} \end{aligned} \tag{4.5}$$

Whereas 360.87 $\mathrm{M_{earth}}$ is the total planetary budget of the solar mass star around the mean of its distribution. 5.3528 $\mathrm{M_{earth}}$ is the terrestrial planetary budget including oxygen for a solar mass star.
That is, the total expected mean combined terrestrial planetary mass should be 1.108 earth mass. Given the probabilistic distribution of planetary mass, the chance of Tau Ceti hosting combined terrestrial planetary mass over 11.08 earth mass is two standard deviations from the mean, that is, 1.5% or less. Therefore, Tau Ceti's planets within its habitable zone have 98.5% chance being mini Neptunes with tiny rocky terrestrial cores shrouded with an extremely thick layer of hydrogen and helium.

The empirical law for planetary mass budget holds an excellent promise for extraterrestrial studies since it sets the upper limit on the combined mass of any systems, which together with metallicity, determines the total budget for the terrestrial planets, their moons, the total water, nitrogen, and CO_2 availability. For future works, as more exoplanet data becomes available, one can continue to refine the parameter of the model to make more precise predictions regarding exoplanets, especially those of the red dwarves, which are outliers in our model in our current regression analysis.

5 Appendix

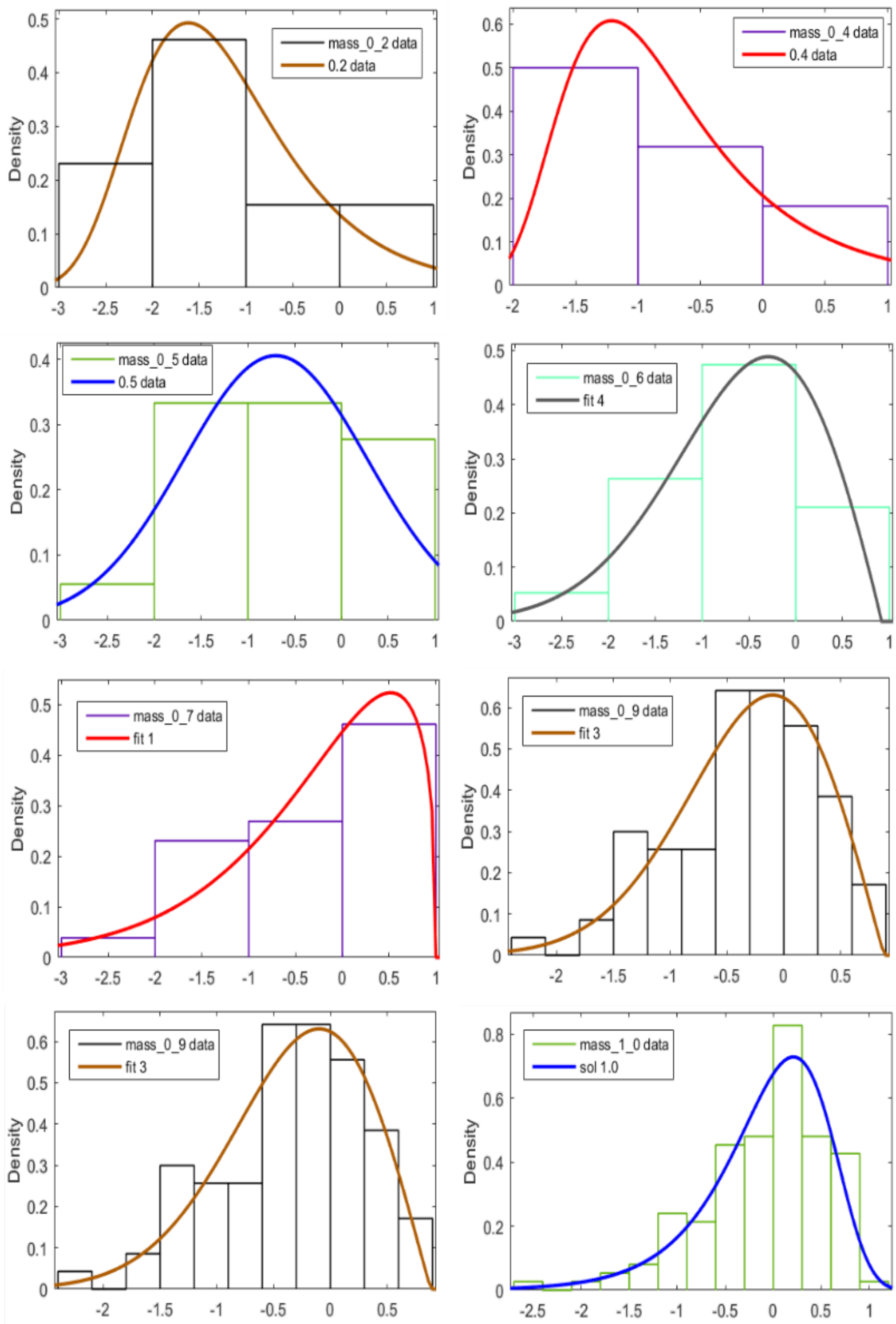

Figure 5.1: Plots for PDF of 0.2 ~ 1.0 solar masses

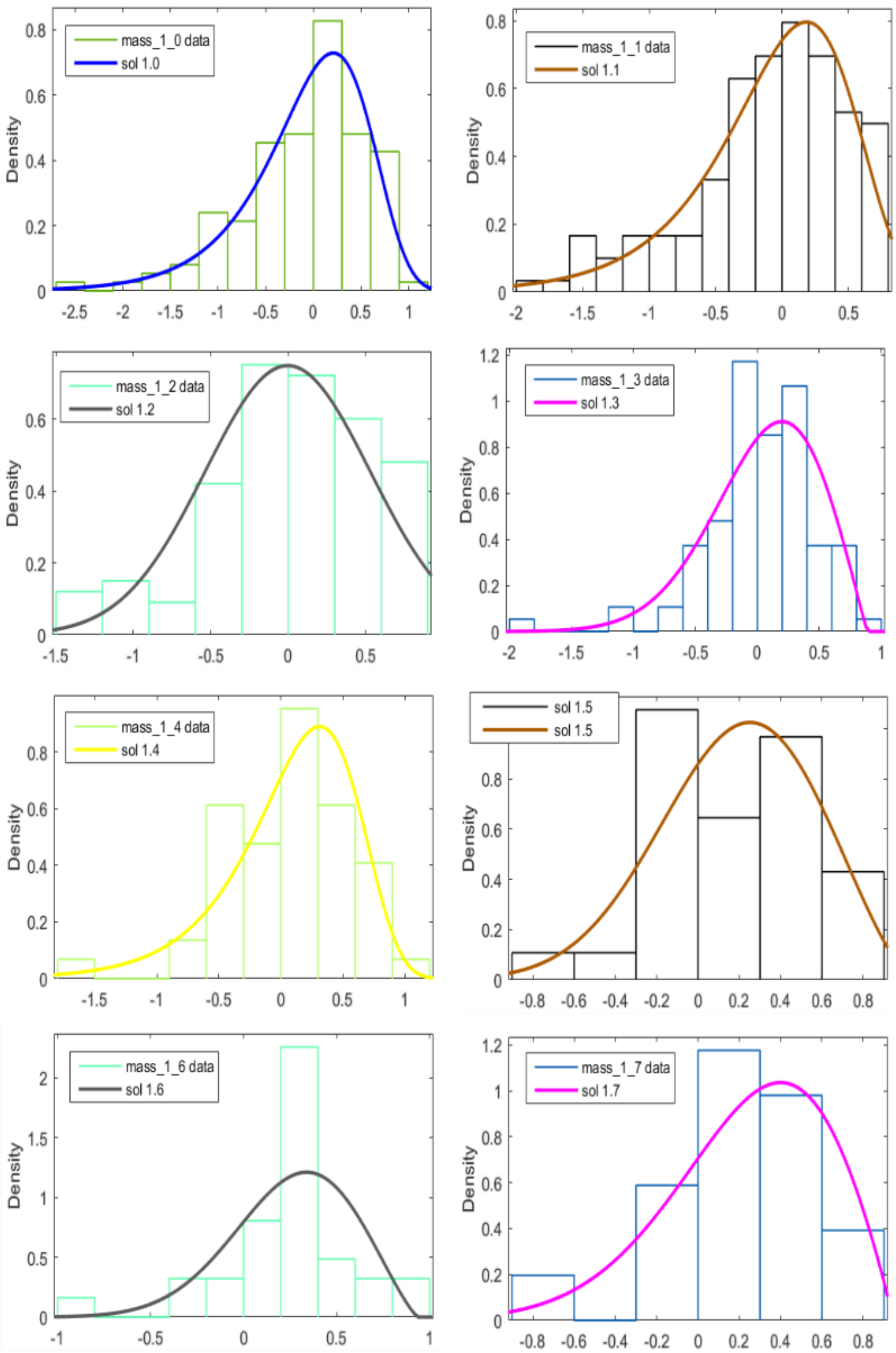

Figure 5.2: Plots for PDF of 1.0 ~ 1.7 solar masses

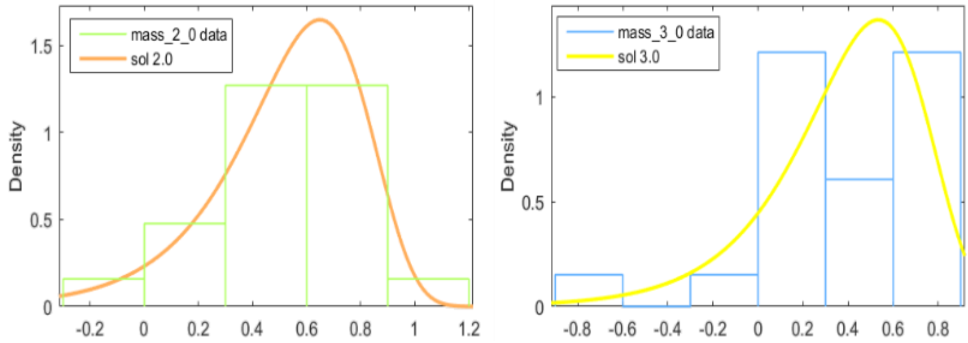

Figure 5.3: Plots for PDF of 2.0 ~ 3.0 solar masses

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