Fibonacci Oscillators and (p, q)-deformed Lorentz Transformations

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Abstract

The two-parameter quantum calculus used in the construction of Fibonacci oscillators is briefly reviewed before presenting the (p, q)-deformed Lorentz transformations which leave invariant the Minkowski spacetime interval $t^2 - x^2 - y^2 - z^2$. Such transformations require the introduction of three different types of exponential functions leading to the (p, q)-analogs of hyperbolic and trigonometric functions. The composition law of two successive Lorentz boosts (rotations) is *no* longer additive $\xi_3 \neq \xi_1 + \xi_2$ ($\theta_3 \neq \theta_1 + \theta_2$). We finalize with a discussion on quantum groups, noncommutative spacetimes, κ-deformed Poincare algebra and quasi-crystals.

Keywords : Fibonacci Oscillators; Quantum Groups; Golden Mean, Noncommutative Geometry.

1 The Fibonacci and (p, q) Oscillators

Before embarking into a discussion of the Fibonacci and (p, q) oscillators we shall follow closely the definitions and results in [8] where many references can be found. The (p, q) number is defined for any number n as

$$[n]_{p,q} = [n]_{q,p} \equiv \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1} \tag{1.1}$$

which is a natural generalization of the q-number

$$[n]_q \equiv \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-2} + q^{n-1} \tag{1.2}$$

The (p,q)-derivative of a function $f(x)$ is

$$D_{p,q}f(x) \equiv \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0 \tag{1.3}$$

A very important function is the (p,q)-Gauss Binomial defined by

$$(x\oplus y)^n_{p,q} = (x+y)\,(px+qy)\,(p^2x+q^2y)\,\ldots\,(p^{n-2}x+q^{n-2}y)\,(p^{n-1}x+q^{n-1}y), \quad n \geq 1 \tag{1.4}$$

$$(x\oplus y)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\, q^{(n-k)(n-k-1)/2}\, x^k\, y^{n-k} \tag{1.5}$$

$(x\oplus y)^n = 1$, for $n = 0$, and the (p,q)-Gauss Binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \equiv \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \quad n \geq k \tag{1.6}$$

$$[n]_{p,q}! = [n]_{p,q}\,[n-1]_{p,q}\,[n-2]_{p,q}\,\ldots\,[2]_{p,q}\,[1]_{p,q}, \quad n \in N \tag{1.7}$$

There are three types of (p,q)-exponential functions

$$e_{p,q}(x) \equiv \sum_{n=0}^{\infty} p^{n(n-1)/2}\,\frac{x^n}{[n]_{p,q}!} \tag{1.8}$$

$$E_{p,q}(x) \equiv \sum_{n=0}^{\infty} q^{n(n-1)/2}\,\frac{x^n}{[n]_{p,q}!} \tag{1.9}$$

$$\widetilde{e_{p,q}}(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!} \tag{1.10}$$

which satisfy the basic identities

$$e_{p,q}(x)\,E_{p,q}(y) = \widetilde{e_{p,q}}(x\oplus y) = \sum_{n=0}^{\infty}\frac{(x\oplus y)^n_{p,q}}{[n]_{p,q}!}, \quad e_{p,q}(x)\,E_{p,q}(-x) = 1 \tag{1.11}$$

$$e_{\frac{1}{p},\frac{1}{q}}(x) = E_{p,q}(x), \quad E_{\frac{1}{p},\frac{1}{q}}(x) = e_{p,q}(x), \tag{1.12}$$

The (p,q) hyperbolic functions are defined by

$$sinh_{p,q}(x) = \frac{e_{p,q}(x)-e_{p,q}(-x)}{2}, \quad SINH_{p,q}(x) = \frac{E_{p,q}(x)-E_{p,q}(-x)}{2},$$

$$\widetilde{sinh}_{p,q}(x) = \frac{\widetilde{e_{p,q}}(x)-\widetilde{e_{p,q}}(-x)}{2} \tag{1.13}$$

$$cosh_{p,q}(x) = \frac{e_{p,q}(x)+e_{p,q}(-x)}{2}, \quad COSH_{p,q}(x) = \frac{E_{p,q}(x)+E_{p,q}(-x)}{2},$$

$$\widetilde{cosh}_{p,q}(x) \;=\; \frac{\widetilde{e_{p,q}}(x) + \widetilde{e_{p,q}}(-x)}{2} \tag{1.14}$$

In particular, they obey the key identity

$$cosh_{p,q}(x)\; COSH_{p,q}(x) \;-\; sinh_{p,q}(x)\; SINH_{p,q}(x) \;=\; 1 \tag{1.15}$$

Similar definitions hold for the trigonometric functions which obey

$$cos_{p,q}(x)\; COS_{p,q}(x) \;+\; sin_{p,q}(x)\; SIN_{p,q}(x) \;=\; 1 \tag{1.16}$$

For further details we refer to [8].

When p, q are given by the Golden Mean, and its Galois conjugate, respectively

$$p \;=\; \tau \;=\; \frac{1+\sqrt{5}}{2}, \quad q \;=\; -\; 1/\tau \;=\; \frac{1-\sqrt{5}}{2} \tag{1.17}$$

the p, q numbers $[n]_{p,q}$ coincide precisely with the Fibonacci numbers as a result of Binet's formula

$$[n]_{p,q} \;=\; [n]_{q,p} \;\equiv\; \frac{\tau^n \;-\; (-1)^n \tau^{-n}}{\sqrt{5}} \;=\; F_n \tag{1.18}$$

Furthermore, the powers of τ^n and τ^{-n} can be expressed themselves in terms of τ and the Fibonacci numbers as follows

$$\tau^n \;=\; F_{n+1} \;+\; \frac{F_n}{\tau}, \quad \tau^{-n} \;=\; (-1)^n\; F_{n-1} \;+\; (-1)^{n+1}\; \frac{F_n}{\tau} \tag{1.19}$$

consequently, the powers of τ are just Dirichlet integers which have the form $m + n\sqrt{5}$, with m, n integers, and the (p, q)-factorial

$$[n]_{p,q}! \;=\; F_n\; F_{n-1}\; F_{n-2}\; \ldots\ldots \tag{1.20}$$

becomes a product of descending Fibonacci numbers. Therefore, all the numerical factors which define the hyperbolic and trigonometric (p, q)-functions will simplify enormously in this special case (1.17).

It has been emphasized by [7] that the Fibonacci oscillators offer a unification of quantum oscillators related to quantum groups [2]. They are the most general oscillators having the property of spectrum degeneracy and invariance under the quantum group. One of the main problems in the theory of quantum groups and algebras is to interpret the physical meaning of the deformation parameters [7]. In this respect, one possible explanation for the deformation parameters was accomplished by a relativistic quantum mechanical model [2]. In such a model, the multi-dimensional Fibonacci oscillator can be interpreted as a relativistic oscillator corresponding to the bound state of two particles with masses m_1, m_2. Therefore, the additional parameter has a physical significance so that it can be related to the mass of the second bosonic particle in a two particle relativistic quantum harmonic oscillator bound state. Although, any quantum algebra with one or more deformation parameters may be mapped onto the standard single-parameter case

[3], it has been argued that the physical results obtained from a two-parameter deformed oscillator system are not the same [4].

An early (p,q) oscillator realization (a la Jordan-Schwinger) of two parameter quantum algebras, $su_{p,q}(2); su_{p,q}(1,1); osp_{p,q}(2|1)$, and the centerless Virasoro algebra was constructed by [5]. Given the creation $A^\dagger$ and annihilation A operators, the spectrum was found to obey

$$AA^\dagger \ = \ [N+1]_{p,q}, \ \ A^\dagger A \ = \ [N]_{p,q}, \ \ [N,\ A] \ = \ -A, \ \ [N,\ A^\dagger] \ = \ A^\dagger \tag{1.21}$$

$$AA^\dagger \ - \ q\ A^\dagger A \ = \ p^N, \ \ AA^\dagger \ - \ p\ A^\dagger A \ = \ q^N \tag{1.22}$$

Furthermore, $[n]_{p,q}$ is the unique solution of the generalized Fibonacci recursion relation [5]

$$[n+1]_{p,q} \ = \ (p+q)\ [n]_{p,q} \ - \ pq\ [n]_{p,q}, \ \ [1]_{p,q} = 1, \ \ [0]_{p,q} = 0, \ \ n \geq 1 \tag{1.23}$$

when $p = \tau, \ q = -\tau^{-1}$, the above equation (1.23) reduces to the standard recursion relation of the Fibonacci numbers [1] $F_{n+1} = F_n + F_{n-1}$. When $q = p(-p)$ the relations (1.22) reduce to the (anti) commutation relations of bosonic (fermionic) q-oscillators. The special case $(q = 0, p \neq 0)$, or $(q \neq 0, p = 0)$ gives a deformation of a single mode of the oscillators exhibiting "infinite statistics" [6]. These hypothetical particles of "infinite-statistics" were coined *quons*. The (p,q) analogs of the fermionic, parafermionic and parabosonic oscillators were also identified [5].

A generating function for the (p,q)-numbers $[n]_{p,q}$ is [5]

$$\sum_{n=0}^{\infty} \ [n]_{p,q}\ z^n \ = \ \frac{z}{(1-qz)\ (1-pz)} \tag{1.24}$$

The $\widetilde{e_{p,q}}(z)$ exponential allows to construct the (p,q)-coherent states, for z complex :

$$|z>_{p.q} \ = \ N(z)\ \widetilde{e_{p,q}}(zA^\dagger)\ |0>, \ \ N(z) \ = \ \frac{1}{\sqrt{\widetilde{e_{p,q}}(|z|^2)}}, \tag{1.24}$$

The inner product is [5]

$$< z_1|z_2 > \ = \ N(z_1)\ N(z_2)\ \widetilde{e_{p,q}}(\bar{z}_1 z_2) \tag{1.25}$$

The non-extensive Tsallis entropy of bosonic Fibonacci oscillators was studied in [7] where connections between the thermo-statistical properties of a gas of the two-parameter deformed bosonic particles called Fibonacci oscillators and the properties of the Tsallis thermostatistics was found. It was shown that the thermo-statistics of the two-parameter deformed bosons can be studied by the formalism of Fibonacci calculus.

[1] The Sanskrit poets Virahanka, Hemachandra, Gopala many centuries before Fibonacci were aware of these numbers. See the Fields Institute Lectures on "Patterns of Numbers in Nature" by Manjul Bhargava

Having presented this brief tour of the (p,q)-oscillator and its connection to the generalized Fibonacci recursion relations we shall proceed with the explicit construction of (p,q)-Lorentz transformations and its role in deformations of Special Relativity.

2 (p,q)-Lorentz transformations

The (p,q)-Lorentz boost transformations along the x-direction in $4D$ that we propose are given by

$$t' = t\sqrt{cosh_{p,q}(\xi)\ COSH_{p,q}(\xi)} - x\sqrt{sinh_{p,q}(\xi)\ SINH_{p,q}(\xi)} \tag{2.1}$$

$$x' = x\sqrt{cosh_{p,q}(\xi)\ COSH_{p,q}(\xi)} - t\sqrt{sinh_{p,q}(\xi)\ SINH_{p,q}(\xi)} \tag{2.2}$$

$$y' = y, \quad z' = z \tag{2.3}$$

due to the identity

$$cosh_{p,q}(\xi)\ COSH_{p,q}(\xi) - sinh_{p,q}(\xi)\ SINH_{p,q}(\xi) = 1 \tag{2.4}$$

it follows that under (p,q)-Lorentz transformations the Minkowski spacetime interval remains invariant

$$(t')^2 - (x')^2 - (y')^2 - (z')^2 = (t)^2 - (x)^2 - (y)^2 - (z)^2 \tag{2.5}$$

Because

$$(\widetilde{cosh}_{p,q}(A))^2 - (\widetilde{sinh}_{p,q}(A))^2 \neq 1 \tag{2.6}$$

the (p,q)-Lorentz transformations do *not* have the form

$$t' = t\ \widetilde{cosh}_{p,q}(\xi) - x\ \widetilde{sinh}_{p,q}(\xi) \tag{2.7a}$$

$$x' = x\ \widetilde{cosh}_{p,q}(\xi) - t\ \widetilde{sinh}_{p,q}(\xi) \tag{2.7b}$$

but must have the form indicated by eqs-(2.1-2.2). Therefore,

$$t' \neq t\ \widetilde{cosh}_{p,q}(\xi) - x\ \widetilde{sinh}_{p,q}(\xi) \tag{2.8a}$$

$$x' \neq x\ \widetilde{cosh}_{p,q}(\xi) - t\ \widetilde{sinh}_{p,q}(\xi) \tag{2.8b}$$

The composition law of two successive (p,q)-Lorentz transformations with boost parameters ξ_1, ξ_2 is given by an ordinary matrix product leading to

$$t'' = t\sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} +$$
$$t\sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} -$$
$$x\sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} -$$

$$x \sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} \tag{2.9}$$

$$x'' = x \sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} + x \sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} - t \sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} - t \sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} \tag{2.10}$$

$$y'' = y, \quad z'' = z \tag{2.11}$$

If the above composition is consistent with a group composition law, one should have

$$t'' = t \sqrt{cosh_{p,q}(\xi_3)\ COSH_{p,q}(\xi_3)} - x \sqrt{sinh_{p,q}(\xi_3)\ SINH_{p,q}(\xi_3)} \tag{2.12}$$

$$x'' = x \sqrt{cosh_{p,q}(\xi_3)\ COSH_{p,q}(\xi_3)} - t \sqrt{sinh_{p,q}(\xi_3)\ SINH_{p,q}(\xi_3)} \tag{2.13}$$

$$y'' = y, \quad z'' = z \tag{2.14}$$

where the resulting boost parameter ξ_3 is now a *complicated* function $\xi_3(\xi_1, \xi_2)$ of ξ_1 and ξ_2 as shown below. It will no longer be given by the naive addition law $\xi_1 + \xi_2$. Once again, from eqs-(2.12-2.14) one can show the invariance of the Minkwoski spacetime interval

$$(t'')^2 - (x'')^2 - (y'')^2 - (z'')^2 = (t)^2 - (x)^2 - (y)^2 - (z)^2 \tag{2.15}$$

Equating eqs-(2.9, 2.10) with eqs-(2.12, 2.13) yields

$$\sqrt{sinh_{p,q}(\xi_3)\ SINH_{p,q}(\xi_3)} = \sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} + \sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} \tag{2.16}$$

$$\sqrt{cosh_{p,q}(\xi_3)\ COSH_{p,q}(\xi_3)} = \sqrt{cosh_{p,q}(\xi_2)\ COSH_{p,q}(\xi_2)\ cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_1)} + \sqrt{sinh_{p,q}(\xi_2)\ SINH_{p,q}(\xi_2)\ sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_1)} \tag{2.17}$$

Dividing eq-(2.16) by eq-(2.17) gives in the left hand side : $\sqrt{tanh_{p,q}(\xi_3)\ TANH_{p,q}(\xi_3)}$. As a result of the identities [8]

$$tanh_{p,q}(A) = TANH_{p,q}(A) \Leftrightarrow sinh_{p,q}(A)\ COSH_{p,q}(A) = cosh_{p,q}(A)\ SINH_{p,q}(A) \tag{2.18}$$

this left-hand side becomes

$$\sqrt{tanh_{p,q}(\xi_3)\ TANH_{p,q}(\xi_3)} = tanh_{p,q}(\xi_3) = TANH_{p,q}(\xi_3) \tag{2.19}$$

The right-handside is of the form

$$\frac{A + B}{C + D} = \frac{(A/C) + (B/C)}{1 + (D/C)} \tag{2.20}$$

where A, B, C, D are the square roots of products of four hyperbolic functions. Due to the identities (2.18) it allows to eliminate the square roots in eq-(2.20), and finally one arrives at

$$tanh_{p,q}(\xi_3) = \frac{tanh_{p,q}(\xi_1) + tanh_{p,q}(\xi_2)}{1 + tanh_{p,q}(\xi_1)\ tanh_{p,q}(\xi_2)} = TANH_{p,q}(\xi_3) =$$

$$\frac{TANH_{p,q}(\xi_1) + TANH_{p,q}(\xi_2)}{1 + TANH_{p,q}(\xi_1)\ TANH_{p,q}(\xi_2)}, \quad \xi_3 \neq \xi_1 + \xi_2 \tag{2.21}$$

It remains to explain that when $(p, q) \neq (1, 1) \Rightarrow \xi_3 \neq \xi_1 + \xi_2$. Therefore, the composition rule for the boost parameters is *no* longer additive. The reason behind this is because now the actual addition laws for the (p, q)-hyperbolic functions are of the form

$$\widetilde{cosh}_{p,q}(\xi_1 \oplus \xi_2) = cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_2) + sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_2) \tag{2.22}$$

$$\widetilde{sinh}_{p,q}(\xi_1 \oplus \xi_2)) = sinh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_2) + cosh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_2) \Rightarrow \tag{2.23}$$

$$\widetilde{tanh}_{p,q}(\xi_1 \oplus \xi_2) = \frac{sinh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_2) + cosh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_2)}{cosh_{p,q}(\xi_1)\ COSH_{p,q}(\xi_2) + sinh_{p,q}(\xi_1)\ SINH_{p,q}(\xi_2)} =$$

$$\frac{tanh_{p,q}(\xi_1) + TANH_{p,q}(\xi_2)}{1 + tanh_{p,q}(\xi_1)\ TANH_{p,q}(\xi_2)} \tag{2.24}$$

The functions $\widetilde{cosh}_{p,q}(\xi_1 \oplus \xi_2), \widetilde{sinh}_{p,q}(\xi_1 \oplus \xi_2), \widetilde{tanh}_{p,q}(\xi_1 \oplus \xi_2)$ admit a power series expansion in terms of the (p, q)-Gauss binomial $(\xi_1 \oplus \xi_2)^n_{p,q}$, and defined by eqs-(1.4,1.5).

Due to the identity $tanh_{p,q}(A) = TANH_{p,q}(A)$, one can see that the expressions in eqs-(2.21, 2.24) are both the same and one ends up with

$$\widetilde{tanh}_{p,q}(\xi_1 \oplus \xi_2) = tanh_{p,q}(\xi_3) = TANH_{p,q}(\xi_3) =$$

$$\frac{tanh_{p,q}(\xi_1) + tanh_{p,q}(\xi_2)}{1 + tanh_{p,q}(\xi_1)\ tanh_{p,q}(\xi_2)} =$$

$$\frac{TANH_{p,q}(\xi_1) + TANH_{p,q}(\xi_2)}{1 + TANH_{p,q}(\xi_1)\ TANH_{p,q}(\xi_2)} \tag{2.25a}$$

Because of the following *inequalities*

$$\widetilde{tanh}_{p,q}(A) \neq tanh_{p,q}(A) = TANH_{p,q}(A) \tag{2.25b}$$

one learns that

$$\xi_3 \neq \xi_1 \oplus \xi_2 = \xi_1 + \xi_2 \tag{2.25c}$$

this last inequality in (2.26b) can be deduced by a simple inspection of the equalities in eq-(2.25). Since the function $\widetilde{tanh}_{p,q}$ appearing in the first term of eq-(2.25a) is *not* the same as $tanh_{p,q}$, and $TANH_{p,q}$, the argument ξ_3 *cannot* be the same as the argument $\xi_1 \oplus \xi_2 = \xi_1 + \xi_2$. Therefore, when $(p,q) \neq (1,1) \Rightarrow \xi_3 \neq \xi_1 + \xi_2$. It is *only* when $p = q = 1$ that the boost parameters are additive $\xi_3 = \xi_1 + \xi_2$.

Concluding, the complicated expression for $\xi_3 = \xi_3(\xi_1, \xi_2)$ is explicitly given by evaluating the $arctanh_{p,q}, ARCTANH_{p,q}$ of the right hand side of eqs-(2.25), respectively. Both results lead to the same ξ_3

$$\xi_3 \;=\; arctanh_{p,q} \left(\frac{tanh_{p,q}(\xi_1) \;+\; tanh_{p,q}(\xi_2)}{1 \;+\; tanh_{p,q}(\xi_1)\; tanh_{p,q}(\xi_2)} \right) \tag{2.26a}$$

$$\xi_3 \;=\; ARCTANH_{p,q} \left(\frac{TANH_{p,q}(\xi_1) \;+\; TANH_{p,q}(\xi_2)}{1 \;+\; TANH_{p,q}(\xi_1)\; TANH_{p,q}(\xi_2)} \right) \tag{2.26b}$$

Furthermore, because

$$(\widetilde{cosh}_{p,q}(A))^2 \;-\; (\widetilde{sinh}_{p,q}(A))^2 \;\neq\; 1 \tag{2.26c}$$

a careful inspection of eqs-(2.8) reveals that

$$\widetilde{sinh}_{p,q}(\xi_1 \oplus \xi_2)) \;\neq\; \sqrt{sinh_{p,q}(\xi_3)\; SINH_{p,q}(\xi_3)} \tag{2.27a}$$

$$\widetilde{cosh}_{p,q}(\xi_1 \oplus \xi_2) \;\neq\; \sqrt{cosh_{p,q}(\xi_3)\; COSH_{p,q}(\xi_3)} \tag{2.27b}$$

bur their ratio is *equal* : $\widetilde{tanh}_{p,q}(\xi_1 \oplus \xi_2) = tanh_{p,q}(\xi_3) = TANH_{p,q}(\xi_3)$.

From eqs-(2.25) one can derive the addition law of velocities as in ordinary Special Relativity. Given

$$\beta_1 \;\equiv\; \frac{v_1}{c} \;\equiv\; tanh_{p,q}(\xi_1) \;=\; TANH_{p,q}(\xi_1) \tag{2.28a}$$

$$\beta_2 \;\equiv\; \frac{v_2}{c} \;\equiv\; tanh_{p,q}(\xi_2) \;=\; TANH_{p,q}(\xi_2) \tag{2.28b}$$

$$\beta_3 \;\equiv\; \frac{v_3}{c} \;\equiv\; tanh_{p,q}(\xi_3) \;=\; TANH_{p,q}(\xi_3), \quad \xi_3 \neq \xi_1 + \xi_2 \tag{2.28c}$$

the addition law is

$$\beta_3 \;=\; \frac{\beta_1 \;+\; \beta_2}{1 \;+\; \beta_1\; \beta_2} \tag{2.29}$$

similarly one can obtain the subtraction law

$$\beta_3 \;=\; \frac{\beta_1 \;-\; \beta_2}{1 \;-\; \beta_1\; \beta_2} \tag{2.30}$$

such that β_3 never exceeds 1 when $\beta_1, \beta_2 \leq 1$.

So far we have studied the (p,q)-Lorentz boosts transformations. A (p,q)-rotation transformation along the z-direction gives

$$x' \; = \; x \; \sqrt{cos_{p,q}(\theta) \; COS_{p,q}(\theta)} \; - \; y \; \sqrt{sin_{p,q}(\theta) \; SIN_{p,q}(\theta)} \tag{2.31a}$$

$$y' \; = \; y \; \sqrt{cos_{p,q}(\theta) \; COS_{p,q}(\theta)} \; + x \; \sqrt{sin_{p,q}(\theta) \; SIN_{p,q}(\theta)} \tag{2.31b}$$

$$t' \; = \; t, \;\; z' \; = \; z \tag{2.31c}$$

and leaves invariant the Minkowski spacetime line interval (2.5) due to the identity

$$cos_{p,q}(\theta) \; COS_{p,q}(\theta) \; + \; sin_{p,q}(\theta) \; SIN_{p,q}(\theta) \; = \; 1 \tag{2.32}$$

The following relations among hyperbolic and trigonometric (p,q) functions [8]

$$sinh_{p,q}(x) \; = \; -i \; sin_{p,q}(ix), \;\; SINH_{p,q}(x) \; = \; -i \; SIN_{p,q}(ix), \;\; \widetilde{sinh}_{p,q}(x) \; = \; -i \; \widetilde{sin}_{p,q}(ix), \tag{2.33}$$

$$cosh_{p,q}(x) \; = \; cos_{p,q}(ix), \;\; COSH_{p,q}(x) \; = \; COS_{p,q}(ix), \;\; \widetilde{cosh}_{p,q}(x) \; = \; \widetilde{cos}_{p,q}(ix), \tag{2.34}$$

will allow to evaluate the composition rule for two successive rotations with angles θ_1, θ_2 about the z-axis. The composition rule for the angles is

$$\widetilde{tan}_{p,q}(\theta_1 \oplus \theta_2) \; = \; tan_{p,q}(\theta_3) \; = \; TAN_{p,q}(\theta_3) \; = \; \frac{tan_{p,q}(\theta_1) \; + \; tan_{p,q}(\theta_2)}{1 \; - \; tan_{p,q}(\theta_1) \; tan_{p,q}(\theta_2)} \; = \; \frac{TAN_{p,q}(\theta_1) \; + \; TAN_{p,q}(\theta_2)}{1 \; - \; TAN_{p,q}(\theta_1) \; TAN_{p,q}(\theta_2)} \tag{2.35}$$

where $\theta_3 \neq \theta_1 \oplus \theta_2 = \theta_1 + \theta_2$.

The composition law of two succesive (p,q)-Lorentz boosts transformations along two *different* axis directions are more complicated. The same occurs with a (p,q)-Lorentz boost transformation along any arbitrary direction. In general, the ordinary Lorentz transformations can be written in terms of the Pauli spin 2×2 matrices $\sigma_1, \sigma_2, \sigma_3$, and the unit matrix $\mathbf{1}$ as follows. Let us firstly define the 2×2 matrix

$$\mathbf{X} \; \equiv \; x^\mu \; \sigma_\mu \; = \; t \; \mathbf{1} \; + \; x \; \sigma_1 \; + \; y \; \sigma_2 \; + \; z \; \sigma_3 \; = \; \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{2.36}$$

One can show that an ordinary Lorentz boost with parameter ξ along *any* direction can be realized in terms of three parameters defined as

$$\vec{\xi} \; = \; (\xi_1, \xi_2, \xi_3); \;\; \xi \; \equiv \; ||\vec{\xi}|| \; = \; \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2} \tag{2.37}$$

and associated with the three directions x, y, z, respectively. The Lorentz boost in this general case is

$$\mathbf{X}' \; = \; exp(\; \frac{\xi_1}{2}\sigma_1 + \frac{\xi_2}{2}\sigma_2 + \frac{\xi_3}{2}\sigma_3 \;) \; \mathbf{X} \; exp(\; -\frac{\xi_1}{2}\sigma_1 - \frac{\xi_2}{2}\sigma_2 - \frac{\xi_3}{2}\sigma_3 \;) \tag{2.38}$$

Due to $exp(\mathbf{A})\ exp(-\mathbf{A}) = \mathbf{1}$, and because the determinant of a product of matrices is equal to the product of the determinants of the matrices, one then has

$$det(\mathbf{X}') \ = \ det[exp(\mathbf{A})]\ det(\mathbf{X})\ det[exp(-\mathbf{A})] \ = \ det(\mathbf{X}) \ =$$

$$t'^2 \ - \ x'^2 \ - \ y'^2 \ - \ z'^2 \ = \ t^2 \ - \ x^2 \ - \ y^2 \ - \ z^2 \qquad (2.39)$$

so that the transformations (2.38) leave the Minkowski spacetime interval invariant as expected. Given the unit vector

$$\hat{\xi} \ \equiv \ (\frac{\xi_1}{\xi}, \frac{\xi_2}{\xi}, \frac{\xi_3}{\xi}),\ \hat{\xi}^i\ \hat{\xi}_i \ = \ 1 \ \Rightarrow \ (\hat{\xi}^i \sigma_i)\ (\hat{\xi}^j \sigma_j) \ = \ \hat{\xi}^i\ \hat{\xi}^j\ (\delta_{ij} \mathbf{1} + i\ \epsilon_{ijk} \sigma_k) \ = \ \hat{\xi}^i\ \hat{\xi}^j \delta_{ij}\ \mathbf{1} \ = \ \mathbf{1},$$

$$(\hat{\xi}^i \sigma_i)\ (\hat{\xi}^j \sigma_j)\ (\hat{\xi}^k \sigma_k) \ = \ (\hat{\xi}^k \sigma_k), \ \ (\hat{\xi}^i \sigma_i)^{2n} \ = \ \mathbf{1}, \ \ (\hat{\xi}^i \sigma_i)^{2n+1} \ = \ \hat{\xi}^i \sigma_i \qquad (2.40)$$

upon performing a Taylor series expansion one arrives at

$$exp(\ \frac{\xi_1}{2}\sigma_1 + \frac{\xi_2}{2}\sigma_2 + \frac{\xi_3}{2}\sigma_3 \ ) \ = \ cosh(\frac{\xi}{2})\ \mathbf{1} \ + \ \hat{\xi}^i \sigma_i\ sinh(\frac{\xi}{2}) \qquad (2.41a)$$

$$exp(\ -\frac{\xi_1}{2}\sigma_1 - \frac{\xi_2}{2}\sigma_2 - \frac{\xi_3}{2}\sigma_3 \ ) \ = \ cosh(\frac{\xi}{2})\ \mathbf{1} \ - \ \hat{\xi}^i \sigma_i\ sinh(\frac{\xi}{2}) \qquad (2.41b)$$

and after evaluating the matrix product (2.38) one can read-off the expressions for t', x', y', z' in terms of t, x, y, z and the boost parameters.

Guided by the above construction, a (p,q)-Lorentz boost along *any* direction can be realized in terms of the (p,q) deformed Pauli spin algebra generators $\sigma_i^{(p,q)}$, and the (p,q) exponentials (1.8-1.10) as follows [2]

$$\mathbf{X}' \ = \ e_{p,q}(\ \xi_1 \sigma_1^{(p,q)} + \xi_2 \sigma_2^{(p,q)} + \xi_3 \sigma_3^{(p,q)} \ )\ \mathbf{X}\ E_{p,q}(\ -\xi_1 \sigma_1^{(p,q)} - \xi_2 \sigma_2^{(p,q)} - \xi_3 \sigma_3^{(p,q)} \ ) \qquad (2.42)$$

Due to the key relations

$$e_{p,q}(\mathbf{A})\ E_{p,q}(-\mathbf{A}) = 1 \ \Rightarrow \ e_{p,q}(\mathbf{A}) \ = \ \mathbf{M}, \ \ E_{p,q}(-\mathbf{A}) \ = \ \mathbf{M}^{-1} \qquad (2.43a)$$

one will have

$$det(\mathbf{X}') \ = \ det(\mathbf{M})\ det(\mathbf{X})\ det(\mathbf{M}^{-1}) \ = \ det(\mathbf{X}) \ =$$

$$t'^2 \ - \ x'^2 \ - \ y'^2 \ - \ z'^2 \ = \ t^2 \ - \ x^2 \ - \ y^2 \ - \ z^2 \qquad (2.43b)$$

and such that the Minkowski spacetime interval remains invariant under the transformations (2.42).

One may notice that in the (p,q)-deformed case the relations in eq-(2.40) are no longer obeyed, $(\hat{\xi}^i \sigma_i^{(p,q)})\ (\hat{\xi}^j \sigma_j^{(p,q)}) \neq \mathbf{1}$, consequently the exponentials of the deformed generators

$$e_{p,q}(\ \frac{\xi^i}{2} \sigma_i^{(p,q)} \ ) \ \neq \ cosh_{p,q}(\frac{\xi}{2})\ \mathbf{1} \ + \ \hat{\xi}^i \sigma_i^{(p,q)}\ sinh_{p,q}(\frac{\xi}{2}) \qquad (2.44a)$$

[2] Alternatively, one could flip the location of the $e_{p,q}, E_{p,q}$ exponentials in eq-(2.42)

$$E_{p,q}(\frac{\xi^i}{2}\sigma_i^{(p,q)}) \neq COSH_{p,q}(\frac{\xi}{2})\, \mathbf{1} + \hat{\xi}^i\sigma_i^{(p,q)}\, SINH_{p,q}(\frac{\xi}{2}) \quad (2.44b)$$

$$\widetilde{e_{p,q}}(\frac{\xi^i}{2}\sigma_i^{(p,q)}) \neq \widetilde{cosh}_{p,q}(\frac{\xi}{2})\, \mathbf{1} + \hat{\xi}^i\sigma_i^{(p,q)}\, \widetilde{sinh}_{p,q}(\frac{\xi}{2}) \quad (2.44c)$$

cannot be written in the Euler form, and this is one of the reasons behind the inequalities in eq-(2.8).

We finalize with a brief discussion on quantum groups, noncommutative spacetimes, κ-deformed Poincare algebra and quasi-crystals. In the case of κ-deformed quantum Poincare algebra it is not the deformation of the algebra that really matters, but the co-algebra (coproduct) and the associated non-commutative spacetime structure [10]. The phase space as a whole does not have the Hopf algebra structure. In order to deform the phase space, one presumably has to make use of more general structures, like the one of Hopf algebroid. The momentum space associated with κ-deformation is curved [10]. It remains to extend this work to the case of noncommutative spacetimes and to find the corresponding co-algebraic structures; i.e. the coproduct, antipode, counit.

It is known that with quantum groups one can introduce a form of coordinate quantization while preserving, continuously, all group symmetries [1]. One can introduce coordinate quantization using discrete lattices, but prior to quantum groups no one could achieve quantization without breaking the continuous spacetime symmetries [1]. We saw earlier that for the special values $p = \tau, q = -\tau^{-1}$, the p, q integers $[n]_{p,q}$ reduce to the Fibonacci numbers. The Golden mean τ is ubiquitous in the construction of quasi-crystals, and their associated non-crystallographic groups. Quasi-crystals (like the Penrose tiling with five-fold symmetry) can be constructed via the cut-and-projection mechanism of higher dimensional regular lattices; i.e. the projection onto lower dimensions is performed along directions with irrational slopes. It is warranted to explore further the results of this work within the context of coordinate quantization and Noncommutative geometry that will help us cast some light into Quantum Gravity.

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