CLIFFORD ALGEBRAS - NEW RESULTS

Jean Claude Dutailly

Abstract

The main purpose of this paper is to present some new results about Clifford Algebras : exponential, real structures, Cartan algebras... As they address different topics and the definitions in Clifford Algebras still differ from one author to another, it seems simpler to give a full coverage of Clifford Algebras, starting from their definition. So the paper can also be a useful introduction to a subject which gains more and more interest in different areas of Physics, Computing Science and Engineering.

1 OPERATIONS IN A CLIFFORD ALGEBRA

1.1 Definition of a Clifford Algebra

Definition 1 *Let F be a vector space over the field K (of characteristic $\neq 2$) endowed with a symmetric bilinear non degenerate form ρ (valued in the field K). The* ***Clifford algebra*** *$Cl(F,\rho)$ and the canonical map $\imath : F \to Cl(F,\rho)$ are defined by the following universal property : for any associative algebra A over K (with internal product $\cdot$ and unit e) and K-linear map $f : F \to A$ such that :*

$$\forall v,w \in F : f(v) \cdot f(w) + f(w) \cdot f(v) = 2\rho(v,w) \cdot e$$

there exists a unique algebra morphism : $\varphi : Cl(F,g) \to A$ such that $f = \varphi \circ \imath$

$$\begin{bmatrix} & & f & & \\ F & \to & \to & \to & A \\ \downarrow & & & \nearrow & \\ \imath & & \nearrow & \varphi & \\ \downarrow & \nearrow & & & \\ Cl(F,g) & & & & \end{bmatrix}$$

The Clifford algebra includes the scalar K, the vectors of F (so we identify $\imath(u)$ with $u \in F$ and $\imath(k)$ with $k \in K$) and all linear combinations of products of vectors by $\cdot$. We will denote the form $\rho(u,v) = \langle u,v \rangle$.

A definition is not a proof of existence, which is proven for any vector space by defining a morphism with the algebra ΛF of antisymmetric tensors, using an orthonormal basis.

Remarks :

i) A common definition is done with a quadratic form. As any quadratic form gives a bilinear symmetric form by polarization, and a bilinear symmetric form is necessary for most of the applications, we can easily jump over this step. There is also the definition $f(v) \cdot f(w) + f(w) \cdot f(v) + 2\rho(v,w) \cdot e = 0$ which sums up to take the opposite for g.

ii) F can be a real or a complex vector space, but g *must be symmetric :* it does not work with a Hermitian sesquilinear form. In the following K will be $\mathbb{R}$ or $\mathbb{C}$.

For each topic we will provide examples related to the Clifford algebra $Cl(\mathbb{C},4)$, which corresponds to $\mathbb{C}^4$ with the canonical form $\langle X,Y\rangle = \sum_{k=1}^{4} X_k Y_k \Leftrightarrow \langle \varepsilon_j, \varepsilon_k\rangle = \delta_{jk}$

1.2 Algebra structure

1.2.1 Vector space structure

A Clifford algebra is a 2^n dimensional vector space with $n = \dim F$.

An orthonormal basis of F will be denoted $(\varepsilon_j)_{j=1}^{n}$. Then :

$$\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = 2\eta_{ij} \text{ where } \eta_{ij} = \langle \varepsilon_i, \varepsilon_j\rangle = 0, \pm 1$$

or any permutation of the *ordered* set of indices

$\{i_1,...,i_n\} : \varepsilon_{\sigma(i_1).} \cdot \varepsilon_{\sigma(i_2)..} \cdot \varepsilon_{\sigma(i_r)} = \epsilon(\sigma)\, \varepsilon_{i_1} \cdot \varepsilon_{i_2 ..} \cdot \varepsilon_{i_r}$

where $\epsilon(\sigma) = \pm 1$ is the signature of the permutation σ.

The set of ordered products $\varepsilon_{j_1} \cdot \varepsilon_{j_2} ... \varepsilon_{j_p}$ of vectors $(\varepsilon_j)_{j=1}^{n}$ of an orthonormal basis and the scalar 1 is a basis of $Cl(F,g)$, which will be denoted $(F_\alpha)_{\alpha=1}^{2^n}$.

The scalar component of $Z \in Cl(F,g)$ is denoted $\langle Z\rangle \in K$

Example with $Cl(\mathbb{C},4)$: It is convenient to use the basis :

$Z = a + v_0\varepsilon_0 + v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3 + w_1\varepsilon_0 \cdot \varepsilon_1 + w_2\varepsilon_0 \cdot \varepsilon_2 + w_3\varepsilon_0 \cdot \varepsilon_3 + r_1\varepsilon_3 \cdot \varepsilon_2 + r_2\varepsilon_1 \cdot \varepsilon_3 + r_3\varepsilon_2 \cdot \varepsilon_1$

$+x_0\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 + x_1\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_2 + x_2\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3 + x_3\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_1 + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

and to represent a vector by the notation :

$Z = (a, v_0, v, w, r, x_0, x, b)$ in $Cl(\mathbb{C},4)$ with the 4 scalars a, v_0, x_0, b and the 4 vectors $v, w, r, x \in \mathbb{C}^3$.

1.2.2 Algebra structure

With the internal product $\cdot$ $Cl(F,\rho)$ is a unital algebra on the field K, with unity element the scalar $1 \in K$

Because of the relation with the scalar product, a Clifford algebra has additional properties and the vectors of F play a special role.

A Clifford algebra is a graded algebra : the homogeneous elements of degree r of $Cl(F,\rho)$ are elements which can be written as product of r vectors of F.

The product of 2 vectors of a basis of the Clifford algebra has the form : $F_\alpha \cdot F_\beta = \epsilon(\alpha,\beta)\, F_\gamma$ where F_γ is another vector of the basis, and $\epsilon(\alpha,\beta) = \pm 1$ depends on both α, β and their order (it is usually not antisymmetric). And the product of 2 elements of $Cl(F,\rho)$ reads :

$Z = X \cdot Y = \sum_{\alpha,\beta} X_\alpha Y_\beta F_\alpha \cdot F_\beta = \sum_\gamma \left(\sum_{\alpha,\beta} \epsilon(\alpha,\beta)\, X_\alpha Y_\beta\right) F_\gamma$

It can be expressed with $2^n \times 2^n$ matrices acting on the components of the elements :

$[\pi_L(X)]\,[Y] = [X \cdot Y] = \sum_{\alpha\beta} [\pi_L(X)]^\alpha_\beta\, [Y]^\beta\, F_\alpha$

$[\pi_R(Y)]\,[X] = [X \cdot Y] = \sum_{\alpha\beta} [\pi_R(Y)]^\alpha_\beta\, [X]^\beta\, F_\alpha$

The map $\pi_L : Cl(F,g) \to L(K, 2^n)$ is an algebra morphism :

$\pi_L(X\cdot Y)=\pi_L(X)\,\pi_L(Y)\,;\pi_L\left(X^{-1}\right)=[\pi_L(X)]^{-1}\,;\pi_L(1)=I_{2^n}$

The map $\pi_R : Cl(F,g)\to L(K,2^n)$ is an algebra antimorphism :

$\pi_R(Y\cdot X)=\pi_R(X)\,\pi_R(Y)\,;\pi_R\left(X^{-1}\right)=\pi_R(X)^{-1}\,;\pi_R(1)=I_{2^n}$

and : $\pi_L(X)\circ\pi_R(X)(Z)=\pi_R(X)\circ\pi_L(X)(Z)=X\cdot Z\cdot X$

$[(X\cdot Y-Y\cdot X)\cdot Z]=([\pi_L(X)]-[\pi_R(Y)])\,[Z]\Leftrightarrow[X,Y]=[\pi_L(X)]-[\pi_R(Y)]$

In Clifford algebras some elements are invertible for the internal product. The set GCl of invertible elements is a Lie group.

Example with $Cl(\mathbb{C},4)$:

$(a,v_0,v,w,r,x_0,x,b)\cdot(a',v_0',v',w',r',x_0',x',b')=(A,V_0,V,W,R,X_0,X,B)$

$A=aa'+v_0v_0'+v^tv'-w^tw'-r^tr'-x_0'x_0-x^tx'+bb'$

$V_0=av_0'+v_0a'-v^tw'+w^tv'-r^tx'-x^tr'+x_0b'-bx_0'$

$V=av'+a'v+v_0w'-v_0'w+x_0'r+x_0r'+b'x-bx'+j(v)\,r'+j(r)\,v'-j(w)\,x'+j(x)\,w'$

$W=aw'+a'w+v_0v'-v_0'v+b'r+br'+x_0'x-x_0x'-j(v)\,x'+j(w)\,r'+j(r)\,w'+j(x)\,v'$

$R=ar'+a'r-x_0'v-x_0v'+b'w+bw'+v_0'x+v_0x'-j(v)\,v'+j(w)\,w'+j(r)\,r'+j(x)\,x'$

$X_0=ax_0'+a'x_0+v_0b'-bv_0'-v^tr'-r^tv'+w^tx'-x^tw'$

$X=ax'+a'x+b'v-bv'-x_0'w+x_0w'+v_0r'+v_0'r+j(v)\,w'-j(w)\,v'+j(r)\,x'+j(x)\,r'$

$B=ab'+a'b+v_0x_0'-v_0'x_0+v^tx'-x^tv'-w^tr'-r^tw'$

with the operator $j:\mathbb{C}^3\to L(\mathbb{C},3):j(z)=\begin{bmatrix}0&-z_3&z_2\\z_3&0&-z_1\\-z_2&z_1&0\end{bmatrix}$

which has many algebraic properties and is very convenient in computations. In particular :

$j(x)\,y=-j(y)\,x$

$[j(x)]^t=[j(-x)]$

$j(x)\,j(y)=yx^t-y^tx$

1.3 Involutions

1.3.1 Graded involution

The graded involution $\imath : Cl(F,\rho)\to Cl(F,\rho)$ is the extension to the Clifford algebra of the operation on $F:\varepsilon_j\to-\varepsilon_j$, so that the homogeneous elements of rank even do not change sign, and the homogeneous elements of rank odd change sign. The graded involution is an algebra automorphism

$\imath(X\cdot Y)=\imath(X)\cdot\imath(Y)$

$\imath^2=Id$

The graded involution splits $Cl(F,\rho)$: $Cl(F,\rho)=Cl_0\oplus Cl_1$ where $Cl_0=\{Z:\imath(Z)=Z\}$ is a Clifford subalgebra and $Cl_1=\{Z:\imath(Z)=-Z\}$ is a vector subspace. Any element of the algebra has a unique decomposition :

$Z=Z_0+Z_1,Z_0\in Cl_0,Z_1\in Cl_1.$

Example with $Cl(\mathbb{C},4)$:

$\imath(a,v_0,v,w,r,x_0,x,b) = (a,-v_0,-v,w,r,-x_0,-x,b)$

$Cl_0 = \{(a,0,0,w,r,0,0,b)\}$

$Cl_1 = \{(0,v_0,v,0,0,x_0,x,0)\}$

1.3.2 Transposition

Transposition, denoted Z^t is the operation which reverses the order of the product : $Z^t = X_p \cdot X_{p-1} ... \cdot X_1 = (-1)^{\frac{1}{2}p(p-1)} X_1 \cdot X_2 ... \cdot X_p$.

It is not an automorphism :

$(Z^t)^t = Z$

$(X \cdot Y)^t = Y^t \cdot X^t$

Transposition acts by a diagonal matrix D_T on the components :

$[Z^t] = [D_T][Z]$, from which one deduces a relation between the matrices $\pi_L, \pi_R : [\pi_R(Y)] = [D_T][\pi_L(Y^t)][D_T]$

Proof. $(X \cdot Y)^t = Y^t \cdot X^t = (\pi_L(X)(Y))^t = \pi_R(X^t)(Y^t) \Leftrightarrow$

$[D_T][\pi_L(X)][Y] = [\pi_R(X^t)][D_T][Y]$

$[D_T][\pi_L(X)] = [\pi_R(X^t)][D_T]$ ∎

Transposition splits $Cl(F,\rho)$: $Cl(F,\rho) = Cl_S \oplus Cl_A$ where $Cl_S = \left\{Z : (Z)^t = Z\right\}$ and $Cl_A = \left\{Z : (Z)^t = -Z\right\}$ are vector subspaces.

Example with $Cl(\mathbb{C},4)$:

$(a,v_0,v,w,r,x_0,x,b)^t = (a,v_0,v,-w,-r,-x_0,-x,b)$

The symmetric elements are $Cl_S = (a,v_0,v,0,0,0,0,b)$, and the antisymmetric $Cl_A = (0,0,0,w,r,x_0,x,0)$

1.3.3 Chirality

The ordered product of all the vectors of a basis of F : $F_{2^n} = \varepsilon_1 \cdot \varepsilon_2 ... \varepsilon_n$, does not depend on the choice of the basis and has specific properties :

$(F_{2^n})^2 = (-1)^{\frac{n(n-1)}{2}} \det[\eta], (F_{2^n})^t = (-1)^{\frac{n(n-1)}{2}} F_{2^n}$

If n is odd Z commutes with all the other elements.

A volume element is an element $\omega \neq \pm 1$ such that $\omega \cdot \omega = 1$. On complex Clifford algebras there is always a volume element : $\omega = \varepsilon_1 \cdot \varepsilon_2 ... \varepsilon_n$ or $\omega = i\varepsilon_1 \cdot \varepsilon_2 ... \varepsilon_n$. If n is even it decomposes the Clifford algebra in a right and left part $Cl(F,g) = Cl_R \oplus Cl_L$:

$Cl_R = \left\{Z = \frac{1}{2}(Z + \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = Z\}$

$Cl_L = \left\{Z = \frac{1}{2}(Z - \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = -Z\}$

Cl_R is a sub Clifford algebra and an ideal : $\forall Z \in Cl_R; Z' \in Cl : Z \cdot Z' \in Cl_R$

$Z \in Cl_R, Cl_L$ are never invertible : $\omega \cdot g = \epsilon g \Leftrightarrow \omega \cdot g \cdot g^{-1} = \epsilon = \omega$

Example with $Cl(\mathbb{C},4)$:

$\omega = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, \omega^2 = 1, \omega^t = \omega$

$Cl_R = \{Z : (a,v_0,v,w,w,-v_0,-v,a)\}; Cl_L = \{Z : (a,v_0,v,w,-w,v_0,v,-a)\}$

1.3.4 Subalgebras of Quaternionic type

Using the 2 involutions one can decompose any Clifford algebra in subspaces of quaternionic type (Shirokov) :

$[Cl^s] = \oplus_{k=s(\mathrm{mod}\,4)} \left\{ \imath(Z) = (-1)^s Z; (Z)^t = (-1)^{\frac{1}{2}s(s-1)} Z \right\}, s = 0..4$

The decomposition does not depend on the choice of the basis.

Example with $Cl(\mathbb{C}, 4)$:

$Cl^0 : s = 0 : \imath(Z) = Z; (Z)^t = Z; \Leftrightarrow Z = (a, 0, 0, 0, 0, 0, 0, b)$

$Cl^1 : s = 1 : \imath(Z) = -Z; (Z)^t = Z \Leftrightarrow Z = (0, v_0, v, 0, 0, 0, 0, 0)$

$Cl^2 : s = 2 : \imath(Z) = Z; (Z)^t = -Z \Leftrightarrow Z = (0, 0, 0, w, r, 0, 0, 0)$

$Cl^3 : s = 3 : \imath(Z) = -Z; (Z)^t = -Z \Leftrightarrow Z = (0, 0, 0, 0, 0, x_0, x, 0)$

1.4 Scalar product

There is a scalar product on the Clifford algebra, defined by extension from homogeneous elements :

$\langle X_1 \cdot X_2 ... X_p, Y_1 \cdot Y_2 ... Y_q \rangle = \delta_{pq} \langle X_1, Y_1 \rangle ... \langle X_p, Y_p \rangle$

such that the basis $(F_\alpha)_{\alpha=1}^{2^n}$ is orthonormal :

$$\left\langle \varepsilon_{i_1} \cdot \varepsilon_{i_2} ... \varepsilon_{i_p}, \varepsilon_{j_1} \cdot \varepsilon_{j_2} ... \varepsilon_{j_q} \right\rangle = \delta_{pq} \langle \varepsilon_{i_1}, \varepsilon_{j_1} \rangle ... \left\langle \varepsilon_{i_p}, \varepsilon_{j_p} \right\rangle$$

In an orthonormal basis :

$$\langle Z, Z' \rangle = [Z]^t [\eta] [Z']$$

where $[\eta]$ is a diagonal real $2^n \times 2^n$ matrix : $\langle F_\alpha, F_\beta \rangle = [\eta]^\alpha_\beta$

For homogeneous elements : $\langle Z \cdot Z', Z \cdot Z' \rangle = \langle Z, Z \rangle \langle Z', Z' \rangle$

Transpose and the graded involution preserve the scalar product :

$$\left\langle X^t, Y^t \right\rangle = \langle X, Y \rangle ; \langle \imath(X), \imath(Y) \rangle = \langle X, Y \rangle$$

The vector subspaces in the quaternionic decomposition are orthogonal.

The scalar component of the product $Z \cdot Z'$ is related to the scalar product $\langle Z, Z' \rangle$:

$$\langle X, Y \rangle = \left\langle X^t \cdot Y \right\rangle \tag{1}$$

As a consequence :

$\forall X, Y, Z : \langle X \cdot Y, Z \rangle = \langle Y, X^t \cdot Z \rangle, \langle Y \cdot X, Z \rangle = \langle Y, Z \cdot X^t \rangle$

A homogeneous element Z is invertible iff its scalar product $\langle Z, Z \rangle \neq 0$. Its inverse is then : $Z^{-1} = \frac{1}{\langle Z, Z \rangle} Z^t$

Example with $Cl(\mathbb{C}, 4)$:

$\langle Z, Z' \rangle = aa' + v_0 v_0' + v^t v' + w^t w' + r^t r' + x_0 x_0' + x^t x + bb'$

1.4.1 Transpose of the matrices $[\pi_L], [\pi_R]$

From these results we have a useful relation between the matrix $[\pi_L(X)]$ and its transpose :

$$\left[\pi_L\left(X^t\right)\right] = [\eta]\,[\pi_L(X)]^t\,[\eta]$$

Proof. $[X\cdot Y] = \sum_{\alpha\beta}[\pi_L(X)]^\alpha_\beta\,[Y]^\beta\,F_\alpha \Rightarrow [X\cdot F_\beta] = \sum_\alpha[\pi_L(X)]^\alpha_\beta\,F_\alpha \Rightarrow \langle X\cdot F_\beta, F_\alpha\rangle = [\eta]^\alpha_\alpha\,[\pi_L(X)]^\alpha_\beta = \left\langle X, F_\alpha\cdot F^t_\beta\right\rangle = [D_T]^\beta_\beta\,\langle X, F_\alpha\cdot F_\beta\rangle$

using $\langle Y\cdot X, Z\rangle = \langle Y, Z\cdot X^t\rangle, F^t_\beta = [D_T]^\beta_\beta\,F_\beta$

$[\pi_L(X)]^\alpha_\beta = [\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\,\langle X, F_\alpha\cdot F_\beta\rangle$

$[\pi_L(X)]^\beta_\alpha = [\eta]^\beta_\beta\,[D_T]^\alpha_\alpha\,\langle X, F_\beta\cdot F_\alpha\rangle$

$F_\alpha\cdot F_\beta = \epsilon(\alpha,\beta)\,F_\gamma$ with a unique γ and $\epsilon(\alpha,\beta) = \pm 1$

$(F_\alpha\cdot F_\beta)^t = F^t_\beta\cdot F^t_\alpha = \epsilon(\alpha,\beta)\,F^t_\gamma = [D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,F_\beta\cdot F_\alpha = [D_T]^\gamma_\gamma\,\epsilon(\alpha,\beta)\,F_\gamma = [D_T]^\gamma_\gamma\,F_\alpha\cdot F_\beta$

$F_\beta\cdot F_\alpha = [D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\gamma_\gamma\,F_\alpha\cdot F_\beta = \epsilon(\beta,\alpha)\,F_\gamma = [D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\gamma_\gamma\,\epsilon(\alpha,\beta)\,F_\gamma$

$\epsilon(\beta,\alpha) = [D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\gamma_\gamma\,\epsilon(\alpha,\beta)$

$[\pi_L(X)]^\beta_\alpha = [\eta]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\gamma_\gamma\,\langle X, F_\alpha\cdot F_\beta\rangle$

$= [\eta]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\beta_\beta\,[D_T]^\alpha_\alpha\,[D_T]^\gamma_\gamma\,[\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\,[\pi_L(X)]^\alpha_\beta$

$= [\eta]^\alpha_\alpha\,[\eta]^\beta_\beta\,[D_T]^\gamma_\gamma\,[\pi_L(X)]^\alpha_\beta$

$[\pi_L(X^t)]^\alpha_\beta = [\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\,\langle X^t, F_\alpha\cdot F_\beta\rangle$

$= [\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\left\langle X, (F_\alpha\cdot F_\beta)^t\right\rangle = [\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\left\langle X, [D_T]^\gamma_\gamma\,F_\alpha\cdot F_\beta\right\rangle$

using $\langle X^t, Y^t\rangle = \langle X, Y\rangle$

$[\pi_L(X^t)]^\alpha_\beta = [\eta]^\alpha_\alpha\,[D_T]^\beta_\beta\,[D_T]^\gamma_\gamma\,\langle X, F_\alpha\cdot F_\beta\rangle = [D_T]^\gamma_\gamma\,[\pi_L(X)]^\alpha_\beta = [\eta]^\alpha_\alpha\,[\eta]^\beta_\beta\,[\pi_L(X)]^\beta_\alpha$

that we can write : $[\pi_L(X^t)] = [\eta]\,[\pi_L(X)]^t\,[\eta]$ ■

and from there :

$$[\pi_R(X)]^t = [\eta]\left[\pi_R\left(X^t\right)\right][\eta]$$

Proof. $[\pi_R(X)] = [D_T]\,[\pi_L(X^t)]\,[D_T]$

$[\pi_R(X)]^t = [D_T]\,[\pi_L(X^t)]^t\,[D_T] = [D_T]\,[\eta]\,[\pi_L(X)]\,[\eta]\,[D_T] = [\eta]\,[D_T]\,[\pi_L(X)]\,[D_T]\,[\eta] = [\eta]\left[\pi_R\left(X^i\right)\right][\eta]$ ■

Example with $Cl(\mathbb{C},4)$ **:**

$[\pi_L(Z^t)] = [\pi_L(Z)]^t\,;[\pi_R(Z^t)] = [\pi_R(Z)]^t$

$[\pi_R(Z)] = [D_T]\,[\pi_L(Z^t)]\,[D_T]$

1.5 Exponential

1.5.1 Definition

On a Clifford algebra one can always define a norm, and it is a finite dimensional Banach vector space.

The exponential of the matrix $\pi_L(T)$ is well defined, as well as

$$\exp T = \sum_{p=0}^{\infty} \frac{1}{p!} T^p$$

then : $\pi_L(\exp T) = \exp \pi_L(T)$

1.5.2 Properties

The map $T \to \exp T$ is smooth, with derivative $\frac{d}{dT} \exp T|_{T=u} = \exp u$ considered as a linear map from u to $\exp u$, that is :

$\left[\frac{d}{dT} \exp T|_{T=u}\right] = [\pi_L(\exp u)]$

$\det[\pi_L(\exp u)] = \exp Tr(\pi_L(u))$

$Tr(\pi_L(u)) = \sum_\alpha [\pi_L(u)]_\alpha^\alpha = 2^n \langle T \rangle$

$\det[\pi_L(\exp u)] = \exp 2^n \langle T \rangle \neq 0$

thus, according to the constant rank theorem exp is a local diffeomorphism on the Clifford algebra.

The map : $Z(\tau) = \exp(\tau T)$ defines a one parameter group with infinitesimal generator T : $Z(\tau + \tau') = Z(\tau) \cdot Z(\tau')$ and $Z(\tau)^{-1} = Z(-\tau)$.

The inverse map $(\exp)^{-1}$, similar to a logarithm, has for derivative

$[\pi_L(\exp u)]^{-1} = \left[\pi_L\left((\exp u)^{-1}\right)\right] = [\pi_L(\exp(-u))]$.

From the definition :

$\exp(T)^t = (\exp T)^t$; $\imath(\exp T) = \exp(\imath(T))$

Not all elements of a Clifford algebra can be written as an exponential. Ex : $Z \in Cl_R = \{Z : \omega \cdot Z = Z\}$: $\forall n > 0 : Z^n \in Cl_R$ but $1 \notin Cl_R$ so there is an exponential but $\exp Z \notin Cl_R$.

1.5.3 Special values of the exponential

In a complex or real Clifford algebra, if $T.T = \lambda \neq 0 \in \mathbb{C}$:

$\exp T = \sum_{p=0}^{\infty} \frac{1}{p!} T^p = \sum_{p=0}^{\infty} \frac{1}{(2p)!} T^{2p} + T \cdot \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} T^{2p}$

$= \sum_{p=0}^{\infty} \frac{1}{(2p)!} \lambda^p + T \cdot \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \lambda^p$

Let us denote $\lambda = \mu^2$ with any square root μ of λ

$\exp T = \sum_{p=0}^{\infty} \frac{1}{(2p)!} \mu^{2p} + T \cdot \frac{1}{\mu} \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \lambda^{2p+1} = \cosh \mu + \frac{1}{\mu} (\sinh \mu) T$

$$T.T \in \mathbb{C} \Rightarrow \exp T = \cosh \mu + \frac{1}{\mu} (\sinh \mu) T; \mu^2 = T \cdot T$$

If $Z.Z = 0$ then $\exp T = 1 + T$

$\cosh \mu, \frac{1}{\mu} (\sinh \mu)$ are always real.

If $\lambda \in \mathbb{R}$:

$\lambda > 0 : \exp T = \cosh \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \left(\sinh \sqrt{\lambda} \right) T$
$\lambda < 0 : \exp T = \cos \sqrt{-\lambda} + \frac{1}{\sqrt{-\lambda}} \left(\sin \sqrt{-\lambda} \right) T$
and $\left(\exp T \right)^{-1} = \exp \left(-T \right) = \cosh \mu + \frac{1}{\mu} \left(\sinh \mu \right) T$

Example with $Cl\left(\mathbb{C}, 4 \right)$:
$T_r = \left(0,0,0,0,R,0,0,0 \right), R \in \mathbb{C}^3 : T_r \cdot T_r = -R^t R$
$\exp T_r = \cosh \mu_r + \frac{\sinh \mu_r}{\mu_r} \left(T_r \right)$ with $\mu_r^2 = -R^t R = T_r \cdot T_r$
$T_w = \left(0,0,0,W,0,0,0,0 \right), W \in \mathbb{C}^3 : T_w \cdot T_w = -W^t W$
$\exp T_w = \cosh \mu_w + \frac{\sinh \mu_w}{\mu_w} \left(T_w \right)$ with $\mu_w^2 = -W^t W = T_w \cdot T_w$
$T_x = \left(0,0,0,0,0,X_0,X,0 \right), X_0 \in \mathbb{R}, X \in \mathbb{C}^3 : T_x \cdot T_x = -X_0^2 - X^t X$
$\exp T_x = \cosh \mu_x + \frac{\sinh \mu_x}{\mu_x} T_x$
$T_v = \left(0,V_0,V,0,0,0,0,B \right), V_0, V, B \in \mathbb{C} : T_v \cdot T_v = V_0^2 + V^t V + B^2$
$\exp T_v = \cosh \mu_v + \left(\frac{\sinh \mu_v}{\mu_v} \right) T_v$

2 MORPHISMS

2.1 Morphisms of Clifford algebras

Definition 2 *A Clifford algebra morphism between the Clifford algebras* $Cl(F_1, \rho_1), Cl\left(F_2, \rho_2 \right)$ *on the same field* K *is a map*
$\Phi : Cl\left(F_1, \rho_1 \right) \to Cl\left(F_2, \rho_2 \right)$
which is an algebra morphism :
$\forall X, Y \in Cl(F_1, \rho_1), \forall k, k' \in K : \Phi \left(kX + k'Y \right) = k\Phi \left(X \right) + k'\Phi(Y),$
$\Phi \left(1 \right) = 1, \Phi \left(X \cdot Y \right) = \Phi \left(X \right) \cdot \Phi \left(Y \right)$
and preserves the scalar product :
$\forall X, Y \in Cl(F_1, \rho_1) : \left\langle \Phi \left(X \right), \Phi \left(Y \right) \right\rangle_{Cl(F_2, \rho_2)} = \left\langle X, Y \right\rangle_{Cl(F_1, \rho_1)}$

Theorem 3 *Let* $\left(F_1, \rho_1 \right), \left(F_2, \rho_2 \right)$ *be 2 vector spaces over the same field, endowed with bilinear symmetric forms. Then any linear map* $\varphi \in \mathcal{L}(F_1; F_2)$ *which preserves the scalar product can be extended to a morphism* Φ *over the Clifford algebras such that the diagram commutes :*

$$\begin{array}{ccc} \left(F_1, g_1 \right) & \overset{\imath_1}{\to} & Cl\left(F_1, g_1 \right) \\ \downarrow & & \downarrow \\ \downarrow \varphi & & \downarrow \Phi \\ \downarrow & & \downarrow \\ \left(F_2, g_2 \right) & \overset{\imath_2}{\to} & Cl\left(F_2, g_2 \right) \end{array}$$

Proof. It suffices to define $\Phi : Cl(F_1, g_1) \to Cl\left(F_2, g_2 \right)$ as follows :
$\forall k, k' \in K, \forall u, v \in F_1 :$
$\Phi \left(k \right) = k, \Phi \left(u \right) = \varphi \left(u \right), \Phi \left(ku + k'v \right) = k\varphi \left(u \right) + k'\varphi \left(v \right),$
$\Phi \left(u \cdot v \right) = \varphi \left(u \right) \cdot \varphi \left(v \right)$

and as a consequence :

$\Phi(u \cdot v + v \cdot u) = \varphi(u) \cdot \varphi(v) + \varphi(v) \cdot \varphi(u) = 2\rho_2(\varphi(u), \varphi(v)) = 2\rho_1(u, v) = \Phi(2\rho_1(u, v))$ ■

An isomorphism of Clifford algebras is a morphism which is also a bijective map. Then F_1, F_2 must have the same dimension.

An automorphism of Clifford algebra is a Clifford isomorphism on the same Clifford algebra.

Theorem 4 *A Clifford isomorphism of Clifford algebras between the Clifford algebras $Cl(F_1, \rho_1), Cl(F_2, \rho_2)$ maps F_1 to F_2*

Proof. Let $(\varepsilon_j)_{j=1}^n$ be an orthonormal basis of F_1 and $f_j = \Phi(\varepsilon_j)$. Define the algebra A generated by the vectors f_j and the map $f : F_1 \to A :: f(u) = \Phi(u)$. Then $\forall v, w \in F_1 : f(v) \cdot f(w) + f(w) \cdot f(v) = 2\rho_2(v, w)$ and by the universal property of Clifford algebra there is a unique map $\varphi : Cl(F_1, \rho_1) \to A$ such that $f = \varphi \circ \imath$ with $\imath : F_1 \to Cl(F_1, \rho_1)$. As an algebra $A \equiv Cl(F_2, \rho_2)$ and $\Phi = \varphi$ is unique. But, from the previous theorem, any map $\varphi : F_1 \to F_2$ which preserves the scalar product can be extended to a Clifford algebra morphism, and it maps F_1 to F_2 so does Φ. ■

As a consequence the only automorphisms on a Clifford algebra are the changes of orthonormal basis : they must map F on itself and preserve the scalar product.

2.2 The Category of Clifford algebras

The product of Clifford algebras morphisms is a Clifford algebra morphism, so Clifford algebras on a field K and their morphisms define a category $\mathfrak{Cl}_K$.

Vector spaces (F, ρ) on the same field K endowed with a symmetric bilinear form ρ, and linear maps φ which preserve this form, define a category, denoted $\mathfrak{V}_B$

$\mathfrak{TCl} : \mathfrak{V}_B \mapsto \mathfrak{Cl}_K$ is a functor from the category of vector spaces over K endowed with a symmetric bilinear form, to the category of Clifford algebras over K.

$\mathfrak{TCl} : \mathfrak{V}_B \mapsto \mathfrak{Cl}_K$ associates to each object (F, ρ) of $\mathfrak{V}_B$ its Clifford algebra $Cl(F, g)$:

$\mathfrak{TCl} : (F, g) \mapsto Cl(F, \rho)$ associates to each morphism of vector spaces a morphism of Clifford algebras :

$\mathfrak{TCl} : \varphi \in \hom_{\mathfrak{V}_B}((F_1, \rho_1), (F_2, \rho_2)) \mapsto \Phi \in \hom_{\mathfrak{Cl}_K}((F_1, \rho_1), (F_2, \rho_2))$

By picking an orthonormal basis in each Clifford algebra one deduces :

All Clifford algebras $Cl(F, \rho)$ where F is a complex n dimensional vector space are isomorphic. The common structure is denoted $Cl(\mathbb{C}, n)$.

All Clifford algebras $Cl(F, \rho)$ where F is a real n dimensional vector space and ρ have the same signature, are isomorphic. The common structure is denoted $Cl(\mathbb{R}, p, q)$, for the signature $(+p, -q)$.

The algebras $Cl(\mathbb{R},p,q)$ and $Cl(\mathbb{R},q,p)$ are *not* isomorphic if $p \neq q$. For any $n,p,q \geq 0$ we have the algebras isomorphisms :
$Cl\left(\mathbb{R},p,q\right) \simeq Cl_0\left(\mathbb{R},p+1,q\right) \simeq Cl_0\left(\mathbb{R},q,p+1\right)$
$Cl_0\left(\mathbb{R},p,q\right) \simeq Cl_0\left(\mathbb{R},q,p\right)$
$Cl(\mathbb{R},0,p) \simeq Cl(\mathbb{R},p,0)$
$Cl_0\left(\mathbb{C},n\right) \simeq Cl(\mathbb{C},n-1)$
with Cl_0 defined with the graded involution.

2.3 Adjoint map

2.3.1 Definition

The adjoint map :

$$Ad : GCl \rightarrow G\mathcal{L}\left(Cl;Cl\right) :: Ad_g Z = g \cdot Z \cdot g^{-1}$$

defines a linear action of the group GCl of invertible elements on $Cl\left(F,\rho\right)$:

$$Ad_{g\cdot g'} = Ad_g \circ Ad_{g'}; Ad_1 = Id$$

and is such that :

$$Ad_g\left(X \cdot Y\right) = Ad_g X \cdot Ad_g Y$$

In any basis F_α of the Clifford algebra :
$\left[Ad_g\right]\left(F_\alpha\right) = \left[Ad_g\right]\left(\varepsilon_{j_1}\cdot .... \cdot \varepsilon_{j_q}\right) = \left[Ad_g\right]\left(\varepsilon_{j_1}\right)\cdot ... \cdot \left[Ad_g\right]\left(\varepsilon_{j_q}\right)$
so the map Ad_g is fully defined by its value for the vectors ε_j of F, that is by its value on F. Moreover $Ad_g 1 = 1$.
This is a projective map, in the meaning : $\forall k \neq 0 \in K : Ad_{kg} = Ad_g$
$\left(Cl\left(F,\rho\right), Ad\right)$ is a representation of the group GCl. So for any group G of a Clifford algebra, by restriction $\left(Cl\left(F,\rho\right), Ad\right)$ is a representation of G on the Clifford algebra.

Its matrix in an orthonormal basis is :
$\left[Ad_g\right]\left[Z\right] = \left[\pi_L\left(g\right)\right]\left(Z\cdot g^{-1}\right) = \left[\pi_L\left(g\right)\right]\left[\pi_R\left(g^{-1}\right)\right]\left[Z\right] = \left[\pi_R\left(g^{-1}\right)\right]\left[\pi_L\left(g\right)\right]\left[Z\right]$
from which :
$\left[Ad_g\right]^t = \left[\pi_R\left(g^{-1}\right)\right]^t\left[\pi_L\left(g\right)\right]^t = \left[\eta\right]\left[\pi_R\left(\left(g^{-1}\right)^t\right)\right]\left[\eta\right]\left[\eta\right]\left[\pi_L\left(g^t\right)\right]\left[\eta\right]$
$= \left[\eta\right]\left[\pi_R\left(\left(g^{-1}\right)^t\right)\right]\left[\pi_L\left(g^t\right)\right]\left[\eta\right] = \left[\eta\right]\left[Ad_{g^t}\right]\left[\eta\right]$

$$\left[Ad_g\right]^t = \left[\eta\right]\left[Ad_{g^t}\right]\left[\eta\right]$$

2.3.2 Orthogonal group

In a Clifford algebra the adjoint map preserves the scalar product if :
$\langle Ad_g X, Ad_g Y\rangle = \langle X, Y\rangle$
$\langle Ad_g X, Ad_g Y\rangle = \left[Ad_g X\right]^t\left[\eta\right]\left[Ad_g Y\right] = \left[X\right]^t\left[\eta\right]\left[Y\right]$
that is if :

$[Ad_g]^t [\eta] [Ad_g] = [\eta] \Leftrightarrow [\eta] [Ad_{g^t}] [\eta] [\eta] = [\eta] \left[Ad_{g^{-1}}\right]$
$\Leftrightarrow [Ad_{g^t}] = \left[Ad_{g^{-1}}\right] \Leftrightarrow g^t \cdot g \in K$
The set of elements of $Cl\,(F, \rho)$ such that $g^t \cdot g \in K$ is a group G.

Then the adjoint map is an automorphism. It maps F to F, its restriction to F has for matrix an orthogonal matrix belonging to $O\,(n)$, and it defines uniquely the matrix of the adjoint map on $Cl\,(F, \rho)$. We have a morphism : $O\,(n) \to G$.

Conversely, because $Ad_{kg} \equiv Ad_g$ any $kg, k \in K, g \in G$ gives the same matrix of $O\,(n)$.

The orthogonal group of a Clifford algebra is the group :

$$O\,(Cl) = \left\{g \in Cl\,(F, \rho) : g^t \cdot g = 1\right\}$$

The Lie algebra of the orthogonal group is given by :
$T_1 O\,(Cl) = \{T : T^t + T = 0\}$
Then the group G of elements of $Cl\,(F, \rho)$ such that Ad_g preserves the scalar product is $K \times O\,(Cl)$.

The equation $g^t \cdot g = 1$ provides, by computing the product, necessary relations between the components of g.

Similarly for G. The group G is a submanifold of $Cl\,(F, \rho)$, not necessarily connected (with $K = \mathbb{R}$ it has 2 connected components for $k > 0$ and $k < 0$) and each of its connected component is the covering group of one of the connected component of the orthogonal group $O\,(n)$.

Example with $Cl\,(\mathbb{C}, 4)$ **:**
$T_1 O\,(Cl) = \{(0, 0, 0, W, R, X_0, X, 0)\}$

2.3.3 Reflection

In any n dimensional *real* vector space (F, ρ) endowed with a non degenerate scalar product (not necessarily definite positive) a reflection of vector $u, \langle u, u\rangle \neq 0$ is the map : $R\,(u)\,v = v - 2\frac{\langle u,v\rangle}{\langle u,u\rangle}u$. Its unique eigen vector is u with eigen value -1 and $\det R(u) = (-1)^n$. It preserves the scalar product and, conversely, any orthogonal map can be written as the product of at most n reflections.

In a *real* Clifford algebra based on a vector space F of dimension n the reflection of vector $u \in F, \langle u, u\rangle \neq 0$ can be written, using
$u \cdot v + v \cdot u = 2\,\langle u, v\rangle\,, u^{-1} = \frac{1}{\langle u,u\rangle}u$:
$R\,(u)\,v = v - 2\frac{\langle u,v\rangle}{\langle u,u\rangle}u = v - (u \cdot v + v \cdot u) \cdot u^{-1} = -Ad_u v \Leftrightarrow Ad_u v = -R\,(u)\,v$
The matrix of the restriction of Ad_u to F has for determinant : $\det\,[Ad_u]_F = (-1)^n \det\,[R\,(u)] = 1$. The map Ad_u can be extended to the Clifford algebra, it preserves the scalar product on $Cl\,(F, \rho)$, thus it is orthogonal and defines an automorphism. More generally $Ad_{u_1 \ldots u_p}$ defines an automorphism.

Conversely a Clifford algebra automorphism ϑ must preserve both the scalar product and be globally invariant on F. Its restriction to F is expressed as the product of $p \leq n$ reflections, that is $[Ad_g]_F = (-1)^p\,[R\,(u_1)] \ldots [R\,(u_p)] =$

$\left[Ad_{u_1...u_p}\right]$. As the map Ad_g is fully defined by its value on F, any automorphism on a Clifford algebra can be expressed as Ad_g where g is the product of at most n vectors of F. And because :

$g^t \cdot g = \langle u_1, u_1\rangle ... \langle u_p, u_p\rangle$

$Ad_{kg} = Ad_g$

up to the product by a scalar $g \in O(Cl)$.

$\det [Ad_g]_F = 1$ so the matrix of the restriction of Ad_u to F belongs to $SO(n)$. It defines uniquely $[Ad_g]$ on the Clifford algebra.

The sets G of vectors of $Cl(F, \rho)$ which can be written as the product of p vectors of F is a group only if :

- $p = 1$: the vectors are multiple of a fixed vector
- p is even

For p odd, they never constitute a group as can be checked with the graded involution :

$\imath(g) = \imath(u_1 \cdot ... \cdot u_{2p+1}) = (-1)^{2p+1}(u_1 \cdot ... \cdot u_{2p+1}) = -g$

$\imath(g \cdot g') = \imath(g) \cdot \imath(g') = g \cdot g'$

3 COMPLEX AND REAL CLIFFORD ALGEBRAS

3.1 Complex and real structures in vector spaces

3.1.1 From complex to real

A real structure on a complex vector space E is a map $\sigma : E \to E$ which is antilinear and an involution :

$\forall z \in \mathbb{C} : \sigma(zV) = \overline{(z)}\sigma(V), \sigma^2 = Id$

A vector V is decomposed in a real and an imaginary part :

$\operatorname{Re} V = \frac{1}{2}(V + \sigma(V))$

$\operatorname{Im} V = \frac{1}{2i}(V - \sigma(V))$

E splits in 2 vector subspaces $\operatorname{Re} E = \{\sigma(V) = V\}, \operatorname{Im} E = \{\sigma(V) = -V\}$: $E = \operatorname{Re} E \oplus i \operatorname{Im} E$ which are real isomorphic and $\operatorname{Re} E$ is said to be a real form of E.

The complex conjugate of any vector is $CC(\operatorname{Re} V + i \operatorname{Im} V) = \operatorname{Re} V - i \operatorname{Im} V$

One can always define a real structure on a complex vector space E. If it is n dimensional the simplest way is to define σ from the components in a fixed basis $(\varepsilon_j)_{j=1}^n$ and a set of indices $J \subset (1, 2, ...n)$

$\forall z \in \mathbb{C}, j \in J : \sigma(z\varepsilon_j) = \overline{(z)}\varepsilon_j$

$\forall z \in \mathbb{C}, j \in J^c : \sigma(z\varepsilon_j) = -\overline{(z)}\varepsilon_j$

σ defines a real structure :

$\forall V \in E : V = \sum_{j=1}^n v_j \varepsilon_j \to \sigma(V) = \sum_{j\in J} \overline{(v_j)}\varepsilon_j - \sum_{j\in J^c} \overline{(v_j)}\varepsilon_j$

$\sigma(kV) = \sum_{j\in J} \overline{(kv_j)}\varepsilon_j - \sum_{j\in J^c} \overline{(kv_j)}\varepsilon_j = \overline{(k)}\sigma(V)$

$\sigma^2(V) = \sum_{j\in J} \sigma\left(\overline{(v_j)}\varepsilon_j\right) - \sum_{j\in J^c} \sigma\left(\overline{(v_j)}\varepsilon_j\right) = V$

$\mathrm{Re}\,E = \{V : \sigma(V) = V\} = \left\{V = \sum_{j\in J} v_j\varepsilon_j + \sum_{j\in J^c} iv_j\varepsilon_j, (v_j)_{j=1}^n \in \mathbb{R}\right\}$

$\mathrm{Im}\,E = \{V : \sigma(V) = -V\} = \left\{V = \sum_{j\in J} iv_j\varepsilon_j + \sum_{j\in J^c} v_j\varepsilon_j, (v_j)_{j=1}^n \in \mathbb{R}\right\}$

The basis of $\mathrm{Re}\,F$ is $\{\varepsilon_j, j \in J, i\varepsilon_j, j \in J^c\}$, the basis of $\mathrm{Im}\,F$ is $\{i\varepsilon_j, j \in J, \varepsilon_j, j \in J^c\}$, they are both n real dimensional, and in this operation the components of a real vector can be complex or pure imaginary. The usual way is to take $J = (1, 2, ...n)$.

With 4 real linear maps on the real part of E one can define a real linear map

$F : E \to E$:

$F(\mathrm{Re}\,V + i\,\mathrm{Im}\,V) = P_1(\mathrm{Re}\,V) + P_2(\mathrm{Im}\,V) + i(Q_1(\mathrm{Re}\,V) + Q_2(\mathrm{Im}\,V))$.

It is complex linear if it meets the Cauchy conditions : $P_2 = -Q_1, P_1 = Q_2$

The complex conjugate of a complex map $\varphi \in \mathcal{L}(E; E)$ is the map

$CC(\varphi) \in \mathcal{L}(E; E) :: CC(\varphi)\,CC(V) = CC(\varphi(CC(V)))$

If $CC(\varphi) = \varphi$ it is said to be real and maps real vectors to real vectors, imaginary vectors to imaginary vectors If $CC(\varphi) = -\varphi$ then it inverses the structures.

If ρ is a bilinear symmetric form on E, the map : $\widetilde{\rho}(u, v) = \rho(CC(u), v)$ is Hermitian.

3.1.2 From real to complex

There are 2 ways to define a complex vector space from a real vector space E.

i) By complexification : the complexified is the complex vector space $\mathbb{C} \otimes E$ defined by the map : $f : E \times E \to \mathbb{C} \otimes E :: f(x, y) = x + iy$

$\dim_{\mathbb{C}} \mathbb{C} \otimes E = \dim_{\mathbb{R}} E$

ii) By a complex structure : E stays the same, if there is a map $J \in \mathcal{L}(E; E)$ such that $J^2 = -Id$. Then the product by i is defined as : $iV = J(V)$ and the complex conjugate $CC(iV) = -J(V)$. This is always possible iff $\dim E$ is even or infinite countable.

3.2 Real and complex structure on Clifford algebras

If (F, ρ) is a real vector space, the Clifford algebra $Cl(\mathbb{C} \otimes F, \rho)$ of its complexified is the complexified $\mathbb{C} \otimes Cl(F, g)$. $Cl(F, g)$ is a real form of $\mathbb{C} \otimes Cl(F, \rho)$. This is a complex Clifford algebra, but the symmetric form is not the usual one : the signature stays the same. All complex Clifford algebras are isomorphic, but the signature of the bilinear symmetric form can be different. Conversely such an isomorphism is a convenient way to define a real structure on a complex Clifford algebra as we will see now.

3.2.1 Morphisms $C : Cl(\mathbb{R}, p, q) \to Cl(\mathbb{C}, p + q)$

Let $F = R^n$ with a bilinear symmetric form of signature (p, q) and orthonormal basis $(e_j)_{j=1}^n$ with $\rho(e_j, e_j) = -1$ for $j \in J^c$.

Let $F_C = \mathbb{C}^n$ with orthonormal basis $(\varepsilon_j)_{j=1}^n$ with the bilinear symmetric form $\rho_c(\varepsilon_j, \varepsilon_k) = \delta_{jk}$ and $Cl(F_C, \rho_c)$ its Clifford algebra with product $\cdot$ and orthonormal basis $F_{j_1...j_r} = \varepsilon_{j_1} \cdot ... \cdot \varepsilon_{j_r}$.

Let σ be the real structure defined on the complex vector space F_C by :

$\forall z \in \mathbb{C}, j \in J : \sigma(z\varepsilon_j) = \overline{(z)\varepsilon_j}$

$\forall z \in \mathbb{C}, j \in J^c : \sigma(z\varepsilon_j) = -\overline{(z)}\varepsilon_j$

F_C is a $2n$ real vector space with real form $F_R = \operatorname{Re} F_C$ which has for basis $\{\varepsilon_j, j \in J, i\varepsilon_j, j \in J^c\}$, and $\operatorname{Im} F_C$ with basis $\{i\varepsilon_j, j \in J, \varepsilon_j, j \in J^c\}$.

On $\operatorname{Re} F_C$ we define the bilinear symmetric form :

$\rho_1 \left(\sum_{j \in J} V_j \varepsilon_j + \sum_{j \in J^c} V_k i\varepsilon_k, \sum_{j \in J} V'_j \varepsilon_j + \sum_{j \in J^c} V'_k i\varepsilon_k \right)$
$= \sum_{j \in J} V_j V'_j - \sum_{j \in J^c} V_k V'_k$

On $\operatorname{Im} F_C$ we define the bilinear symmetric form :

$\rho_2 \left(\sum_{j \in J} V_j i\varepsilon_j + \sum_{j \in J^c} V_k \varepsilon_k, \sum_{j \in J} V'_j i\varepsilon_j + \sum_{j \in J^c} V'_k \varepsilon_k \right)$
$= \sum_{j \in J} V_j V'_j - \sum_{j \in J^c} V_k V'_k$

ρ_1, ρ_2 are symmetric, real valued, and have the same signature $(+p, -q)$.

The real Clifford algebras $Cl(\operatorname{Re} F_C, \rho_1), Cl(\operatorname{Im} F_C, \rho_2)$, are isomorphic because the signature of the form is the same, and are isomorphic to $Cl(F, \rho)$.

As a vector space the Clifford algebra $Cl(F_C, \rho)$ is the sum of the real algebras :

$Cl(F_C, \rho) = Cl(\operatorname{Re} F_C, \rho_1) \oplus iCl(\operatorname{Im} F_C, \rho_2)$

so that $Cl(\operatorname{Re} F_C, \rho_1)$, and by extension $Cl(F, \rho)$, are a real form of $Cl(F_C, \rho)$.

In the real and imaginary parts of $Cl(\mathbb{C}, n)$ the components of a vector $Z \in Cl(\mathbb{C}, n)$, expressed in the usual orthonormal basis of $Cl(\mathbb{C}, n)$, can be real or pure imaginary.

The isomorphism $C : Cl(F, \rho) \to Cl(\operatorname{Re} F_C, \rho_1)$ is defined through the bases

$C : F \to \operatorname{Re} F_C :: C(e_j) = \varepsilon_j$ for $j \in J; C(e_j) = i\varepsilon_j$ for $j \in J^c$

It defines an isomorphism of vector spaces which preserves the symmetric form. It can be extended to an isomorphism between the Clifford algebras as seen above.

So we have a real Clifford algebra morphism $C : Cl(\mathbb{R}^n, p, q) \to Cl(\mathbb{C}, n)$ such that its image $C(Cl(\mathbb{R}^n, p, q))$ is $\operatorname{Re} Cl(\mathbb{C}, n)$ which is a real Clifford algebra. And similarly we can define $C' : F \to \operatorname{Im} F_C :: C'(e_j) = i\varepsilon_j$ for $j \in J; C(e_j) = \varepsilon_j$ for $j \in J^c$ which can be extended to a Clifford algebra morphism $C' : Cl(\mathbb{R}^n, p, q) \to Cl(\mathbb{C}, n)$ such that its image $C(Cl(\mathbb{R}^n, p, q))$ is $\operatorname{Im} Cl(\mathbb{C}, n)$ which is a real Clifford algebra.

Example with $Cl(\mathbb{C}, 4)$:

$C : Cl(3,1) \to Cl(\mathbb{C}, 4) :: C([a, v_0, v, w, r, x_0, x, b]) = (a, iv_0, v, iw, r, x_0, ix, ib)$

$\operatorname{Re}(a, v_0, v, w, r, x_0, x, b) = (\operatorname{Re} a, i \operatorname{Im} v_0, \operatorname{Re} v, i \operatorname{Im} w, \operatorname{Re} r, \operatorname{Re} x_0, i \operatorname{Im} x, i \operatorname{Im} b)$

$\operatorname{Im}(a, v_0, v, w, r, x_0, x, b) = (\operatorname{Im} a, -i \operatorname{Re} v_0, \operatorname{Im} v, -i \operatorname{Re} w, \operatorname{Im} r, \operatorname{Im} x_0, -i \operatorname{Re} x, -i \operatorname{Re} b)$

3.2.2 Complex conjugation

The map $C : Cl(\mathbb{R}^n, p, q) \to Cl(\mathbb{C}, n)$ has many interesting properties :

$$\forall\alpha,\beta\in\mathbb{R}: C(\alpha Z+\beta Z')=\alpha C(Z)+\beta C(Z')$$
$$C(Z\cdot Z')=C(Z)\cdot C(Z')$$
$$C(Z)^t=C(Z^t)$$
$$\langle C(Z),C(Z')\rangle_{Cl(\mathbb{C},n)}=\langle Z,Z'\rangle_{Cl(\mathbb{R}^n,p,q)}$$

In the orthonormal bases the map C is represented by a diagonal matrix with entries equal to $\pm i$ and $\underline{[C]}^2=[\eta]$ where $[\eta]$ is a diagonal matrix with entries equal to ± 1, such that $\overline{[C]}=[\eta][C]$.

For any $Z\in Cl(\mathbb{C},n)$ there are $Z_1,Z_2\in Cl(\mathbb{R}^n,p,q)$ such that

$Z=C(Z_1)+iC(Z_2)\Leftrightarrow[Z]=[C][Z_1]+i[C][Z_2]$

$\Rightarrow\overline{[Z]}=\overline{[C]}[Z_1]-i\overline{[C]}[Z_2]=[\eta][C][Z_1]+i[\eta][C][Z_2]$

The real and imaginary part of a vector $Z\in Cl(\mathbb{C},n)$ are then defined by :

$$\operatorname{Re}Z=\frac{1}{2}\left([Z]+[\eta]\overline{[Z]}\right);\operatorname{Im}Z=\frac{1}{2i}\left([Z]-[\eta]\overline{[Z]}\right)$$

Complex conjugation is then defined on $Cl(\mathbb{C},n)$ by :

$$CC(\operatorname{Re}Z+i\operatorname{Im}Z)=\operatorname{Re}Z-i\operatorname{Im}Z$$

With the components in the orthonormal basis : $[CC(Z)]=[\eta]\overline{[Z]}$

The operation is antilinear, an involution and it commutes with transposition and the principal involution. Moreover :

$CC(Z\cdot Z')=CC(Z)\cdot CC(Z')$

The adjoint of $Z\in Cl(\mathbb{C},n)$ is $Z^*=CC(Z^t)$

The complex conjugate of the map :

$\pi_L(X):Cl(\mathbb{C},n)\to Cl(\mathbb{C},n)::\pi_L(X)(Z)=X\cdot Z$

is :

$CC(\pi_L(X)(CC(Z)))=CC(\pi_L(X)CC(Z))=CC(X)\cdot CC(Z)$

that is $CC(\pi_L(X))=\pi_L(CC(X))$ and similarly $CC(\pi_R(X))=\pi_R(CC(X))$

With $Ad_g,g\in Cl(\mathbb{C},n)$:

$CC(Ad_g)(Z)=CC(Ad_gCC(Z))=CC\left(g\cdot CC(Z)\cdot g^{-1}\right)$

$=CC(g)\cdot Z\cdot CC\left(g^{-1}\right)=Ad_{CC(g)}Z$

$$CC(Ad_g)=Ad_{CC(g)}$$

A map $\varphi\in\mathcal{L}(Cl(\mathbb{C},n);Cl(\mathbb{C},n))$ is real if $CC(\varphi)=\varphi$: it maps real vectors to real vectors and imaginary vectors to imaginary vectors. If $CC(\varphi)=-\varphi$ then it inverses the structures. $\pi_L(X),\pi_R(X)$ are real if X is real.

The map Ad_g is real if $g\in\operatorname{Re}Cl(\mathbb{C},n)$ or $g\in\operatorname{Im}Cl(\mathbb{C},n)$ because $Ad_{-g}\equiv Ad_g$.

The vectors of the basis $(\varepsilon_j)_{j=1}^n$ of $Cl(\mathbb{C},n)$ belong to $\operatorname{Re}Cl(\mathbb{C},n)$ if $j\in J$, or to $\operatorname{Im}Cl(\mathbb{C},n)$ if $j\in J^c$.

The vectors $F_{j_1\dots j_r}=\varepsilon_{j_1}\cdot\ldots\cdot\varepsilon_{j_r}$ of an orthonormal basis of $Cl(\mathbb{C},n)$ belong to $\operatorname{Re}Cl(\mathbb{C},n)$ or $\operatorname{Im}Cl(\mathbb{C},n)$ according to :

$CC(F_{j_1\dots j_r})=\pm F_{j_1\dots j_r}=CC(\varepsilon_{j_1})\cdot\ldots\cdot CC(\varepsilon_{j_r})$.

Example with $Cl(\mathbb{C},4)$:

$CC(a,v_0,v,w,r,x_0,x,b)=\left(\overline{(a)},-\overline{(v_0)},\overline{(v)},-\overline{(w)},\overline{(r)},\overline{(x_0)},-\overline{(x)},-\overline{(b)}\right)$

3.2.3 Hermitian scalar product

The Hermitian scalar product on $Cl(\mathbb{C},n)$ is defined by :

$$\langle X,Y\rangle_H=\langle CC(X),Y\rangle_{Cl(\mathbb{C},n)}$$

$\langle X,Y\rangle_H=[CC(X)]^t[Y]=\overline{[X]}^t[\eta][Y]$

The usual basis $(F_\alpha)_{\alpha=0}^{2^n}$ of $Cl(\mathbb{C},n)$ is orthonormal for the Hermitian product with a signature, depending on (p,q), given by the value of $\eta_{\alpha\beta}$ in the matrix $[\eta]$

$\langle X,Y\rangle_H=\langle \operatorname{Re}X-i\operatorname{Im}X,\operatorname{Re}Y+i\operatorname{Im}Y\rangle_{Cl(\mathbb{C},n)}$

$=\langle \operatorname{Re}X,\operatorname{Re}Y\rangle_{Cl(\mathbb{C},n)}+\langle \operatorname{Im}X,\operatorname{Im}Y\rangle_{Cl(\mathbb{C},n)}$

$-i\langle \operatorname{Im}X,\operatorname{Re}Y\rangle_{Cl(\mathbb{C},n)}+i\langle \operatorname{Re}X,\operatorname{Im}Y\rangle_{Cl(\mathbb{C},n)}$

The components of the vectors $\operatorname{Re}X,\operatorname{Re}Y,\operatorname{Im}X,\operatorname{Im}Y$ can be real or complex and on the real and imaginary parts of the Clifford algebra the signature is (p,q).

Some of the usual identities are generalized :

$\forall u,v\in F=Span(\varepsilon_j)_{j=1}^n:2\langle u,v\rangle_H=u^*\cdot v+v\cdot u^*$

Proof. $2\langle u,v\rangle_H=2\langle CC(u),v\rangle_{Cl(\mathbb{C},n)}=CC(u)\cdot v+v\cdot CC(u)=CC(u^t)\cdot v+v\cdot CC(u^t)=u^*\cdot v+v\cdot u^*$ ■

$\langle X_1\cdot X_2...X_p,Y_1\cdot Y_2...Y_q\rangle_H=\delta_{pq}\langle X_1,Y_1\rangle_H...\langle X_p,Y_p\rangle_H$

Proof. $\langle X_1\cdot X_2...X_p,Y_1\cdot Y_2...Y_q\rangle_H=\langle CC(X_1)\cdot CC(X_2)...CC(X_p),Y_1\cdot Y_2...Y_q\rangle_{Cl(\mathbb{C},n)}$

$=\delta_{pq}\langle CC(X_1),Y_1\rangle_{Cl(\mathbb{C},n)}...\langle CC(X_p),Y_p\rangle_{Cl(\mathbb{C},n)}=\delta_{pq}\langle X_1,Y_1\rangle_H...\langle X_p,Y_p\rangle_H$

■

The Hermitian product is preserved by the graded involution and by transpose. It is preserved by a map φ if :

$\langle X,Y\rangle_H=\langle\varphi(X),\varphi(Y)\rangle_H=\langle CC(\varphi(X)),\varphi(Y)\rangle_{Cl(\mathbb{C},n)}$

$=\langle CC\varphi(CC(X)),\varphi(Y)\rangle_{Cl(\mathbb{C},n)}=[CC(X)]^t[CC\varphi]^t[\varphi][Y]=[CC(X)]^t[Y]$

That is if : $[CC\varphi]^t[\varphi]=I$

With $\varphi=Ad_g$ if $[CC(Ad_g)]^t[Ad_g]=\left[Ad_{CC(g)}\right]^t[Ad_g]=\left[Ad_{CC(g^t)}\right][Ad_g]=\left[Ad_{CC(g^t)\cdot g}\right]=I\Leftrightarrow CC(g^t)\cdot g\in\mathbb{C}\Leftrightarrow g^*\cdot g\in\mathbb{C}$

The unitary group of $Cl(\mathbb{C},n)$ is then defined as

$$U(Cl(\mathbb{C},n))=\left\{g\in Cl(\mathbb{C},n):CC\left(g^t\right)\cdot g=1\right\}$$

It depends on the complex conjugation, and there is a group for each signature.

Example with $Cl(\mathbb{C},4)$:

With $C : Cl(\mathbb{R},3,1) \to Cl(\mathbb{C},4)$

$\langle (a, v_0, v, w, r, x_0, x, b), (a', v_0', v', w', r', x_0', x', b') \rangle_R$

$= \overline{(a)}a' - \overline{(v_0)}v_0' + \overline{(v)}^t v' - \overline{(w)}^t w' + \overline{(r)}^t r' + \overline{(x_0)}x_0' - \overline{(x)}^t x' - \overline{(b)}b'$

3.2.4 Reflections

We have an extension of the theorem on reflections.

On a n dimensional complex vector space F, endowed with a bilinear symmetric form and a real structure, one can define a Hermitian product. A linear map which preserves the Hermitian product is represented by a unitary matrix, with the appropriate signature. Such a map is also an orthogonal map on the $2n$ dimensional real vector space. Indeed $U(n,p,q) \subset O(2n,p,q) \cap GL(\mathbb{C},n)$. Then it can be expressed as the product of at most $2n$ real reflections.

On $Cl(\mathbb{C}, p+q)$ a real reflection is a map :

$R(u) : \operatorname{Re} Cl(\mathbb{C}, p+q) \to \operatorname{Re} Cl(\mathbb{C}, p+q) :: R(u)z = z - 2\frac{\langle u,z\rangle_{Cl(\mathbb{C},p+q)}}{\langle u,u\rangle_F} u$

where u, z are vectors of the real part of $Span(\varepsilon_i)_{i=1}^n$

Writing $u = C(u_1), z = C(z_1)$:

$R(u)z = C(z_1) - 2\frac{\langle C(u_1),C(z_1)\rangle_{Cl(\mathbb{C},p+q)}}{\langle C(u_1),C(u_1)\rangle_{Cl(\mathbb{C},p+q)}} C(u_1) = C(z_1) - 2\frac{\langle u_1,z_1\rangle_{Cl(\mathbb{R};p,q)}}{\langle u_1,u_1\rangle_{Cl(\mathbb{R},p,q)}} C(u_1)$

$= C\left(z_1 - 2\frac{\langle u_1,z_1\rangle_{Cl(\mathbb{R};p,q)}}{\langle u_1,u_1\rangle_{Cl(\mathbb{R},p,q)}} u_1\right) = C(R(u_1)z_1)$

and :

$R(u_1)z_1 = -Ad_{u_1} z_1$

$R(u)z = -C(Ad_{u_1} z_1) = -Ad_{C(u_1)} C(z_1)$

As $Ad_{ig} \equiv Ad_g$ the vectors u can belong to $\operatorname{Re}(\mathbb{C}^n)$ or $i \operatorname{Re}(\mathbb{C}^n)$.

Then $Ad_{u_1 \dots u_p}$ preserves the Hermitian product :

$\left\langle Ad_{u_1 \dots u_p} Z, Ad_{u_1 \dots u_p} Z' \right\rangle_H = \left\langle Ad_{u_1 \dots u_p} CC(Z), Ad_{u_1 \dots u_p} Z' \right\rangle_{Cl(\mathbb{C},p+q)}$

$= \langle CC(Z), Z' \rangle_{Cl(\mathbb{C},p+q)} = \langle Z, Z' \rangle_H$

Any map on F can be extended over the Clifford algebra by

$[Ad_g](F_\alpha) = [Ad_g]\left(\varepsilon_{j_1} \cdot \dots \cdot \varepsilon_{j_q}\right) = [Ad_g](\varepsilon_{j_1}) \cdot \dots \cdot [Ad_g]\left(\varepsilon_{j_q}\right)$

So any map on $Cl(\mathbb{C},n)$ which preserves both the Hermitian product and the vector space F is necessarily of the form $Ad_{u_1 \dots u_p}$ where u_j are at most $2n$ vectors of $\operatorname{Re}(Cl(\mathbb{C},n))$ or $\operatorname{Im}(Cl(\mathbb{C},n))$.

4 LIE ALGEBRAS AND LIE GROUPS

4.1 Lie algebra

As any algebra a Clifford algebra is a Lie algebra with the bracket

$$[Z, Z'] = Z \cdot Z' - Z' \cdot Z$$

The principal involution $\imath$ preserves the bracket : $\imath([Z, Z']) = [\imath(Z), \imath(Z')]$

Transposition gives the opposite value : $[Z^t, Z'^t] = -[Z, Z']^t$

The map $ad(Z) : Cl \to Cl :: ad(Z)(Z') = [Z, Z']$ is linear and represented in matrix by $[ad(Z)] = \pi_L(Z) - \pi_R(Z)$

$[ad(Z)]^t = [\pi_L(Z)]^t - [\pi_R(Z)]^t = [\eta][\pi_L(Z^t)][\eta] - [\eta]\left[\pi_R\left(Z^i\right)\right][\eta]$

$$[ad(Z)]^t = [\eta]\left[ad\left(Z^t\right)\right][\eta]$$

The radical is the center Z_{Cl}, composed of the scalars if n is even, of the scalars and the multiple of the element $F_{2^n} = \varepsilon_1 \cdot \varepsilon_2 ... \varepsilon_n$ if n is odd. The quotient Cl/Z_{Cl} is then a semi-simple Lie algebra.

Example with $Cl(\mathbb{C}, 4)$:

$[(a, v_0, v, w, r, x_0, x, b), (a', v_0', v', w', r', x_0', x', b')] = (A, V_0, V, W, R, X_0, X, B)$

$A = 0$

$\frac{1}{2}V_0 = -v^t w' + w^t v' + x_0 b' - b x_0'$

$\frac{1}{2}V = v_0 w' - v_0' w + b' x - b x' + j(v) r' + j(r) v'$

$\frac{1}{2}W = v_0 v' - v_0' v + x_0' x - x_0 x' + j(w) r' + j(r) w'$

$\frac{1}{2}R = -j(v) v' + j(w) w' + j(r) r' + j(x) x'$

$\frac{1}{2}X_0 = v_0 b' - b v_0' + w^t x' - x^t w'$

$\frac{1}{2}X = b' v - b v' - x_0' w + x_0 w' + j(r) x' + j(x) r'$

$\frac{1}{2}B = v_0 x_0' - v_0' x_0 + v^t x' - x^t v'$

4.2 Killing form

The Killing form is the bilinear map

$$B(Z, Z') = Tr(ad(Z) \circ ad(Z'))$$

It is preserved by all automorphisms on the Lie algebra.

Moreover :

$$B(X, [Y, Z]) = B([X, Y], Z)$$

The Killing form is degenerate : it is null on the radical, and non degenerate on $Cl(F, \rho)/rad$.

Example with $Cl(\mathbb{C}, 4)$:

$B(Z, Z') = 32(v_0 v_0' + v^t v' - w^t w' - r^t r' - x_0 x_0' - x^t x' + b b') = 32(\langle Z^t, Z' \rangle - a a')$

4.3 Lie subalgebras

Any vector subspace of a Clifford algebra which is closed for the bracket is a Lie subalgebra. There are many subalgebras (see Shirokov for a partial list). Among them :

the homogeneous elements of order k are such that $[Cl_k, Cl_k] \subset Cl_2$ so that the homogeneous elements of order 2 constitute a Lie subalgebra.

the Lie subalgebra $Cl_0 = \{Z \in Cl(F, \rho) : \imath(Z) = Z\}$

the Lie subalgebra $T_1O(Cl) = \left\{ Z \in Cl(F,\rho) : (Z)^t = -Z \right\}$ which is the Lie algebra of the orthogonal group.

On a complex Clifford algebra, endowed with a real structure, we can have a real Lie subalgebra. With the morphisms $C : Cl(\mathbb{R},p,q) \to Cl(\mathbb{C},p+q)$, if $L \subset Cl(\mathbb{R},p,q)$ is a Lie algebra, then $C(L)$ is a real Lie algebra in $Cl(\mathbb{C},p+q)$. $T_1U(\mathbb{C},p+q) = \left\{ Z \in Cl(\mathbb{C},p+q) : CC(Z)^t = -Z \right\}$ is the Lie algebra of the unitary group and is a real form of $T_1O(Cl(\mathbb{C},n))$.

Examples with $Cl(\mathbb{C},4)$:

Are Lie subalgebras :

$Cl^2(\mathbb{C},4) : \{(0,0,0,W,R,0,0,0)\}$

$Cl_0(\mathbb{C},4) = \{(A,0,0,W,R,0,0,B)\}$

$Cl_A(\mathbb{C},4) = \{(0,0,0,W,R,X_0,X,0)\}$

$Cl_R(\mathbb{C},4) = \{(A,V_0,V,W,W,-V_0,-V,A)\}$

$\{(A,0,V,0,R,X_0,0,0)\}$

$\{(A,0,V,\epsilon V,R,X_0,-V,\epsilon X_0)\}$ with $\epsilon = \pm 1$

4.3.1 Cartan algebra

In any semi-simple complex Lie algebra L there is a Cartan algebra H which has the properties :

i) it is abelian : $\forall h,h' \in H : [h,h'] = 0$

ii) there is a set $\{Y_j\}$ of vectors of L such that

$\forall h \in H : ad(h)Y_j = \alpha_j(h)Y_j$ where α_j is a linear form on L

iii) $L = H \oplus Span(Y_j)$

$Cl(\mathbb{C},n)/Z_{Cl}$ is semi-simple and has a Cartan algebra, which can be found through a representation (see below).

Example with $Cl(\mathbb{C},4)$:

The Cartan algebra is 4 dimensional :

$T_1\Gamma = \{A + W_1\varepsilon_0 \cdot \varepsilon_1 + R_1\varepsilon_3 \cdot \varepsilon_2 + B\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, A, W_1, R_1, B \in \mathbb{C}\}$

We have a similar result by selecting the components W_2, R_2 or W_3, R_3.

There are 12 vectors

$Y_1(\epsilon_{11},\epsilon_{12}) = i(\varepsilon_0) + \epsilon_{11}(\varepsilon_1) + i\epsilon_{12}(\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) + \epsilon_{11}\epsilon_{12}(\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_2), \epsilon_{ij} = \pm 1$

$Y_2(\epsilon_{21},\epsilon_{22}) = i(\varepsilon_2) + \epsilon_{21}(\varepsilon_3) + i\epsilon_{22}(\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3) + \epsilon_{21}\epsilon_{22}(\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_1), \epsilon_{ij} = \pm 1$

$Y_3(\epsilon_{31},\epsilon_{32}) = i(\varepsilon_0 \cdot \varepsilon_2) + \epsilon_{31}(\varepsilon_0 \cdot \varepsilon_3) + i\epsilon_{32}(\varepsilon_1 \cdot \varepsilon_3) + \epsilon_{31}\epsilon_{32}(\varepsilon_2 \cdot \varepsilon_1), \epsilon_{ij} = \pm 1$

$ad(h)(Y_1(\epsilon_{11},\epsilon_{12})) = -(iW_1\epsilon_{11} + B\epsilon_{12})Y_1(\epsilon_{11},\epsilon_{12})$

$ad(h)(Y_2(\epsilon_{21},\epsilon_{22})) = (-B\epsilon_{22} + iR_1\epsilon_{21})Y_2(\epsilon_{21},\epsilon_{22})$

$ad(h)(Y_3(\epsilon_{31},\epsilon_{32})) = i(R_1\epsilon_{31} + W_1\epsilon_{31}\epsilon_{32})Y_3(\epsilon_{31},\epsilon_{32})$

4.4 Lie groups

Any subset of a Clifford algebra, closed for the product, is a Lie group, subgroup of the group GCl of its invertible elements.

The orthonormal group $O(Cl)$ is a Lie group.

On a complex Clifford algebra, endowed with a real structure, we can have a real Lie group. With the morphisms $C : Cl(\mathbb{R}, p, q) \to Cl(\mathbb{C}, p+q)$ if $G \subset Cl(\mathbb{R}, p, q)$ is a Lie group, then $C(G)$ is a real Lie group in $Cl(\mathbb{C}, p+q)$. The unitary group $U(Cl)$ is a real Lie group, real form of the orthogonal group.

4.4.1 Lie algebra of a Lie group on a Clifford algebra

A Clifford algebra is the Lie algebra of the group GCl of its invertible elements.

The Lie algebra denoted T_1G of a group G is defined as the set of its left invariant vector fields. The tangent vector space to a group belongs to the Clifford algebra. Let $Z : [0, \infty] \to G :: Z(\tau)$ be a path in G, its tangent vector is $T(\theta) = \frac{dZ}{d\tau}|_{\tau=\theta} \in Cl(F, \rho)$. It is left invariant if :

$T(\tau) = L'_Z 1(T(0)) = Z(\tau) \cdot T(0)$ which gives the differential equation : $\frac{dZ}{d\tau} = Z(\tau) \cdot T(0), Z(0) = T(0)$

The left invariant vector fields of G are then characterized by the differential equation : $\frac{dZ}{d\tau} = Z(\tau) \cdot T; Z(\tau) = 1$ which holds whatever the element $T \in T_1G$.

The differential equation reads in coordinates :

$\left[\frac{dZ}{d\tau}\right] = [Z \cdot T] = [\pi_R(T)][Z(\tau)]; Z(0) = 1$

with a fixed matrix $[\pi_R(T)]$ so the solution is given by the exponential of a matrix :

$[Z] = [\exp[\pi_R(T)]][1] = [1 \cdot \exp T] = [\exp T]$

$$Z : [0, \infty] \to G :: Z(\tau) = \exp \tau T \Leftrightarrow \frac{dZ}{d\tau} = Z(\tau) \cdot T$$

Which gives the rule to compute the Lie algebra of a group defined by a relation on its elements. For instance $g^t \cdot g = 1$: take $g = Z(\tau)$ and by differentiation : $\left(\frac{dZ}{d\tau}\right)^t \cdot Z(\tau) + (Z(\tau))^t \cdot \left(\frac{dZ}{d\tau}\right) = 0$ and at $Z(0) = 1 : T^t + T = 0$.

The exponential on a Lie algebra has well known general properties in particular :

$\forall T \in Cl(F, \rho)$:

$\exp(ad(T)) = Ad_{\exp T}$

$\frac{d}{d\tau}(Ad_{\exp \tau T} X) = Ad_{\exp \tau T}[T, X]$

from where we have :

$g \cdot \exp T \cdot g^{-1} = Ad_g \exp T = \exp(Ad_g T)$

4.4.2 Compact Lie groups

A Lie group is compact if it is compact as a manifold, then its Lie algebra is compact. The simplest criterion for a real group is that, if its Killing form is definite negative, then it is compact.

From the definition : $B(Z, Z') = Tr(ad(Z) \circ ad(Z'))$

$B(Z, Z) = Tr(ad(Z) \circ ad(Z)) = \sum_{i,j=1}^{n^2} [ad(Z)]_j^i [ad(Z)]_i^j$

$= \sum_{i,j=1}^{n^2} [ad(Z)]_j^i \left([ad(Z)]^t\right)_j^i$

$[ad(Z)]^t = [\eta][ad(Z^t)][\eta]$

For the orthogonal group : $Z^t + Z = 0 \Rightarrow [ad(Z)]^t = -[ad(Z)]$.

On $Cl(\mathbb{R}, n, 0), Cl(\mathbb{R}, 0, n)$ the orthogonal group $O(Cl)$ is compact.

On $Cl(\mathbb{C}, n)$ with a morphism C, the unitary group $U(Cl)$ is a real Lie group $CC(Z^t) + Z = 0.$

$CC(ad(Z)) = CC(\pi_L(Z)) - CC(\pi_R(Z)) = ad(CC(Z))$

$[ad(Z)]^t = [ad(Z^t)] = -[ad(CC(Z))]$

If $J = (1, 2, ..n)$, that is for the morphism $Cl(\mathbb{R}, n) \to Cl(\mathbb{C}, n)$ with $p = n, q = 0$, then $[ad(CC(Z))] = \overline{[ad(Z)]}$ and $B(Z, Z) = -\sum_{i,j=1}^{n^2} \overline{[ad(Z)]}_j^i [ad(Z)]_j^i$ is definite negative, and the unitary group is compact. Then the Cartan algebra is a maximal torus.

4.4.3 Computing a Lie group from its Lie algebra

If L is a Lie subalgebra of a group G then the map : $\exp : L \to G :: g = \exp T$ is well defined, but not onto : some elements of the group cannot be written this way (usually they can be written $\pm \exp T$). The exponential is onto if the group is compact.

A Lie group is a manifold, and a group G in a Clifford algebra is a manifold embedded in a vector space, it has a chart :

$\varphi : Cl(F, \rho) \to G :: \varphi(x_1, .., x_\alpha) = g$

where x_α are coordinates in the basis of $Cl(F, \rho)$.

When the Lie algebra of a group can be written : $T_1 G = T_1 H \oplus E$ where H is the Lie algebra of a subgroup H and E a vector subspace, and the exponential is onto H, there is a chart :

$\varphi : H \times E \to G :: g = h \cdot \exp T$

which is convenient when $T \cdot T$ is a scalar. The chart is differentiable, but usually we do not have $g \cdot g' = h \cdot h' \cdot \exp T \cdot \exp T'$.

4.4.4 Spin group

The Spin group $Spin(F, \rho)$ of $Cl(F, \rho)$ is the subset of $Cl(F, \rho)$ whose elements can be written as the product $g = u_1 \cdot ... \cdot u_{2p}$ of an even number of vectors of F of norm $\langle u_k, u_k \rangle = 1$.

As a consequence : $\langle g, g \rangle = 1, g^t \cdot g = 1$ and $Spin(F, \rho) \subset O(Cl)$.

The scalars ± 1 belong to the Spin group. The identity is $+1$. $Spin(F, \rho)$ is a connected Lie group.

The Lie algebra is $T_1 Spin(F, \rho) = \{T^t + T = 0\}$ as the orthogonal group. Because $(\varepsilon_1 \cdot \varepsilon_2 ... \cdot \varepsilon_p)^t = (-1)^{\frac{1}{2}p(p-1)} \varepsilon_1 \cdot \varepsilon_2 ... \cdot \varepsilon_p$ the components of order odd must be null.

The map : $Ad : Spin(F, \rho) \to \mathcal{L}(Cl(F, \rho); Cl(F, \rho))$ is an action and defines a group of automorphisms.

The adjoint map Ad_g preserves the scalar product and maps F to F. The matrix of $[Ad_g]$ on F belongs to $SO(n)$, it defines uniquely $[Ad_g]$ on $Cl(F, \rho)$ and there is a subjective group morphism $Spin(F, \rho) \to SO(n)$. But $+g$ and $-g$ gives the same matrix, and $Spin(F, \rho)$ is the double cover of $SO(n)$.

Example with $Cl(\mathbb{C},4)$:

The group $Spin(\mathbb{C},4)$ is a 6 dimensional complex semi-simple Lie group with Lie algebra :

$T_1 Spin(\mathbb{C},4) = \left\{ T = (0,0,0,W,R,0,0,0), W,R \in \mathbb{C}^3 \right\}$

$T_1 Spin(\mathbb{C},3) = \left\{ T_r = (0,0,0,0,R,0,0,0), R \in \mathbb{C}^3 \right\}$ is the Lie algebra of the Lie group $Spin(\mathbb{C},3)$

$T_r \cdot T_r = -R^t R$ and the elements of the group read :

$\exp T_r = \cosh \mu_r + \frac{\sinh \mu_r}{\mu_r} (T_r)$ with $\mu_r^2 = -R^t R = T_r \cdot T_r$

The vector space $\left\{ T_w = (0,0,0,W,0,0,0,0), W \in \mathbb{C}^3 \right\}$ is not a Lie algebra.

$T_w \cdot T_w = -W^t W$ and $\exp T_w = \cosh \mu_w + \frac{\sinh \mu_w}{\mu_w} (T_w)$ with $\mu_w^2 = -W^t W = T_w \cdot T_w$

The elements of the group $Spin(\mathbb{C},4)$ read :

$g = \exp T_w \cdot \exp T_r$ with $T_w \cdot T_r = (0,0,0,j(W)R,0,0,0,-W^t R)$

or $g = (a,0,0,w,r,0,0,b)$

with

$a = \cosh \mu_w \cosh \mu_r$

$w = \frac{\sinh \mu_w}{\mu_w} \left(\cosh \mu_r - \frac{\sinh \mu_r}{\mu_r} j(R) \right) W$

$r = \cosh \mu_w \frac{\sinh \mu_r}{\mu_r} R$

$b = -\frac{\sinh \mu_w}{\mu_w} \frac{\sinh \mu_r}{\mu_r} (W^t R)$

and :

$w^t r = -ab$

$a^2 + b^2 + w^t w + r^t r = 1$

$g^{-1} = (a,0,0,-w,-r,0,0,b)$

5 REPRESENTATION OF CLIFFORD ALGEBRAS

5.1 Definitions

An algebraic representation of a Clifford algebra $Cl(F,\rho)$ over a field K is the couple (A,γ) of a unital algebra $(A,\circ)$ on the field K and a map : $\gamma : Cl(F,\rho) \to A$ which is an algebra morphism :

$\forall X,Y \in Cl(F,\rho), k,k' \in K$:

$\gamma(kX + k'Y) = k\gamma(X) + k'\gamma(Y)$,

$\gamma(X \cdot Y) = \gamma(X) \circ \gamma(Y), \gamma(1) = I_A$

A geometric representation of a Clifford algebra $Cl(F,\rho)$ over a field K is a couple (V,ϑ) of a vector space V on the field K and a map : $\vartheta : Cl(F,\rho) \to G\mathcal{L}(V;V)$ which is an algebra morphism :

$\forall X,Y \in Cl(F,\rho), k,k' \in K$:

$\vartheta(kX + k'Y) = k\vartheta(X) + k'\vartheta(Y)$,

$\vartheta(X \cdot Y) = \vartheta(X) \circ \vartheta(Y), \vartheta(1) = Id_V$

If (A,γ) is a representation of $Cl(\mathbb{C},n)$ then $\gamma \circ C$ is a real representation of $Cl(\mathbb{R},p,q)$.

5.1.1 The generators of a representation

The generators of an algebraic representation (A, γ) of the Clifford algebra $Cl(F, g)$ are : $(\gamma_i)_{i=0}^n : \gamma_i = \gamma(\varepsilon_j), j = 1..n, \gamma_0 = \gamma(1)$ where $(\varepsilon_j)_{j=1}^n$ is an orthonormal basis of F. They meet necessarily the relation :

$$\forall j, k = 1...n : \gamma_j \gamma_k + \gamma_k \gamma_j = 2 \langle \varepsilon_j, \varepsilon_k \rangle_F \gamma_0$$

Conversely a set of generators, which are invertible and $\gamma_0 = 1_A$ defines uniquely an algebraic representation.

5.1.2 Equivalence of representations

Two algebraic representations $(A_1, \vartheta_1), (A_2, \vartheta_2)$ of a Clifford algebra $Cl(F, \rho)$ are said to be equivalent if there are :

i) a bijective algebra morphism $\phi : A_1 \to A_2$

ii) an automorphism $\tau : Cl(F, \rho) \to Cl(F, \rho)$

such that : $\phi \circ \vartheta_1 = \vartheta_2 \circ \tau$

$$\begin{array}{ccccc} & & \tau & & \\ & Cl(F,g) & \to & Cl(F,g) & \\ \vartheta_1 & \downarrow & & \downarrow & \vartheta_2 \\ & A_1 & \to & A_2 & \\ & & \phi & & \end{array}$$

The automorphisms on a Clifford algebra correspond to a change of orthonormal basis on F. On the same algebra A, all the equivalent representations are defined by conjugation with a fixed invertible element U : $\widetilde{A} = U \circ A \circ U^{-1}$.

If (V, ϑ) is a geometric representation of $Cl(F, \rho)$ then (V^*, ϑ^*) with V^* the dual of V and ϑ^* the transpose of ϑ, is another representation, which usually is not equivalent.

If $Cl(F, \rho)$ is a complex Clifford algebra, with real structure C, A a complex algebra endowed with a real structure σ, then to any algebraic representation (A, γ) is associated the contragredient representation : $(A, \widetilde{\gamma})$ with $\widetilde{\gamma} = \sigma \circ \gamma \circ C$ which, usually, is not equivalent.

5.1.3 Representation on the exterior algebra

A Clifford algebra $Cl(F, \rho)$ has a geometric representation on the algebra ΛF^* of linear forms on F.

Consider the maps with $u \in F$:

$\lambda(u) : \Lambda_r F^* \to \Lambda_{r+1} F^* :: \lambda(u) \mu = u \wedge \mu$

$i_u : \Lambda_r F^* \to \Lambda_{r-1} F^* :: i_u(\mu) = \mu(u)$

The map : $\Lambda F^* \to \Lambda F^* :: \widetilde{\vartheta}(u) = \lambda(u) - i_u$ is such that :

$\widetilde{\vartheta}(u) \circ \widetilde{\vartheta}(v) + \widetilde{\vartheta}(v) \circ \widetilde{\vartheta}(u) = 2\rho(u, v) Id$

thus there is a map : $\vartheta : Cl(F, g) \to \Lambda F^*$ such that : $\vartheta \cdot \imath = \widetilde{\vartheta}$ and $(\Lambda F^*, \vartheta)$ is a geometric representation of $Cl(F, \rho)$. It is reducible.

5.2 Representations on algebras of matrices

5.2.1 Complex Clifford algebras

The unique faithful, irreducible, algebraic representation of the complex Clifford algebra $Cl(\mathbb{C}, n)$ is over an algebra $L(\mathbb{C}, m)$ of matrices of complex numbers.

The algebra $L(\mathbb{C}, m)$ depends on n :

If $n = 2p$: $m = 2^p$: the square matrices $2^p \times 2^p$ (we get the dimension 2^{2p} as vector space)

If $n = 2p + 1$: $4p \times 4p$ complex matrices of the form :

$[M] = \begin{bmatrix} [A]_{2p\times 2p} & 0 \\ 0 & [B]_{2p\times 2p} \end{bmatrix}_{4p\times 4p}$

(the vector space has the dimension 2^{2p+1}).

The representation is faithful : there is a bijective correspondence between elements of the Clifford algebra and matrices.

There is always a representation such that the generators are Hermitian, then they are also unitary (see Shirokov).

Representation of $Cl(\mathbb{C}, 4)$

The representations are built around the Dirac's matrices :

$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

which are such that : $\sigma_j = \sigma_j^*; \sigma_j\sigma_k + \sigma_k\sigma_j = \delta_{jk} I_2$

A convenient representation is with :

$\gamma_4 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; j = 1, 2, 3 : \gamma_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$

The generators have the property that : $j = 1...4 : \gamma_j = (\gamma_j)^* = (\gamma_j)^{-1}$

5.2.2 Real Clifford algebras

The unique faithful irreducible algebraic representation of the Clifford algebra $Cl(\mathbb{R}, p, q)$ is over an algebra of matrices. The matrices algebras are over a field K' $(\mathbb{C}, \mathbb{R})$ or the division ring H of quaternions with the following rules :

$(p-q) \bmod 8$	$Matrices$	$(p-q) \bmod 8$	$Matrices$
0	$\mathbb{R}(2^m)$	0	$\mathbb{R}(2^m)$
1	$\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$	−1	$\mathbb{C}(2^m)$
2	$\mathbb{R}(2^m)$	−2	$H(2^{m-1})$
3	$\mathbb{C}(2^m)$	−3	$H(2^{m-1}) \oplus H(2^{m-1})$
4	$H(2^{m-1})$	−4	$H(2^{m-1})$
5	$H(2^{m-1}) \oplus H(2^{m-1})$	−5	$\mathbb{C}(2^m)$
6	$H(2^{m-1})$	−6	$\mathbb{R}(2^m)$
7	$\mathbb{C}(2^m)$	−7	$\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$

The division ring of quaternions can be built as $Cl_0(\mathbb{R},0,3)$

When the Clifford algebra is real and represented by a set of real $2^m \times 2^m$ matrices there is a geometric representation on $\mathbb{R}^{2m}$. The vectors of $\mathbb{R}^{2m}$ in such a representation are the Majorana spinors.

5.2.3 Equivalence between the adjoint representation on the Clifford algebra and the representation of the Clifford Algebra

To keep it simple let us consider $Cl(\mathbb{C},2n)$ with its representation $(L(\mathbb{C},2^n),\gamma)$.

Let (T_1G, Ad) be a representation of a group $G \subset Cl(\mathbb{C},2n)$ on the Clifford algebra itself with the adjoint map..The Lie algebra $T_1G \subset Cl(\mathbb{C},2n)$

Let us consider the action : $\Theta : G \to \mathcal{L}(L(\mathbb{C},2^n);L(\mathbb{C},2^n)) :: \Theta(g)(M) = [\gamma(g)][M][\gamma(g)]^{-1}$

It has the properties :

$\Theta(g\cdot g')(M) = [\gamma(g\cdot g')][M][\gamma(g\cdot g')]^{-1} = \Theta(g)\circ\Theta(g')(M)$

$\forall [M] \in L(\mathbb{C},2^n), \exists Z \in L(\mathbb{C},2^n) : [M] = [\gamma(Z)]$

$\Theta(g)(\gamma(Z)) = [\gamma(g)][\gamma(Z)][\gamma(g)]^{-1} = \left[\gamma\left(g\cdot Z\cdot g^{-1}\right)\right] = [\gamma(Ad_g Z)] \Leftrightarrow \Theta(g)\circ\gamma = \gamma\circ Ad_g \Leftrightarrow \Theta(g) = \gamma\circ Ad_g\circ\gamma^{-1}$

We have the commuting diagram :

$$\begin{array}{ccccc}
Cl(\mathbb{C},2n) & & Ad_g & & Cl(\mathbb{C},2n) \\
Z & \to & \to & \to & Ad_g(Z) \\
\downarrow & & & & \downarrow \\
\gamma & & & & \gamma \\
\downarrow & & & & \downarrow \\
\gamma(Z) & \to & \to & \to & \Theta(g)(\gamma(Z)) \\
L(\mathbb{C},2^n) & & \Theta(g) & & L(\mathbb{C},2^n)
\end{array}$$

The representation $(Cl(\mathbb{C},2n), Ad)$ of G is equivalent to the representation $(L(\mathbb{C},2^n),\Theta)$ of G by $\Theta(g) = \gamma\circ Ad_g\circ\gamma^{-1}$ and the morphism is an isomorphism because γ is bijective. The action Θ is just the adjoint action on matrices and the representation $(L(\mathbb{C},2^n),\Theta)$ of G is a subrepresentation of the adjoint representation $(L(\mathbb{C},2^n),\Theta)$ of $GL(\mathbb{C},2n)$, as $(Cl(\mathbb{C},2n),Ad)$ is a subrepresentation of the group $GCl(\mathbb{C},2n)$ of invertible elements of $Cl(\mathbb{C},2n)$.

The 2^n matrices $\gamma(F_\alpha)$ are linearly independent because F_α are independent, thus they constitute a basis of $L(\mathbb{C},2^n)$. In this basis the matrix of $\Theta(g)$ is the same as Ad_g in the orthonormal basis of $Cl(\mathbb{C},2n)$:

$\Theta(g)(M) = \Theta(g)\left(\sum_\alpha \kappa^\alpha[\gamma(F_\alpha)]\right) = \sum_\alpha \kappa^\alpha[\gamma(g)][\gamma(F_\alpha)][\gamma(g)]^{-1}$

$= \sum_\alpha \kappa^\alpha[\gamma(Ad_g(F_\alpha))] = \sum_\alpha \kappa^\alpha\left[\gamma\left(\sum_\beta [Ad_g]^\beta_\alpha F_\beta\right)\right] = \sum_{\alpha,\beta}[Ad_g]^\beta_\alpha \kappa^\alpha\gamma(F_\beta)$

Whenever the group G is defined by a condition on the matrix Ad_g the same condition applies on the representation $(L(\mathbb{C},2^n),\Theta)$.

The map γ depends on a choice of generators but it is faithful. To each $2^n\times 2^n$ matrix representing $[\Theta(g)]$ corresponds a unique matrix Ad_g and thus a unique g, up to the product by a constant.

$(L(\mathbb{C},2^n),\Theta)$ is the adjoint representation of $GL(\mathbb{C},2^n)$ on its Lie algebra. Similarly $(Cl(\mathbb{C},n),Ad)$ is the adjoint representation of GCl on its Lie algebra. The two representations are equivalent, as well as their derivative : the representation $(L(\mathbb{C},2^n),ad)$ of $L(\mathbb{C},2^n)$ and $(Cl(\mathbb{C},n),ad)$ of $Cl(\mathbb{C},n)$. The root spaces decomposition of the representation $(sl(\mathbb{C},2^n),ad)$ is based on the Cartan algebra of diagonal matrices, then the Cartan algebra of $Cl(\mathbb{C},2n)$ is given by the 2^n-1 elements F_α of the basis which are represented by diagonal matrices.

These results can be extended at any complex Clifford algebra.

6 REFERENCES

R.Coquereaux *Clifford algebra, spinors and fundamental interactions : twenty years after* arXiv:math-ph/0509040v1 (16 sep 2005)

P.Dechant *Clifford algebra is the natural framework for roots systems and Coxeter groups* arXiv:math-ph/1602-06003v1 (18 feb 2016)

J.C.Dutailly *Mathematics for theoretical physics* (2016) Amazon E-book

A.W.Knapp *Lie groups beyond an introduction* Birkhäuser (2005)

I.Kolar, P.Michor, J.Slovak *Natural operations in differential geometry* Springer-Verlag (1991)

D.S.Shirokov *On some Lie groups containing spin groups in Clifford algebras* arXiv:math-ph:1607-07363v2 (19 Aug 2017)