Calculus 1

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1 Introduction

Calculus is about solving various problems quickly and easily. The word *calculus* is related to the word *calculation* and despite the feeling that it is a hard subject to learn, everyone who learns it performs calculations easily and eventually almost automatically.

Perhaps the most fun calculations are those that show how to derive classic geometric formulas such as the circumference of a circle, the area of a circle, and the volume of a sphere (cf. the inside back cover of your book). Other types of problems give laws of physics and still others model engineering phenomenon and can predict the success and failure of some dynamic systems: browse the inside back and front covers of your book.

Another attractive feature of calculus is that it is all built on a single block. The cornerstone is the *function* and the building is comprised of lots of functions that can be combined and used to model lots of phenomenon. Properties of these functions can be calculated with calculus.

The two big calculations are *integration* and *differentiation*. The first combines or sums many things to tell a story and other differentiates things or pulls things apart to reveal properties. We will look at two of the classic examples of this in the next section and then give a broad roadmap for learning calculus in the last section of this introduction.

1.1 Two classic problems

1.1.1 Area under a curve

Everyone knows the area of a rectangle is length times width and that of a triangle is one half base times height. Let's put a constant function $f(x) = 3$ on the Cartesian coordinate system and calculate the area using an integral

– say the area under $f(x)$ between the origin and four. Here's the look of this calculus problem:

$$\int_0^4 3 \text{ dx}. \tag{1}$$

This is pronounced the integral from 0 to 4 of 3 dx. There is a rule for calculuating such integrals, it is given by

$$\int_a^b cx^k \text{ dx} = \frac{cx^{k+1}}{k+1}\Big|_a^b = \frac{cb^{k+1} - ca^{k+1}}{k+1}. \tag{2}$$

This says that the integral of a power of x is found by adding one to the power of x and dividing by that new power. We get a new function given a function and that new function is the indefinite integral. If there is a constant involved, just keep the constant. In integral calculus we frequently want to evaluate that function at two points and find the difference. That's the definite integral. So here, what is the power of x in (1)? And what is the constant c? We have

$$\int_0^4 3x^0 \text{ dx} = \frac{3x^{0+1}}{0+1}\Big|_0^4 = 3 \cdot 4 - 3 \cdot 0 = 12 \tag{3}$$

and that is the area under the curve. Did we know this already?

Let's try calculating the area of a triangle given by the area under the curve $f(x) = x$ from the origin to three. Here we have

$$\int_0^3 x \text{ dx} \tag{4}$$

as the definite integral and we can use the formula given in (6) to calculate the area of the triangle. We have

$$\int_0^3 x \text{ dx} = \frac{x^2}{2}\Big|_0^3 = \frac{3^2 - 0^2}{2} = \frac{9}{2}. \tag{5}$$

Is this correct?

Congratulations you are using the integral calculus to find the area under curves. So far one could say that the areas calculated don't need such complicated machinery as the integral calculus, but we are not limited to confirming easy areas that we know already. We can explore functions and the areas associated with them for which we have no rule. So, for example, what is the area under the quadratic $f(x) = x^2$ between the origin and three? It's not going to be hard to find:

$$\int_0^3 x^2 \text{ dx} = \frac{x^{2+1}}{2+1}\bigg|_0^3 = \frac{3^3 - 0^3}{3} = \frac{27}{3} = 9. \tag{6}$$

Finding such an area is easy now, but it took a couple of thousand years to make it easy. Can we find the area under x^3, $5x^5$, etc.? How about $x^2 + 2x$? What would you guess the answer is? Of course: you just add the results of two integrals. Are we limited to just the areas of polynomials? Not at all. We will review all the major types of functions calculus can calculate in the next section: our roadmap. Now we turn to the other classic problem calculus solves.

1.1.2 Find the maximum

Hopefully you have taken college algebra and trigonometry (or precalculus) and are familiar with quadratic functions given by $f(x) = ax^2 + bx + c$. You recall that the maximum or minimum is given by

$$x = -\frac{b}{2a}. \tag{7}$$

It's a maximum if $a < 0$ and a minimum if $a > 0$. Draw a parabola and consider the slope of the tangent at such maximums and minimums? What will it be? We are differentiating points in the domain of the function when we find which ones have different tangent line slopes. There is a way to calculate such slopes. One takes the derivative at a specific value. The derivative of cx^k is given by

$$\frac{d}{dx}cx^k = kcx^{k-1}. \tag{8}$$

Whereas before with the integral we increased the power of x by one and divided by it, now we multiply by the current power and diminish the power by one. So what is the derivative of x? It is just $1x^0$ or, that's right, one. What is the slope of the line tangent to $f(x) = x$ at all points? It is, of course, just one. How about $f(x) = bx$? It is just b. And $f(x) = x^2$? It is just $2x$. At $x = 0$ what is the slope of the line tangent to this curve?

Let's try $f(x) = -(x-2)^2 + 2$. This quadratic uses the standard form of the quadratic. It's vertex occurs at $(2, 2)$ and its vertex gives the maximum. Converting this form into the $ax^2 + bx + c$ form we have

$$f(x) = -(x^2 - 4x + 4) + 2 = -x^2 + 4x - 2 \tag{9}$$

and the formula $-b/2a$ confirms $x = 2$ is the maximum, but using calculus we can take the derivative and get the slope of the tangent lines and set it equal to zero. So

$$\begin{aligned} \frac{d}{dx}\left(-x^2 + 4x - 2\right) &= \frac{d}{dx}\left(-x^2\right) + \frac{d}{dx}4x + \frac{d}{dx}\left(-2\right) && (10) \\ &= -2x + 4 && (11) \end{aligned}$$

and $-2x + 4 = 0$ gives $x = 2$, as we already knew. What about the general formula? We have

$$\frac{d}{dx}\left(ax^2 + bx + c\right) = 2ax + b \tag{12}$$

and $2ax + b = 0$ is solved by, you guessed it, $-b/2a$.

Are we limited to quadratics? Not at all. Where does the maximum and minimum of a third degree polynomial occur? We don't have a formula for that in college algebra, but now we can determine such things by routine calculations? So, for example, where does the max and min occur in $f(x) = x^3 - 3x^2 + 3$? We can use our calculators to find this, but with calculus we take the derivative, set to zero, and solve for x:

$$\begin{aligned} \frac{d}{dx}\left(x^3 - 3x^2 + 3\right) = 3x^2 - 6x = 3x(x-2) && (13) \\ 3x(x-2) = 0 && (14) \end{aligned}$$

and the last equation gives roots 0 and 2. At $x = 0$ we get 3 and at 2 we get $8 - 12 + 3 = -1$, so $x = 0$ gives a relative maximum and $x = 2$ gives a relative minimum. This is problem 23, page 215. Looking at the problems on this page, one senses that any function we can generate with arithmetic operations we can differentiate.

We will review all such functions and a few integral and differentiation formulas in our roadmap. Right now let's consider the integral and derivative of x^k. Is there a relationship? The derivative is kx^{k-1} and the integral of this gives back x^k. Does this remind one of anything? Consider $\ln e^{f(x)}$ it is just $f(x)$ because the natural logarithm and the exponential function are inverses. So what is the relationship between the derivative and the integral? Yes, they are inverses. So everytime we get a derivative formula we should look for a corresponding integral formula.

Problems 1

1. *Given the derivative of* cos *is* $-$sin *and the derivative of the* sin *is* cos, *what is the area under the* cos *curve between* 0 *and* $\pi/2$*? What other integral can you solve?*

2. *Given the derivative of* e^x *is* e^x *what is*
$$\int_1^2 e^x \; dx? \tag{15}$$

3. *Looking at the inner cover of your book, the cutout entitled* Derivatives and Integrals, *speculate on what u and v are. Hint: think of composition and functions.*

4. *Try using your calculators* calc *menu to explore integration and differentiation. You need a TI-83 or better for this course. You will generally not be allowed to use a TI-89, but we will talk about this calculator.*

5. *Sections 1.5 and 5.3 are pure college algebra (or precalculus) sections. Take a look at them.*

6. *Section 3.6 gives a summary of curve sketching. This should be conceptually clear and help motivate the reason why differential calculus is closely linked with analytical geometry.*

1.1.3 Roadmap to calculus

The question a student should be asking at this point is *how did you get the formula for the area under a curve and the slope of the line tangent to a curve?* Those answers come in the form of considering various limit problems: adding up an infinite number of rectangles and making ever smaller secants for a curve, respectively. Looking at your book, consider the table of contents and chapters P and 1 through 5. These are the chapters we will be covering in

	properties/limits/formulas	d/dx	$\int$	inverses	series	apps
poly						
rational						
exp						
ln						
trigs						
hyper						

Table 1: Roadmap for Calculus functions

this course. We start with limit ideas and then consider differential calculus and then integral calculus. We derive the formulas mentioned using limits. We extend these polynomial formulas to a set of functions and combinations of functions. That's the roadmap. Browse your book and note on page 284, example 3 one sees an area problem and on page 213, example 5 one sees a curve sketching problem (finding slopes of tangent lines).

In Table 1 there is a list of the functions we will consider. This list applies for other two courses in the calculus sequence. In calculus 2 an important column is added: series. That is how do you express transcendental functions as infinite series. In calculus 3 we extend the number of independent variables considered. Don't worry if you don't understand what infinite series or independent variables are right now. That's for subsequent courses. For each function family we want to consider the column topics: what are its properties, its derivatives, and its integrals. We then wish to extend these functions by composing with them. So, for example, what is the derivative of $\sin(e^x)$? What is its integral? There are formulas for such things and techniques. Easier than composition, we can add functions: e.g. $\sin x + x^2$. In general, we can use any number of arithmetic operations to build new functions from old.

The formulas for differentiating say the product or quotient of two functions is built up slowly: limit theorems lead to continuity theorems which lead to differentiation theorems and then integration theorems. Table 2 shows the key Theorems (formal math statements with proofs). Some Theorems are also formulas – procedures for finding the derivatives of products and quotients, for example.

Theorems/Formulas	$bf(x), f(x) \pm g(x), f(x)g(x), f(x)/g(x), f^n(x), f(g(x))$
Limits	T1.2, p. 59; T1.5, p.61
Continuity	T1.11, p. 75, T1.15, p. 87
Derivatives	p. 136 Summary
Integrals	p. 250 Basic Integration Formulas
Integrals	inverses, change of variable

Table 2: Differential and Integral Single Variable Theory

Problems 2

1. *Some proofs are fairly easy. Try looking at the proof of Theorem 2.5 on page 111 and express the general statement of the theorem.*
2. *The product rule, Theorem 2.7, is considerably harder. It uses a common theme: express something you don't know using things you do know.*
3. *A hard and important theorem is given on page 65. Try to understand the geometric argument given in Theorem 1.9. It is crucial to finding the derivatives of* sin *and* cos*.*
4. *Consider the pictures and captions supplied at the beginning of each chapter of the book (or chapters 1-5). Use your imagination to speculate on the problem posed. Try doing the same for a few items in the* Index of Applications *in the front cover of your book (try the items that start with a* p*), like* pendulum*.*

2 Coming Attractions

I will be making a video that shows how typical differentiation and integration problems are done. All people who learn calculus can always do such problems, even after years of inactivitity. The routines become automatic for everyone who masters calculus.

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