The Optimization Principle for the Riemann Hypothesis

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Abstract. Based on the observation that several physical, biological and social proceesses seem to be optimizing an objective function such as an action or a utility, the Central Principle of Science was deemed to be Optimization. Indeed, optimization proved to be an efficient tool for uncovering several scientific laws and proving some scientific theories. In this paper, we use this paradigm to identify the location of the nontrivial zeros of the Riemann Zeta function (RZF).This approach enabled the formulation of this problem as a constrained optimization problem where a simple objective function referred to here as the "Push-Pull Action" is maximized. The solution of the resulting constrained nonlinear optimization problem proved that nontrivial zeros of RZF are located on the critical line. In addition to proving the Riemann Hypothesis, this approach unveiled a plausible law of "Maximum Action of Push-Pull" that seems to be driving RZF to its equilibrium states at the different heights where it reaches its nontrivial zeros. We also show that this law applies to functions exhibiting the same properties as RZF.

Keywords: Zeta function, Riemann Hypothesis, Constrained Optimization

Introduction

A great deal of research has been and still is being devoted to the zeros of the Riemann Zeta function (RZF) that are located on the critical strip[1] of the complex plane, commonly known as the nontrivial zeros of RZF. The Riemann Hypothesis (RH) states that the location of RZF's nontrivial zeros is the critical line[2]. Although billions of nontrivial zeros have proved to be located on the critical line through numerical computational methods, no general analytical proof or disproof of RH have been found since its conjecture by Riemann in 1859.

In this paper, we propose an analytical approach to RH based on optimization. This approach proved efficient for deriving some important scientific laws and theories. In the paper "A Central Principle of Science: Optimization" [1], it was argued that the structure of scientific theories is based on the implicit assumption of the consistency underlying them. This hypothesis led to the belief that scientific consistency can be maintained by using an optimization framework where systems and processes are derived from the optimization of an objective function such as minimizing an action or maximizing a utility. Indeed, the formulation of the appropriate objective functions enabled the proof of theories and the uncovering of laws in such scientific fields as physics, economics and psychology, among other scientific areas [2].

Such a solid scientific support motivated us to apply the paradigm of optimization believed to be the Central Principle of Science, to identify the location of the nontrivial zeros of the Riemann Zeta Function (RZF). By formulating and solving the appropriate optimization

[1] Denoting RZF by ζ ($s= \sigma +it$), the critical strip is the defined by $0 < \sigma < 1$

[2] The critical line is the line $\sigma = ½$

problem, we derive evidence in support of the Riemann Hypothesis and uncover a law that underlines a plausible cause of this location.

This paper is organized in two sections. In Section I, we first present the analytic continuation of RZF in the critical strip as an integral functional, and list some of its properties that are useful in formulating the objective function of the optimization problem of interest. We then list the properties of RZF's nontrivial zeros that are relevant to the formulation of the constraints on the objective function. In Section II, we define the optimization-based approach, formulate and solve the resulting optimization problem, then derive support for RH as well as a plausible law that is underlying RZF's drive toward its states of nontrivial zero values. In the concluding section, the results of our analyses are summarized.

I. Properties of RZF and its nontrivial zeros

Ia. Some Relevant Facts about RZF

As in the traditional notation, the Riemann Zeta Function is denoted $\zeta(s) = U(s) + iV(s)$, s being a complex variable defined as $s = \sigma + it$, with σ and t. being real variables, U and V are real functions of σ and t. The strip in the complex plane between the lines $\sigma = 0$ and $\sigma = 1$ is referred to as the critical strip. The line $\sigma = ½$ is called the critical line.

For notation simplification, the values σ^* and t^* where a nontrivial zero of RZF is achieved will often be implicit in the functions and expressions used in the analyses. In addition, for simplicity of carrying out our analysis, the following notation will be used:

The partial derivative $\partial U/\partial\sigma$ and $\partial V/\partial\sigma$ will be denoted by U_σ and V_σ, respectively. Quantities where RZF vanishes are upper-scripted by a star, thus the notation U^*, V^*, U^*_σ, V^*_σ, and $s^* = (\sigma^* + it^*)$ denoting a nontrivial zero of RZF at $\sigma = \sigma^*$ *and* $t = t^*$. The value σ^* in the critical strip where RZF vanishes at some height $t = t^*$ will be referred to as the location of a nontrivial zero of RZF for $t = t^*$, which is conjectured to be the critical line according to RH.

The following integral functional representing the analytic continuation of RZF in the critical strip is used to provide exact expressions of the real part U and the imaginary part V of $\zeta(s)$, and to present some of their relevant properties. This integral functional form of RZF [3] is given by:

$$\zeta(s) = -s\int_0^\infty \frac{\{x\}}{x^{s+1}}dx \ ; \quad s = \sigma + it; \ \{x\} = x - integer(x)$$

Letting $I = \int_0^\infty \frac{\{x\}}{x^{s+1}} dx$, then one has:

$$I = \int_0^\infty \frac{\{x\}}{x^{\sigma+1}} x^{-it} dx = \int_0^\infty \frac{\{x\}}{x^{\sigma+1}} [\cos(t\log(x))]dx - i\int_0^\infty \frac{\{x\}}{x^{\sigma+1}} [\sin(t\log(x))]dx$$

Defining F and G as:

$$F = \int_0^\infty \frac{\{x\}}{x^{\sigma+1}} [\cos(t\log(x))]dx \ ; \ G = \int_0^\infty \frac{\{x\}}{x^{\sigma+1}} [\sin(t\log(x))]dx$$

One gets $I = F – iG$, and $\zeta(s) = - (\sigma + it)(F - iG) = - (\sigma F + tG) + i(\sigma G – tF)$

So that $U = - (\sigma F + tG)$ and $V = (\sigma G – tF)$.

Since $\zeta(\sigma + it)$ is analytic in the critical strip, the functions F and G are convergent and (infinitly) differentiable in the critical strip. This implies that U and V are also convergent and (infinitily) differentiable, hence $U(\sigma;\ t^*)$ and $V(\sigma;\ t^*)$ are also differentiable real functions of σ, and their derivatives U_σ ,V_σ , $U_{\sigma\sigma}$ and $V_{\sigma\sigma}$ exist, therefore, at $s^* = (\sigma^* + i\ t^*)$ where RZF vanishes , one has $U^* = 0$, and $V^* = 0$, so that:

$$U^*U^*_\sigma + V^*V^*_\sigma = 0, \text{ and } U^*U^*_{\sigma\sigma} + V^*V^*_{\sigma\sigma} = 0 \qquad (1)$$

Ib: Some relevant properties of RZF's nontrivial zeros

The most important and relevant properties of RZF [3] are listed below:

1. RZF has an infinite number of nontrivial zeros
2. The nontrivial zeros are located in the critical strip at different heights $t = t^*$
3. The nontrivial zeros are symmetric about the critical line $\sigma = 1/2$, and quite a few of them proved to be on this line.
4. If σ^* is a location of a nontrivial zero at $t = t^*$, then $(1- \sigma^*)$ is also a location of a nontrivial zero at $t = t^*$.
5. RZF has no zeros on the line $\sigma = 1$. Thus by symmetry about the critical line, RZF has no zero on the line $\sigma = 0$

These properties enable limiting the search for the location of RZF's nontrivial zeros to the left half of the critical strip. The nontrivial zeros in the right half of the critical strip can then be obtained by symmetry. Hence, constraint (2) below will be relevant to the proposed optimization approach:

$$0 < \sigma \leq ½ \qquad (2)$$

II. Application of the Central Principle of Science - Optimization

The search for the location of the nontrivial zeros that is relevant to RH entails finding the value σ^* where $\zeta(\sigma + it)$ vanish at some height $t = t^*$. To implement this search, we apply the Optimization Principle approach using a simple objective function which is assumed to be the driver of the process which is "pushing" RZF toward reaching a nontrivial zero at height $t = t^*$, for some value of σ in $(0,1/2]$. This function is derived from the symmetry of the nontrivial zeros about the critical line. As reported above, this symmetry implies that if σ^* is a location of a nontrivial zero for RZF at some $t = t^*$ then $(1-\sigma^*)$ is also a location of a nontrivial zero at $t = t^*$. This fact suggests a plausible action process where a push, LP, is exerted on RZF from the left side of the critical line, and a pull, RP, is exerted on RZF from the right side, driving RZF to reach an equilibrium state where $\zeta(\sigma^*;\ t^*) = 0$, as σ increases from 0 to $1/2$. These two actions are assumed to be proportional to σ and $(1- \sigma)$, respectively, so that:

$LP = k\sigma$, and $RP = k(1- \sigma)$, with k being a positive constant.

The joint push-pull action, *JP* driving RZF toward its equilibrium states of zero value is assumed to be equal to the product of the push and pull actions, i.e.

$$JP = k^2\sigma\,(1-\sigma) = K\sigma\,(1-\sigma);\; K = k^2 > 0.$$

The aim of the proposed analytical approach is to show that a necessary condition for *JP* to reach its maximum is that RZF vanishes in the critical strip at $t = t^*$, that is when $\zeta\,|(\sigma; t^*)|^2 = 0$, for $0 < \sigma \leq ½$, through solving the following optimization problem:

$$\begin{aligned} &\text{Maximize } JP = K\sigma(1-\sigma) \\ &\text{Subject to:} \quad h_1(\sigma) = |\zeta(\sigma; t^*)|^2 = U^2 + V^2 = 0 \qquad \text{(P1)} \\ &\qquad\qquad\quad h_2(\sigma) = \sigma \leq 1/2 \\ &\qquad\qquad\quad h_3(\sigma) = \sigma > 0, \end{aligned}$$

To solve the small nonlinear constrained problem (P1), it is best to replace the inequality constraint $h_2(\sigma)$ by an equality constraint through the use of a "slack" parameter and then apply the Lagrange Multipliers Method [4] as discussed below. The simple positivity constraint $h_3(\sigma)$, is not transformed into an equality constraint, and is used as a feasibility condition on any identified solution to the resulting equality constrained problem. Hence, the reduced version of problem (P1) is to:

$$\begin{aligned} &\text{Maximize } JP = K\sigma\,(1-\sigma) \\ &\text{Subject to: } h_1(\sigma) = |\zeta(\sigma; t^*)|^2 = U^2 + V^2 = 0 \qquad \text{(P2)} \\ &\qquad\qquad h_2(\sigma) = 1/2 - \sigma - r^2) = 0, \; r \text{ is the slack variable} \end{aligned}$$

To solve equality-constrained optimization problems such as (P2) the equality constraints, weighted by multipliers, are incorporated in the objective function as a penalty on any solution that does not meet the corresponding constraint, to "penalize" infeasible solutions. This method is the so-called Lagrange Multipliers Method and the resulting new objective function to be optimized is the Lagrangian function $\mathscr{L}(S)$ defined below.

$$\mathscr{L}[S(\sigma, \lambda, \mu, r)] = K\sigma(1-\sigma) - \lambda(U^2 + V^2) - \mu\,(1/2 - \sigma - r^2)$$

This procedure enables transforming the constrained problem (P2) into the unconstrained problem (P3) below:

$$\text{Maximize } \mathscr{L}(S) = K\sigma(1-\sigma) - \lambda(U^2 + V^2) - \mu\,(1/2 - \sigma - r^2) \qquad \text{(P3)}$$

Where λ and μ are the Lagrange multipliers associated with the equality constraints, and r is a slack parameter. Any solution of (P3) will have to meet the feasibility constraint $\sigma > 0$ in order to be a feasible candidate solution for P(2) and (P1).

Problem (P3) is a classical optimization of a multivariate function. The necessary conditions for optimality of problem (P1) and (P2) are derived from the following necessary conditions for a point S to be an optimizer [4] for (P3):

1. The functions involved are continuously differentiable at the solution point S^* under consideration. In our case all the functions involved are differentiable.

2. There exists a stationary point $S^*(\sigma^*, \lambda^*, \mu^*, r^*)$ for $\mathscr{L}(S)$, i.e., a point where the gradient of the Lagrangian function vanishes at S^*, i.e. $\nabla\mathscr{L}(S^*) = 0$. Hence, the necessary conditions for S^* to be an optimal point for $\mathscr{L}$(S) are the following:

$$\partial\mathscr{L}(S^*)/\partial\sigma = K(1 - 2\sigma^*) - 2\lambda^*(U^* U^*_\sigma + V^*V^*_\sigma) + \mu^* = 0 \qquad (3)$$
$$\partial\mathscr{L}(S^*)/\partial\lambda = (U^{*2} + V^{*2}) = 0 \qquad (4)$$
$$\partial\mathscr{L}(S^*)/\partial\mu = (1/2 - \sigma^* - r^{*2}) = 0 \qquad (5)$$
$$\partial\mathscr{L}(S^*)/\partial r = 2\mu^* r^* = 0 \qquad (6)$$

Solving the above system of equations will enable computing the stationary points which identify the required locations of RZF's nontrivial zero for $t = t^*$ where RZF is expected to vanish. Hence, the variable t will be a constant equal to t^* throughout the following analyses, using the properties and conditions discussed above, with $\mathscr{L}(S)$ being a function of one variable, namely σ.

Condition (6) above is the starting point used to identify the stationary points S^*. For a given value of σ^* in $(0,1/2]$, inequality constraint $h_2(\sigma^*)$ is either not binding, whereby $r^* \neq 0$, or binding whereby $r^* = 0$. The implications of these two possibilities are presented below:

a. $r^* \neq 0$, (7)
   Then condition (6) and (7) imply $\mu^* = 0$ (8)
   Condition (5) and (7) imply $\sigma^* < ½$. , thus $1 - 2\sigma^* > 0$ (9)
   Then conditions (3), (1) and (8) require that $K(1 - 2\sigma^*) = 0$ (10)
   Conditions (9) and (10) we get $K = 0$, which is not possible, since by definition $K > 0$.
   Hence, the only possibility is (b): $r^* = 0$

b. $r^* = 0$. (11)
   Then condition (5) and (11) require that $\sigma^* = ½$, (12)
   From conditions (3) , (1) and (12), we get $\mu^* = 0$ (13)
   From conditions (12), (13) and (3) we get:

$$2\lambda^*(U^*U_\sigma^* + V^*V^*_\sigma) = 0 \qquad (14)$$

   From Property (1): $(U^*U_\sigma^* + V^*V^*_\sigma) = 0$, and condition (14) we see that λ^* can take on any finite value. For simplicity, let us set it equal to zero. Thus, there exists a solution for the system of necessary conditions above, which is:

$$S^* = (\sigma^* = ½, \lambda^* = 0, \mu^* = 0, r^* = 0). \qquad (15)$$

Point S^* also meets the necessary positivity conditions since $\sigma^* = ½$. Hence, it is a feasible stationary point, therefore a candidate solution that meets the necessary optimality conditions for (P1) and (P2), which include condition (4): $(U^{*2}(\sigma^*; t^*) + V^{*2}(\sigma^*; t^*) = 0$. Thus assuming that RZF vanishes at $t = t*$, it is necessarily that RZF vanishes at $\sigma = \sigma^* = ½$. This result shows that a nontrivial zero of RZF at $t = t^*$ is necessarily located on the critical line, thus proving the Riemann Hypothesis.

The sufficiency condition for S^* to be a maximum [6] for problem (P3), also known as the second order condition, is for the Hessian of $\mathscr{L}(S)$ to be negative definite. Since $\mathscr{L}(S)$ is a function of only one variable, namely σ, this condition reduces to $[\partial^2\mathscr{L}(S) /\partial\sigma^2][S^*] < 0$. Indeed, we have:

$$[\partial^2\mathscr{L}(S) /\partial\sigma^2][S^*]= \partial[K(1 - 2\sigma) - 2\lambda(U\, U_\sigma + VV_\sigma) + \mu]/\partial\sigma][S^*]$$
$$= -2K – [2\lambda(U^*{}_\sigma{}^2 + U^*\, U^*{}_{\sigma\sigma} + U^*{}_\sigma{}^2 + V^*\, V^*{}_{\sigma\sigma})][S^*]$$

From property (1) we have $U^*\, U^*{}_{\sigma\sigma} + V^*\, V^*{}_{\sigma\sigma} = 0$, so that:

$$[\partial^2\mathscr{L}(S) /\partial\sigma^2][S^*] = -2K - 2\lambda^*(U^*{}_\sigma{}^2 + V_\sigma{}^{*2})] < 0 \ (\text{for } \lambda^* \geq 0\ )$$

Meeting the necessary and sufficient optimality conditions, solution S^* is a maximum point for (P3). The unique value $\sigma^* = ½$ is therefore a necessary and sufficient condition for maximizing $JP(\sigma)$ under constraints $h_3(\sigma,)\ h_2(\sigma)$, and $h_1(\sigma) = \zeta\,|(\sigma^*; t^*)|^2 = 0$. Hence, the necessary optimality conditions for $JP(\sigma)$ require that the condition $h_1(\sigma^*) = 0$ be met, hence for RZF to vanish at $t = t*$, for $\sigma^* = ½$. Hence the location of the nontrivial zero of RZF *at* $t = t^*$ is necessarily on the critical line when $JP(\sigma)$ reaches it maximum. This result achieves our proposed aim and proves the Riemann Hypothesis.

It is important to note that σ is required to be strictly positive, since there will be no push action when $\sigma^*= 0$. In addition, if $\sigma^* = 0$, then condition (5) implies $r^* \neq 0$, hence from (6) we get $\mu^* = 0$, and $1- 2\sigma > 0$. Then (1) and (13) imply that $K(1\text{-}2\sigma) = K = 0$. This is not possible since K is by definition greater than zero. This condition is excluded in the case of RZF, according to property (2): RZF has no zeros on the line $\sigma = 0$.

In addition to the constraint $\sigma > 0$, it should be proven that σ can reach its upper bound of ½ at least once, hence the constraint $\sigma \leq 12$. This constraint is a property of RZF since quite a few nontrivial zeros have proved to be located on the critical line.

As a result, these two constraints on σ, the push-pull action will drive RZF to reach its nontrivial zeros on the critical line is active ($K > 0$).

The above results can be generalized to other functions that have the same properites as RZF. Indeed in the proposed approach to identifing the location of RZF's nontrivial zeros, the analysis did not require the use of a closed form expression of RZF, but used instead a set of its properities which were sufficient to show that RZF's nontrivial zeros are necessarily located on the critical line. Hence the same result is valid for any function

$Z(\sigma, t)=H(\sigma, t)+iW(\sigma, t)$ satisfying the same properties as RZF. These properties are listed again below:

a. $Z(\sigma, t)$ has multiple or inifinite number of nontrivial at different heights t^*
b. $Z(\sigma\ t)$ is twice differentiable in the critical region defined by $0 \leq \sigma < 1$
c. The zeros of $Z(\sigma, t)$ are located in the critical strip at different heights t^*
d. The zeros of $Z(\sigma, t)$ are symetric about the critical line, i.e on the line $\sigma = ½$
e. At least one zero is proven to be on the critical line at some height t^* so that $\sigma \leq 1/2$
f. $Z(\sigma; t^*)$ does not vanish at $\sigma = 0$, so that $\sigma > 0$

As in the case of RZF, these properties help formulate and solve the constrained optimization problems (P1), (P2) and (P3) used to find the zeros of $Z(\sigma; t^*)$. The roles of these properties are as follows:

- Property (a) ensures the function $Z(\sigma; t)$ has multiple zeros, the locations of which cannot all be easily identified through analysis or numerical computation, especially if there is an infinite number of zeros.
- Property (b) establishes the existence of the derivatives of $H(\sigma; t^*)$ and $W(\sigma; t^*)$ so that $HH^*_\sigma + W^*W^*_\sigma = 0$, and $H^*H^*_{\sigma\sigma} + W^*W^*_{\sigma\sigma} = 0$. This property circumvents the need for using a closed form expression of the function $Z(\sigma; t^*)$ in order to derive its gradient which is needed for the computation of stationary points and optimal solutions of the optimization problem associated with the solution approach.
- Properties (c), (d) and (e) reduce the search for the location of the zeros to the interval $(0, 1/2]$, thus defining two constraints on σ, namely; $\sigma > 0$ and $\sigma \leq ½$. As discussed earlier, these two constraints are most critical for identifying the zeros of $Z(\sigma; t^*)$. Thus, it is important to ascertain, using numerical analysis if needed, that $\sigma > 0$ and that σ can reach its upper bound of ½, at least in one instance where $Z(\sigma =1/2., t^*) = 0$, so that $\sigma \leq ½$.. As shown above, these two constraints on σ are required in order to make sure that $K \neq 0$ in the *JP* formula in order to have an active push-pull action.

As an example, the Riemann $\xi(s)$ function [5] has the properties above, and its zeros are therefore located on the critical line.

For counter examples, consider the following functions, $Z_1(\sigma,t)$ and $Z_2(\sigma,t)$ which do not meet the required conditions on either the lower and upper bounds on σ , or both, although they meet all of the other required properties.

$$Z_1(\sigma,t) = \sigma (\sigma - 1) + i(t-1)(t-2))(t-3)....(t - n)$$
$$Z_2(\sigma,t) = (\sigma - 1/3)(\sigma - 2/3) + i(t-1)(t-2)....(t - n)$$

Function Z_1 does not meet the condition $\sigma \neq 0$, while for Z_2 , $\sigma < ½$. Clearly the zeros of Z_1 and Z_2 are not located on the critical line at any height t^* since they vanish for some values of $\sigma \neq ½$. Rather, these zeros are at $\sigma_{1,1}^* = 0$ and $\sigma_{1,2}^* = 1$ *for* Z_1, and $\sigma^*_{2,1} = 1/3$, and $\sigma_{2.2}^* = 2/3$ *for* Z_2.

Conclusion

In this paper, the search for the location of the nontrivial zeros of the Riemann Zeta function is implemented using an optimization approach which, as the Central Principle of Science, served as the basis for proving several scientific laws and theories. This approach enabled the formulation of locating RZF's nontrivial zeros as a constrained optimization problem where a simple objective function referred to as the "Push-Pull Action" driving RZF to reach its equilibrium states at its nontrivial zeros, is maximized The solution of the resulting constrained nonlinear programming problem proved that any nontrivial zero of RZF is unique and is necessarily located on the critical line, thus proving the conjecture stated in the Riemann Hypothesis. The paradigm of Optimization as the Central Principle of Science also showed that the law of "Maximum Action of Push-Pull (MAPP)" would be driving RZF to its equilibrim states at the different heights where it vanishes in the critical strip. This law is also proven to be valid for functions that have the same properties as RZF whereby the same results regarding the location of their zeros on the critical line as well as the law driving this process.

References

[1] Robert F. Bordley. *A Central Principle of Science: Optimization*. Behavioral Science, Wiley Online Library, 1983 p 53

[2] Robert F. Bordley. *A Central Principle of Science: Optimization*. Behavioral Science, Wiley Online Library, 1983 p.54

[3] Chandrasekhar K. *Lectures on The Riemann Zeta–Function* [PDF document] URL: https://julianoliver.com/share/free-science-books/tifr01.pdf, pp. 94-110, 1956.

[4] David M. Himmelblau. Applied Nonlinear Programming. McGraw-Hill Inc., p. 32, 1972

[5] Chandrasekhar K. *Lectures on The Riemann Zeta–Function* [PDF document] URL: https://julianoliver.com/share/free-science-books/tifr01.pdf, pp. 106, 1956.

[6] David M. Himmelblau. Applied Nonlinear Programming. McGraw-Hill Inc., p. 32, 1972.