The Schwartzschild Solution and the Embedding of Special Coordinates in 4D Space

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Abstract

We present some formulas and calculations worked out while studying the Schwartzschild Solution. Nothing new!

Key Words: Gravity.

1 Proper Distance Coordinates

If we use units where the speed of light is $c = 1$ and the Schwartzschild radius $R_s = 1$, the Schwartzschild metric can be written as follows:

$$d\tau^2 = \left(1 - \frac{1}{r}\right) dt^2 - \left(1 - \frac{1}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{1}$$

with $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. Following the classical steps to derive the Kruskal metric, we want to perform a change of coordinates and use proper distance ρ from the event horizon, with $dt = 0$, as a new radial coordinate. We have:

$$\rho(r) = \int_0^r \sqrt{\left(1 - \frac{1}{v}\right)^{-1}} dv = \sqrt{r(r-1)} + \frac{1}{2} ln \left(\frac{r + \sqrt{r(r-1)}}{r - \sqrt{r(r-1)}} \right) \tag{2}$$

where $\rho(r)$ is invertible and its inverse is:

$$r(\rho) = \sqrt{\rho(\rho-1)} + \frac{1}{2} ln \left(\rho + \sqrt{\frac{\rho(\rho-1)}{\rho - \sqrt{\rho(\rho-1)}}} \right) \tag{3}$$

We say a few words on the behaviour of $r(\rho)$ at infinity. It is possible to show that:

$$\forall \epsilon > 0, \exists M : x > M \Rightarrow x < r(\rho) < x^{1+\epsilon} \tag{4}$$

and therefore $r(\rho)$ is not asymptotic to any line:

$$\lim_{r \to \infty} \rho(r) - x = \infty \tag{5}$$

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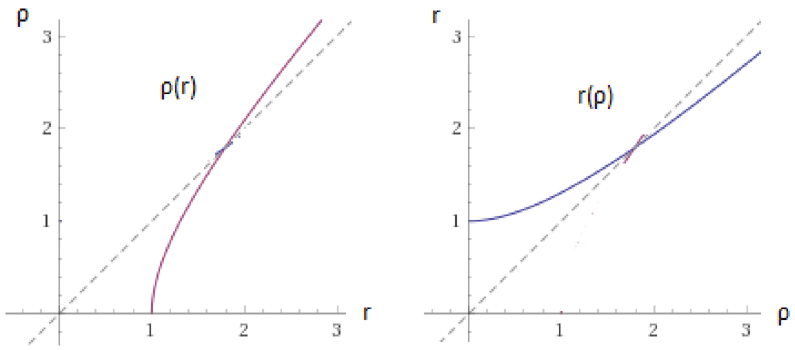

Figure 1: Functions $\rho(r)$ and $r(\rho)$

However we have:

$$\lim_{r\to\infty}\frac{\rho(r)}{x} = 1 \tag{6}$$

With this new radial coordinate the coefficient of dt^2 can be written as:

$$G(\rho) = \left(1 - \frac{1}{r(\rho)}\right) = \left(\frac{r(\rho)-1}{r(\rho)}\right) \tag{7}$$

Moreover, by definition we have:

$$\left(1 - \frac{1}{r}\right)^{-1} dr = d\rho \tag{8}$$

Given the above, in the proper space coordinates, the metric became:

$$d\tau^2 = G(\rho)dt^2 - d\rho^2 - r^2(\rho)d\Omega^2 \tag{9}$$

Nothing new since the above metric is basically the original form of the Kruskal metric that make the discontinuity of the metric at the event horizon to disappear.

Now, if we consider a sphere S of radius $1+\rho$, the area of the sphere is equal to:

$$A = \int_S r^2 d\Omega = r^2 \int_{S'} d\Omega = 4\pi r^2(\rho) \tag{10}$$

where S' is the unit sphere. Since the effect of the mass became negligible at great distance from the singularity and the space became flat, given (6) we have:

$$\lim_{\rho\to\infty}\frac{4\pi r^2(\rho)}{(1+\rho)^2} = 4\pi \tag{11}$$

as expected although we expected also $\rho(r)$ to be asymptotic to a line and this does not happen as shown by (5). However, this is what the maths tells us.

2 Embedding in 4D Space

We want to embed the special 3 coordinates of the Schwartzschild solution (for t=0) in a 4D Space as a 3D manifold that inherit its metric from the canonical metric of the ambient space. This is the classical funnel shaped image, with goes to a plane when the radius increases, which shows how matter curves spaces in popular books on gravity.

To do that we start from the classical metric (1) and we add an additional h coordinates to the existing (r, θ, ϕ) coordinates. For each r we give a value to h so that a curve moving radially on the manifold from a distance r_1 to a distance r_2 and with θ and ϕ constant, has a length equal to the proper space ρ defined in the paragraph above and integrated between r_1 and r_2. This ensure that the radial metric is correct where the coefficient of the $d\Omega^2$, which is equal to r^2 in the (1), ensure that the metric is correct for path with r constant.

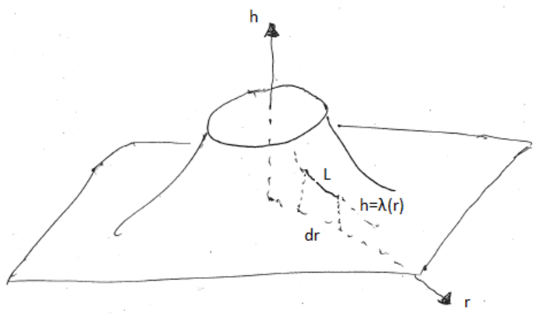

Figure 2: Embedding of the Solution in $\mathbb{R}^4$

We are basically looking for a function $h = \lambda(r)$ which length L between two value of r is equal to the proper space distance of them (i.e. $\rho(r_2) - \rho(r_1)$ see Fig. 2). The classical formula for evaluating the length of a curve is:

$$L = \int_{r_1}^{r_2} \sqrt{1 + (\lambda')^2}\, dr \tag{12}$$

Using the function proper space (2), we have:

$$\rho(r) = \int_{1}^{r} \sqrt{1 + (\lambda(v)')^2}\, dv \tag{13}$$

Taking the derivative of both sides we have:

$$\rho'(r) = \sqrt{1 + (\lambda')^2} \tag{14}$$

and from the definition of ρ given in (2):

$$\sqrt{\left(1 - \frac{1}{r}\right)^{-1}} = \sqrt{1 + (\lambda')^2} \tag{15}$$

from which, by choosing the negative sign for the square root (we like the funnel shape of the manifold to point up and not down) we get easily:

$$\lambda'(r) = -\sqrt{\frac{1}{r - 1}} \tag{16}$$

and integrating:

$$h = \lambda(r) = -2\sqrt{r-1} + const \tag{17}$$

the funnel shape we get does not go to a plane (see Fig. 2). This embedding is called Flamm's paraboloid.

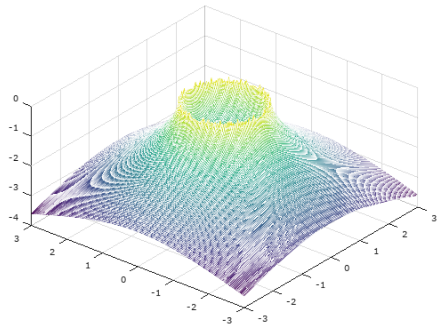

Figure 3: Plot of the Embedding