FURTHER DISCUSSIONS ON GEOMETRIC INTERACTIONS

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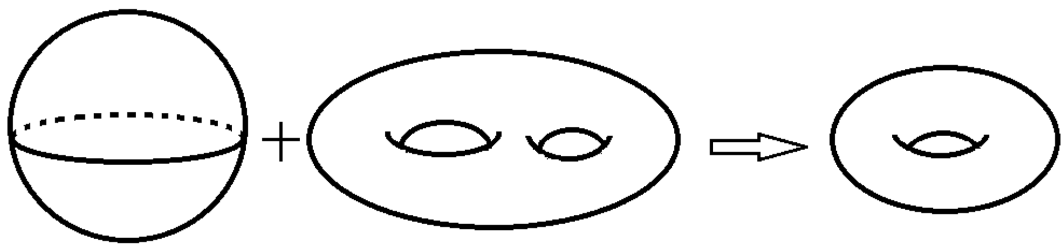
Abstract: In this work we discuss further the geometric interactions resulted from the coupling of n-cells that are decomposed from CW complexes. As an illustration, we discuss whether a gravitational field can be considered as the result of a geometric and topological coupling of two decomposed sub-manifolds from CW complexes in terms of Gaussian curvatures of two-dimensional manifolds.

In our previous works we have investigated the possibility to associate an elementary particle with a differentiable manifold which may be endowed with geometric and topological structures of a CW complex [1,2]. Since elementary particles behave differently therefore they should have different geometric and topological structures and the problem that arises from the speculation is how to determine these structures, which in turns give rise to different corresponding physical fields. However, if an elementary particle has the geometric and topological structure of a CW complex then in order to determine its mathematical structure we need to determine the forms of n-cells that are decomposed from it. For example, of the three types of prime manifolds, which are the spherical types, $S^2 \times S^1$ and $K(\pi, 1)$ [3,4], only embedded tori can be decomposed from the prime manifold $K(\pi, 1)$, therefore if physical fields are associated with decomposed n-cells then the three types of prime manifolds represent three different types of elementary particles and the force carriers of physical fields may possess the geometric and topological structures of the two-spheres and the n-tori. This consideration leads to a more profound speculation that physical properties assigned to an elementary particle, such as charge, are in fact manifestations due to the force carriers rather than physical quantities that are assigned to the elementary particle. If this is the case then the analysis of physical interactions will be reduced to the analysis of the geometric processes that are related to the geometric and topological structures of the force carriers and the study of physical dynamics associated with physical interactions reduces to the study of the geometric evolution of two-dimensional Riemannian surfaces. It is remarkable that we can actually formulate a physical theory based only on the postulate of the existence of these sub-manifolds without regard to why and how they exist. If we simply assume physical interactions are due to two-dimensional surfaces then the Feynman's method of sum over random paths can be extended to higher-dimensional spaces to formulate physical theories in which the transition amplitude between states of a quantum mechanical system is the sum over random hypersurfaces. This generalisation of the path integral method in quantum mechanics has been developed and applied to other areas of physics, such as condensed matter physics, quantum field theories and quantum gravity theories, mainly for the purpose of field quantisation [1]. This statistical approach to formulating quantum theories leaves out the question of how the surfaces are actually formed and how they give rise to physical interactions in terms of geometric and topological formations. In the following we will address this problem. It may be assumed that at the beginning, by contact forces associated with 0-cells, mass points formed elementary particles in the form of prime manifolds from which embedded 2-spheres, tori and 2-tori can be decomposed. As will be discussed later, an interaction of these decomposed cells could give rise to an overall effect of a type of van der Waals attractive interaction that may be identified with gravity. An outstanding feature of the gravitational field is that it is universal and does not depend on any physical properties of elementary particles except for their mass. In order to form other fundamental systems the sub-manifolds that are decomposed from CW complexes break themselves and then reunite to form other structures that can be used to build a more sophisticated system. For example, a torus can break itself by decomposing a sphere to form a double torus in which the sphere represents one physical field and the double torus a different physical field. This situation is to some extent similar to that of protein metabolism in a biological system in which proteins break down to amino acids so that the latter can synthesise to form a new biological system. As an illustration, we now discuss topologically and algebraically how to form a torus from a sphere and a double torus. Our approach to this problem is however rather suggestive that requires a more rigorous formulation within the framework of homological and homotopical topology.

We assume that one charged particle has a particular topological structure so that it can only emit 2-cells in the form of a 2-sphere while another charged particle has a different topological structure that only allows it to emit 2-cells in the form of a 2-torus. When they interact the 2-sphere and the 2-torus will join to form a torus, which may be assumed to be associated with a gravitational field. As shown in the figure below, topologically, a torus can be formed from a sphere and a double torus by the process of continuous deformation following a surgery of cutting and gluing



We now show that the joining of a sphere and a double torus to form a torus can be represented algebraically. Assume that the double torus is described by a function $f(x,y,c,a)$ in the form $z^2 = f(x,y,c,a)$. Furthermore, the function $f(x,y,c,a)$ is assumed to be symmetrical with respect to the reflection in the (x,y)-plane and one of the two holes has its centre at the origin and can be described by the Cartesian equation of a torus given by $\left(c-\sqrt{x^2+y^2}\right)^2 + z^2 = a^2$ up to where it is joined to the second hole. The parameter c is the radius from the centre of the hole to the centre of the torus tube and a is the radius of the tube. Let the sphere of radius r with its centre also at the origin be written in the form

$x^2+y^2+z^2=r^2$. Now we split the double torus and the sphere so they can be expressed as $z=\pm\sqrt{f(x,y,c,a)}$ and $z=\pm\sqrt{r^2-x^2-y^2}$, respectively. If we assume that the double torus and the sphere have the same z-value at the points (x_C,y_C) on the circle ${x_C}^2+{y_C}^2=b^2$ with $c-a<b<c+a$, i.e., $\sqrt{f(x_C,y_C,c,a)}=\sqrt{r^2-b^2}$, then the radius of the circle is found as $r=\sqrt{b^2+f(x_C,y_C,c,a)}$. With these conditions, the sphere and the double torus can be joined to form a torus which can be expressed as a hybrid function as follows

$$z^2=\begin{cases} f(x,y,c,a), & x^2+y^2>b^2 \\ r^2-x^2-y^2, & x^2+y^2\le b^2 \end{cases} \tag{1}$$

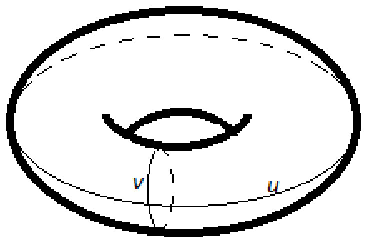
It should be mentioned here that despite a precise equation for a double torus has not been established the function $z^2=f(x,y,c,a)$ for a double torus can be generated from the implicit equation of a lemniscate. For example, an equation for a double torus that is formed from the lemniscate of Bernoulli can be given in the form $((x^2+y^2)^2-x^2+y^2)^2+z^2=\varepsilon$, where ε is a small number [5]. A more profound problem that we are facing is the problem of how to represent physical interactions in terms of the geometric and topological structures of n-cells that are decomposed from CW complexes, as well as the n-cells that are synthesised from them. It is seen that even though a torus can be obtained topologically and algebraically by joining a sphere and a double torus, the physical fields that are represented by these cells cannot be simply added. This can be illustrated as follows. It is shown in differential geometry that the Gaussian curvature K of a surface embedded in the three-dimensional Euclidean space R^3 can be calculated from the formula

$$K=\frac{1}{2}R=\kappa_1\kappa_2=\frac{1}{R_1R_2} \tag{2}$$

where R is the Ricci scalar curvature and the principal radii of curvature R_1 and R_2 are given in terms of the principal curvatures κ_1 and κ_2 as $R_1=1/\kappa_1$ and $R_2=1/\kappa_2$. From Equation (2), the Gaussian curvature K_S of a sphere of radius r can be obtained as

$$K_S=\frac{1}{r^2} \tag{3}$$

On the other hand, the Gaussian curvature K_T of a torus can be calculated from the parametric equations [6]



$$x=(c+a\cos v)\cos u \tag{4}$$

$$y=(c+a\cos v)\sin u \tag{5}$$

$$z = a\sin v \quad (6)$$

From the above parametric equations of the torus, the line element is found as

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 dv^2 \quad (7)$$

With the metric $g_{\alpha\beta}$ obtained from the line element given in Equation (7), the non-zero components of the connection $\Gamma^{\gamma}_{\alpha\beta}$, the non-zero components of the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R can be calculated

$$\Gamma^{u}_{uv} = \Gamma^{u}_{vu} = -\frac{a\sin v}{c + a\cos v} \qquad \Gamma^{v}_{uu} = \frac{1}{a}\sin v(c + a\cos v) \quad (8)$$

$$R_{uu} = \frac{1}{a}\cos v(c + a\cos v) \qquad R_{vv} = \frac{a\cos v}{(c + a\cos v)} \quad (9)$$

$$R = g^{\alpha\beta} R_{\alpha\beta} = \frac{2\cos v}{a(c + a\cos v)} \quad (10)$$

Since the Ricci scalar R is twice the Gaussian curvature K_T as given in Equation (2), the Gaussian curvature K_T of the torus is obtained

$$K_T = \frac{\cos v}{a(c + a\cos v)} \quad (11)$$

In spite of the fact that the main physical forces are additive and as shown in our works that they may be associated with the Gaussian curvatures, it is clear from the above results that we cannot simply add the Gaussian curvature of a sphere to the Gaussian curvature of a double torus to obtain the Gaussian curvature of a torus, such as $K_S + K_D = K_T$, even though the formation of the Gaussian curvature of a torus can be achieved by a homotopical process of continuous deformation after a surgery of cutting and gluing a sphere to a double torus. However, it is observed that the total effect of adding a sphere to a double torus to give a similar effect from a torus can be obtained by the Gauss Bonnet theorem which states that the total Gaussian curvature of such a closed surface is equal to 2π times the Euler characteristic of the surface [7,8]

$$\int K dA = 2\pi(2 - 2g) \quad (12)$$

where g is the genus which counts the number of holes of the surface. For a two-dimensional sphere $g = 0$, a torus $g = 1$ and a double torus $g = 2$, therefore we obtain

$$\int (K_S + K_D) dA = \int K_S dA + \int K_D dA = 0 = \int K_T dA \quad (13)$$

If we consider physical interactions between two physical systems, which are assumed to be endowed with geometric and topological structures of CW complexes, as geometric processes then a geometric evolution which is associated with synthesised tori must be formulated in terms of torus connection. How can synthesised tori be connected to give the effect of a geometric evolution that represents a physical interaction, such as gravity? Or in fact the collective effect is simply a statistical effect similar to dispersive London-van der Waals forces that bound neutral microscopic particles. For a torus we have $\int K_T dA = 0$ therefore we may consider a torus as a neutral particle. Can a collection of tori give rise to the effect of a gravitational field? Even though we cannot give a definite answer to this question but it is remarkable to observe that van der Waals force derived from a collection of neutral particles takes the form which is similar to Newton's law of gravitation, which is an inverse square law. For macroscopic bodies with known volumes and densities, the total van der Waals force is calculated by integrating over the total volume of the object. For example, the van der Waals interaction energy between spherical bodies of radii R_1 and R_2 with smooth surface can be calculated as [9,10]

$$U(r) = -\frac{A}{6}\left(\frac{2R_1R_2}{(R_1+R_2+r)^2-(R_1+R_2)^2}+\frac{2R_1R_2}{(R_1+R_2+r)^2-(R_1-R_2)^2}\right.$$
$$\left.+\ln\left(\frac{(R_1+R_2+r)^2-(R_1+R_2)^2}{(R_1+R_2+r)^2-(R_1-R_2)^2}\right)\right) \tag{14}$$

where A is the Hamaker coefficient and r is the distance between the surfaces. In the limit $r \ll R_1$ or R_2, Equation (14) is simplified to

$$U(r) = -\frac{AR_1R_2}{6(R_1+R_2)r} \tag{15}$$

The van der Waals force F can be found from the relation $F = -dU/dr$ as

$$F = -\frac{AR_1R_2}{6(R_1+R_2)r^2} \tag{16}$$

The results given in Equations (14-16) were obtained by assuming a form of energy which is a physical entity rather than a mathematical object which is the Gaussian curvature that we want to use to formulate physical interactions as geometric processes. However, as shown in our works on the spacetime structures of quantum particles [12], a potential energy can be identified with the Ricci scalar curvature, therefore a Gaussian curvature, as indicated in Equation (2). As an example, we showed that Schrödinger wavefunction ψ can be used to construct the spacetime structures of the quantum states of a hydrogen atom. By using the relations $L = dS/dt$, $dS/dt = \partial_t S + \sum_{\mu=1}^{3}\partial_\mu S(dx^\mu/dt)$, $T = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2$ and $V = T - L$, we obtain

$$V = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2 - \partial_t S + \sum_{\mu=1}^{3}\partial_\mu S(dx^\mu/dt) \tag{17}$$

In terms of the Schrödinger wavefunction ψ, Equation (17) is rewritten as

$$V = m\sum_{\mu=1}^{3}(dx^{\mu}/dt)^{2} - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^{\mu}/dt)}{\psi} \tag{18}$$

Since $V = kR = 2kK$, we obtain a relation between the Schrödinger wavefunction ψ and the Gaussian curvature K

$$K = \frac{1}{2k}\left(m\sum_{\mu=1}^{3}(dx^{\mu}/dt)^{2} - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^{\mu}/dt)}{\psi}\right) \tag{19}$$

For the case of a hydrogen atom, the wavefunction ψ satisfies the Schrödinger wave equation

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(E + \frac{kq^2}{r}\right)\psi = 0 \tag{20}$$

As a further discussion, we want to mention here that the coupling of two different types of 2-cells to form a gravitational field can be formulated in a different manner in terms of Riemannian manifold. As formulated in our previous work [11], the gravitational field can be described as the result of a coupling of two electromagnetic fields of equal magnitude but opposite direction and the coupling can be formulated in terms of the general theory of relativity. Within the framework of what have been discussed above, this is similar to the coupling of two two-dimensional manifolds of $g = 0$ and $g = 2$ to form a two-dimensional manifold of $g = 1$. The formulation is given as follows. Consider an asymmetric connection of the form $\Gamma^{\sigma}_{\mu\nu} = \Lambda^{\sigma}_{\mu}\Phi_\nu$. The quantity Φ_ν will be identified with the four-vector potential of one electromagnetic field and the quantity Λ^{σ}_{μ} with the field strength of the second opposing field. The affine connection of the particular form $\Gamma^{\sigma}_{\mu\nu} = \Lambda^{\sigma}_{\mu}\Phi_\nu$ reduces the Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu} = \partial_\mu\Gamma^{\alpha}_{\beta\nu} - \partial_\nu\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu}$ to the simpler form as

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial\left(\Lambda^{\alpha}_{\beta}\Phi_\nu\right)}{\partial x^{\mu}} - \frac{\partial\left(\Lambda^{\alpha}_{\beta}\Phi_\mu\right)}{\partial x^{\nu}} \tag{21}$$

The Ricci tensor defined by the relation $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ becomes

$$R_{\mu\nu} = \left(\frac{\partial\Phi_\nu}{\partial x^{\sigma}} - \frac{\partial\Phi_\sigma}{\partial x^{\nu}}\right)\Lambda^{\sigma}_{\mu} + \Phi_\nu\frac{\partial\Lambda^{\sigma}_{\mu}}{\partial x^{\sigma}} - \Phi_\sigma\frac{\partial\Lambda^{\sigma}_{\mu}}{\partial x^{\nu}} \tag{22}$$

The Ricci tensor in this form can be reduced to a symmetric form if the quantities Λ^{σ}_{μ} satisfy the relation $\partial\Lambda^{\sigma}_{\mu}/\partial x^{\nu} = \eta\Lambda^{\sigma}_{\mu}\Phi_\nu$, where η are arbitrary functions of the coordinate variables. The Ricci tensor then becomes

$$R_{\mu\nu} = \Lambda^{\sigma}_{\mu}F_{\sigma\nu} \tag{23}$$

where we have defined $F_{\mu\nu} = \partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu$. In terms of the field strengths Λ^{σ}_{μ} and $F_{\sigma\nu}$, the reduced Ricci tensor given in Equation (23) takes the explicit form as follows

$$\Lambda_{\mu}^{\sigma} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{24}$$

$$F_{\sigma\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{26}$$

$$R_{\mu\nu} = \begin{pmatrix} E_1^2 + E_2^2 + E_3^2 & E_2B_3 - E_3B_2 & E_3B_1 - E_1B_3 & E_1B_2 - E_2B_1 \\ E_2B_3 - E_3B_2 & E_1^2 + B_2^2 + B_3^2 & E_1E_2 - B_1B_2 & E_1E_3 - B_1B_3 \\ E_3B_1 - E_1B_3 & E_1E_2 - B_1B_2 & B_1^2 + E_2^2 + B_3^2 & E_2E_3 - B_2B_3 \\ E_1B_2 - E_2B_1 & E_1E_3 - B_1B_3 & E_2E_3 - B_2B_3 & B_1^2 + B_2^2 + E_3^2 \end{pmatrix} \tag{27}$$

In order to determine the dynamical aspects of the particle in the spirit of general relativity, a new symmetrical metric tensor $g_{\mu\nu}$ is introduced according to the defining relation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. With the introduction of this symmetric metric tensor into the spacetime structure, it is now possible to construct field equations for the gravitational field in terms of differential geometry that reflects the structure of the physical quantity that determines the Ricci tensor as specified in Equation (27). It is possible to adopt Einstein field equations of general relativity as a postulating physical formulation to describe gravity, however, as shown in our other works on general relativity, in order to formulate gravity purely in terms of differential geometry, we can postulate field equations of general relativity by adopting Bianchi identities instead [12,13]. It is shown that the Ricci tensor satisfies the Bianchi identities

$$\nabla_{\mu}R^{\mu\nu} = \frac{1}{2}g^{\mu\nu}\nabla_{\mu}R \tag{28}$$

Even though Equation (28) is purely geometrical, it has a form of Maxwell field equations of the electromagnetic tensor, $\partial_{\mu}F^{\mu\nu} = \mu j^{\nu}$. If the quantity $\frac{1}{2}g^{\mu\nu}\nabla_{\mu}R$ can be perceived as a physical entity, such as a four-current of gravitational matter, then Equation (28) has the status of a dynamical law of a physical theory. With the assumption that the quantity $\frac{1}{2}g^{\mu\nu}\nabla_{\mu}R$ to be identified with a four-current of gravitational matter then a four-current $j^{\nu} = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as follows

$$j^{\nu} = \frac{1}{2}g^{\mu\nu}\nabla_{\mu}R \tag{29}$$

For a purely gravitational field, Equation (28) reduces to

$$\nabla_{\mu}R^{\mu\nu} = 0 \tag{30}$$

Using the identity $\nabla_{\mu}g^{\alpha\beta} \equiv 0$, Equation (30) implies

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{31}$$

where Λ is an undetermined constant. With the new purely geometrical formulation of gravity, Einstein field equations given by the relation $R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$ can be interpreted as a definition of an energy-momentum tensor, as that of Maxwell theory of the electromagnetic field. From Equation (31), we obtain

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa} g_{\mu\nu} \tag{32}$$

If we assume that the effect of the coupling of two opposing electromagnetic fields can be interpreted as a gravitational field, then the affine connection $\Gamma^{\sigma}_{\mu\nu} = \Lambda^{\sigma}_{\mu} \Phi_{\nu}$ gives rise to a geodesic equation of the form

$$\frac{d^2 x^{\mu}}{ds^2} + \Lambda^{\mu}_{\nu} \Phi_{\sigma} \frac{dx^{\sigma}}{ds} \frac{dx^{\nu}}{ds} = 0 \tag{33}$$

Equation (33) admits a linear first integral of the form [14]

$$\Phi_{\sigma} \frac{dx^{\sigma}}{ds} = -\frac{q}{m} \tag{34}$$

provided the quantities Φ_{σ} satisfy the condition $\nabla_{\mu} \Phi_{\nu} + \nabla_{\nu} \Phi_{\mu} = 0$ and we have set the constant in the first integral equal to $-q/m$ for convenience. This condition identifies Φ_{σ} as a Killing vector field, which defines a direction of symmetry along which the motion leaves the spacetime geometry unchanged. The geodesic equation then has the form of the Lorentz force law

$$\frac{d^2 x^{\mu}}{ds^2} = \frac{q}{m} \Lambda^{\mu}_{\nu} \frac{dx^{\nu}}{ds} \tag{35}$$

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