The Dirac Operator for Lie Groups

A.Balan

June 9, 2018

1 Recalls of Lie group theory

A Lie group G is a differentiable manifold with differentiable group structure [W]. The tangent fiber bundle at any point is the Lie algebra, due to the product of the Lie group. So a vector field is a map $m : G \to \mathfrak{g}$ of G in the tangent space at unity. The Killing form is an invariant Riemann metric over the group.

2 The Dirac operator over a Lie group

Let be an orthonormal basis E_i of the Lie algebra $\mathfrak{g}$.

Definition 1 *The Dirac operator $\mathcal{D}$ for the Lie group G is acting over the vector fields:*

$$\mathcal{D}(m) = \sum_i [E_i, \nabla_{E_i}(m)]$$

with ∇ the Levi-Civita connection over the Riemann manifold G and $[,]$ the Lie bracket of vector fields.

Theorem 1 *The definition is independant of the choice of the basis.*

Demonstration 1

The choice of another basis E'_i define an orthogonal matrice, so:

$$E'_i = \sum_j a_{ij} E_i$$

and as $\sum_j a_{ij} a_{kj} = \delta_i^k$, the Kronecker symbol, the Dirac operators are identical. □

References

[F] T.Friedrich, ”Dirac operators in Riemannian Geometry”, Graduate Studies in Mathematics vol 25, AMS, 2000.

[W] F.Warner, ”Foundations of Differentiable Manifold and Lie Groups, Springer Verlag, 1983.