# Grounding the Kaluza-Klein Fifth Dimension in Observed Physical Reality, Using Dirac's Quantum Theory of the Electron to Ensure Five-Dimensional General Covariance

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*Abstract: We require all components of the Kaluza-Klein metric tensor to be generally-covariant across all five dimensions by deconstructing the metric tensor into Dirac-type square root operators. This decouples the fifth dimension from the Kaluza-Klein scalar, makes this dimension timelike not spacelike, makes the metric tensor inverse non-singular, covariantly reveals the quantum fields of the photon, makes Kaluza-Klein fully compatible with Dirac theory, and roots this fifth dimension in the physical reality of the chiral, pseudo-scalar and pseudo-vector particles abundantly observed in particle physics based on Dirac's gamma-5 operator, thereby "fixing" all of the most perplexing problem in Kaluza-Klein theory. Albeit with additional new dynamics expected, all the benefits of Kaluza-Klein theory are retained, insofar as providing a geometrodynamic foundation for Maxwell's equations, the Lorentz Force motion and the Maxwell-Stress energy tensor, and insofar as supporting the viewpoint that the fifth dimension is, at bottom, the matter dimension. We find that the Kaluza-Klein scalar must be a massless, luminous field quantum to solve long-standing problems arising from a non-zero scalar field gradient, and we suggest multiple pathways for continued development.*

## Contents

## 1. Introduction – The Incompatibility of Kaluza-Klein and Dirac Theories

About a century ago with the 1920s approaching, much of the physics community was trying to understand the quantum reality that Planck had first uncovered almost two decades prior [1]. But with the General Theory of Relativity [2] having recently placed gravitation and the dynamical behavior of gravitating objects onto an entirely geometric and geodesic foundation (which several decades later Wheeler would dub "geometrodynamics" [3]), a few scientists were trying to scale the next logical hill, which – with weak and strong interactions not yet known – was to obtain a geometrodynamic theory of electromagnetism. Besides Einstein's own work on this which continued for the rest of his life [4], the two most notable efforts were those of Hermann Weyl [5], [6] who was just starting to develop his U(1) gauge theory in four dimensions (which turned out to be a theory of "phase" invariance [7] that still retains the original moniker "gauge"), and Kaluza [8] then Klein [9], [10] who quite successfully used a fifth dimension to geometrize the Lorentz Force motion and the Maxwell Stress-Energy tensor (see, e.g., [11] and [12]). This is a very attractive aspect of Kaluza-Klein theory, and it remains so because even today, despite almost a century of efforts to do so, U(1) gauge theory has not yet successfully been able to place the Lorentz Force dynamics and the Maxwell Stress Energy on an entirely-geometrodynamic foundation. And as will be appreciated by anyone who has studied this problem seriously, it is the inequivalence of electrical mass (a.k.a. charge) and inertial mass which has been the prime hindrance to being able to do so.

Notwithstanding these Kaluza "miracles" of geometrizing the Lorentz Force motion and the Maxwell Stress-Energy, this fifth dimension and an associated scalar field known as the graviscalar or radion or dilaton raised its own new challenges, many of which will be reviewed here. These have been a legitimate hurdle to the widespread acceptance of Kaluza-Klein theory as a theory of what is observed in the natural world. It is important to keep this historical sequencing in mind, because Kaluza's work in particular predated what we now know to be modern gauge theory and so was the "first" geometrodynamic theory of electrodynamics. And it of course predated any substantial knowledge about re weak and strong interactions. Of special interest in this paper, Kaluza-Klein also preceded Dirac's seminal Quantum Theory of the Electron [13] which today is the foundation of how we understand fermion behavior.

Now in Kaluza-Klein theory, the metric tensor which we denote by $G_{\mathrm{MN}}$ and its inverse $G^{\mathrm{MN}}$ obtained by $G^{\mathrm{MA}}G_{\mathrm{AN}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ are specified in five dimensions with an index $\mathrm{M} = 0,1,2,3,5$, and may be represented in the 2x2 matrix format:

$$G_{\mathrm{MN}} = \begin{pmatrix} g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu & \phi^2 k A_\mu \\ \phi^2 k A_\nu & \phi^2 \end{pmatrix}; \quad G^{\mathrm{MN}} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & g_{\alpha\beta}A^\alpha A^\beta + 1/\phi^2 \end{pmatrix}. \tag{1.1}$$

In the above $g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$ transforms as a 4x4 tensor symmetric in spacetime. This is because $g_{\mu\nu} = g_{\nu\mu}$ is a symmetric tensor, and because electrodynamics is an abelian gauge theory with a commutator $\left[A_\mu, A_\nu\right] = 0$. The components $G_{\mu 5} = \phi^2 k A_\mu$ and $G_{5\mathrm{N}} = \phi^2 k A_\nu$ transform as covariant (lower-indexed) vectors in spacetime. And the component $G_{55} = \phi^2$ transforms as a scalar in spacetime. If we regard $\phi$ to be a dimensionless scalar, then the constant $k$ must have dimensions of charge/energy because the metric tensor is dimensionless and because the gauge field $A_\mu$ has dimensions of energy/charge.

It is very important to understand that when we turn off all electromagnetism by setting $A_\mu = 0$ and $\phi = 0$, $G^{\mathrm{MN}}$ in (1.1) becomes singular. This is indicated from the fact that in this situation $\mathrm{diag}\left(G_{\mathrm{MN}}\right) = \left(g_{00}, g_{11}, g_{22}, g_{33}, 0\right)$ with a determinant $\left|G_{\mathrm{MN}}\right| = 0$, and is seen directly from the fact that $G^{55} = g_{\alpha\beta}A^\alpha A^\beta + 1/\phi^2 = 0 + \infty$. Therefore, (1.1) relies upon $\phi$ being non-zero to avoid the degeneracy of a metric inverse singularity when $\phi = 0$.

We also note that following identifying the Maxwell tensor in the Kaluza-Klein fields via a five-dimensional the Einstein field equation, again with $\phi$ taken to be dimensionless, the constant $k$ is found to be:

$$\frac{k^2}{2} \equiv \frac{2G}{c^4} 4\pi\epsilon_0 = \frac{2}{c^4}\frac{G}{k_e} \quad \text{i.e.,} \quad k = \frac{2}{c^2}\sqrt{\frac{G}{k_e}}, \tag{1.2}$$

where $k_e = 1/4\pi\varepsilon_0 = \mu_0 c^2 / 4\pi$ is Coulomb's constant and $G$ is Newton's gravitational constant.

Now, as noted above, Kaluza-Klein theory predated Dirac's Quantum Theory of the Electron [13]. Dirac's later theory begins with taking an operator square root of the Minkowski metric tensor $\mathrm{diag}\left(\eta^{\mu\nu}\right) = \left(+1, -1, -1, -1\right)$ by defining ("$\equiv$") a set of four operator matrices $\gamma^\mu$ according to the anticommutator relation $\tfrac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} = \tfrac{1}{2}\left\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right\} \equiv \eta^{\mu\nu}$. The lower-indexed gamma operators are likewise defined such that $\tfrac{1}{2}\left\{\gamma_\mu, \gamma_\nu\right\} \equiv \eta_{\mu\nu}$. To generalize to curved spacetime thus to gravitation which employs the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ defined such that $g^{\mu\alpha}g_{\alpha\nu} \equiv \delta^\mu{}_\nu$ and we define a set of $\Gamma^\mu$ with a parallel definition $\tfrac{1}{2}\left\{\Gamma^\mu, \Gamma^\nu\right\} \equiv g^{\mu\nu}$. We simultaneously define a vierbein a.k.a. tetrad $e^\mu_a$ with both a superscripted Greek "spacetime / world" index and a subscripted Latin "local / Lorentz / Minkowski" index using the relation $e^\mu_a \gamma^a \equiv \Gamma^\mu$. Thus, we deduce that $g^{\mu\nu} = \tfrac{1}{2}\left\{\Gamma^\mu, \Gamma^\nu\right\} = \tfrac{1}{2}\left\{\gamma^a\gamma^b + \gamma^b\gamma^a\right\} e^\mu_a e^\nu_b = \eta^{ab} e^\mu_a e^\nu_b$. So just as the metric tensor $g^{\mu\nu}$ transforms in four-dimensional spacetime as a contravariant (upper-indexed) tensor, these $\Gamma^\mu$ operators likewise transform in spacetime as a contravariant four-vector.

One might presume in view of Dirac theory that the five-dimensional $G_{\mathrm{MN}}$ and $G^{\mathrm{MN}}$ in the Kaluza-Klein metric tensor (1.1) can be likewise deconstructed into square root operators defined using the anticommutator relations:

$$\tfrac{1}{2}\left\{\Gamma_{\mathrm{M}}, \Gamma_{\mathrm{N}}\right\} = \tfrac{1}{2}\left\{\Gamma_{\mathrm{M}}\Gamma_{\mathrm{N}} + \Gamma_{\mathrm{N}}\Gamma_{\mathrm{M}}\right\} \equiv G_{\mathrm{MN}}; \quad \tfrac{1}{2}\left\{\Gamma^{\mathrm{M}}, \Gamma^{\mathrm{N}}\right\} = \tfrac{1}{2}\left\{\Gamma^{\mathrm{M}}\Gamma^{\mathrm{N}} + \Gamma^{\mathrm{N}}\Gamma^{\mathrm{M}}\right\} \equiv G^{\mathrm{MN}}, \tag{1.3}$$

where $\Gamma_{\mathrm{M}}$ and $\Gamma^{\mathrm{M}}$ transform as *five-dimensional vectors* in five-dimensional spacetime. This would presumably include a five-dimensional definition $\varepsilon_A^{\mathrm{M}}\gamma^A \equiv \Gamma^{\mathrm{M}}$ for a tetrad $\varepsilon_A^{\mathrm{M}}$, where $\mathrm{M}=0,1,2,3,5$ is a world index and $A=0,1,2,3,5$ is a local index, and where $\gamma^5$ is a fifth operator matrix which may or may not be associated with Dirac's $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, depending upon the detailed mathematical calculations which determine this $\gamma^5$.

However, as we shall now demonstrate, the Kaluza-Klein metric tensors in (1.1) *cannot* be deconstructed into $\Gamma_{\mathrm{M}}$ and $\Gamma^{\mathrm{M}}$ in the manner of (1.3) without modification to their $G_{05}=G_{50}$ and $G_{55}$ components, and without imposing certain constraints on the gauge fields $A^\mu$ which remove two degrees of freedom and fix the gauge of these fields to that of a photon. We represent these latter constraints by $A^\mu = A_\gamma{}^\mu$, with *a subscripted $\gamma$ which denotes a photon and which is not a spacetime index*. This means that in fact, in view of Dirac theory which was developed afterwards, the Kaluza-Klein metric tensors (1.3) are really *not* generally-covariant in five dimensions. Rather, they only have a four-dimensional spacetime covariance represented in the components of $G_{\mu\nu}=g_{\mu\nu}+\phi^2k^2A_\mu A_\nu$ and $G^{\mu\nu}=g^{\mu\nu}$, and of $G_{\mu 5}=\phi^2kA_\mu$ and $G^{\mu 5}=-A^\mu$, which are all patched together with fifth-dimensional components with which they are not generally-covariant. Moreover, even the spacetime components of (1.1) alone are not generally covariant even in the four spacetime dimensions alone, unless the gauge symmetry of the gauge field $A_\mu$ is broken to remove two degrees of freedom and fixed to that of a photon, $A^\mu = A_\gamma{}^\mu$.

In today's era when the General Theory of Relativity [2] is now a few years past its centenary, and when at least in classical field theory general covariance is firmly-established as a required principle for the laws of nature, it would seem essential that any theory of nature which purports to operate in five dimensions that include the four dimensions of spacetime, ought to manifest general covariance *across all five dimensions*, and ought to be wholly consistent at the "operator square root" level with Dirac theory. Accordingly, it is necessary to "repair" Kaluza-Klein theory to make certain that it adheres to such five-dimensional covariance. In so doing, many of the most-nagging, century-old difficulties of Kaluza-Klein theory are immediately resolved, including those related to the scalar field in $G_{55}=\phi^2$ and the degeneracy of the metric tensor when this field is zeroed out, as well as the large-magnitude terms which arise when the scalar field has a non-zero gradient. Moreover, the fourth spacelike dimension of Kaluza-Klein is instead revealed to be a second timelike dimension. And of extreme importance, this Kaluza-Klein fifth dimension which has spent a century looking for direct observational grounding, may be tied directly to the clear observational physics built around the Dirac $\gamma^5$, and the multitude of observed chiral and pseudoscalar and axial vector particle states that are centered about this $\gamma^5$. Finally, importantly, all of this happens without sacrificing the Kaluza "miracle" of placing electrodynamics onto a geometrodynamic footing. This is what will now be demonstrated.

## 2. The Kaluza-Klein Tetrad and Dirac Operators in Four Dimensional Spacetime, and the Covariant Fixing of Gauge Fields to the Photon

The first step to ensure that Kaluza-Klein theory is covariant in five dimensions using the operator deconstruction (1.3), is to obtain the four-dimensional spacetime deconstruction:

$$\tfrac{1}{2}\left\{\Gamma_\mu,\Gamma_\nu\right\}=\tfrac{1}{2}\left\{\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu\right\}=\tfrac{1}{2}\varepsilon_{\mu a}\varepsilon_{\nu b}\left\{\gamma^a\gamma^b+\gamma^b\gamma^a\right\}=\eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b}\equiv G_{\mu\nu}=g_{\mu\nu}+\phi^2k^2A_\mu A_\nu \tag{2.1}$$

using a four-dimensional tetrad $\varepsilon_{\mu a}$ defined by $\varepsilon_{\mu a}\gamma^a\equiv\Gamma_\mu$, where $\mu=0,1,2,3$ is a spacetime world index raised and lowered with $G^{\mu\nu}$ and $G_{\mu\nu}$, and $a=0,1,2,3$ is a local Lorentz / Minkowski tangent spacetime index raised and lowered with $\eta^{ab}$ and $\eta_{ab}$. To simplify calculation, we set $g_{\mu\nu}=\eta_{\mu\nu}$ thus $G_{\mu\nu}=\eta_{\mu\nu}+\phi^2k^2A_\mu A_\nu$. Later on, we will use the minimal-coupling principle to generalize back from $\eta_{\mu\nu}\mapsto g_{\mu\nu}$. In this circumstance, the spacetime is "flat" *except for* the curvature in $G_{\mu\nu}$ brought about by the electrodynamic terms $\phi^2k^2A_\mu A_\nu$. We can further simplify calculation by defining an $\varepsilon'_{\mu a}$ such that $\delta_{\mu a}+\varepsilon'_{\mu a}\equiv\varepsilon_{\mu a}$, which represents the degree to which $\varepsilon_{\mu a}$ differs from the unit matrix $\delta_{\mu a}$. We may then write the salient portion of (2.1) as:

$$\begin{aligned}&\eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b}=\eta^{ab}\left(\delta_{\mu a}+\varepsilon'_{\mu a}\right)\left(\delta_{\nu b}+\varepsilon'_{\nu b}\right)=\eta^{ab}\delta_{\mu a}\delta_{\nu b}+\delta_{\nu b}\eta^{ab}\varepsilon'_{\mu a}+\delta_{\mu a}\eta^{ab}\varepsilon'_{\nu b}+\eta^{ab}\varepsilon'_{\mu a}\varepsilon'_{\nu b}\\&=\eta_{\mu\nu}+\eta_{a\nu}\varepsilon'^{\,a}_{\mu}+\eta_{\mu b}\varepsilon'^{\,b}_{\nu}+\eta_{ab}\varepsilon'^{\,a}_{\mu}\varepsilon'^{\,b}_{\nu}=\eta_{\mu\nu}+\phi^2k^2A_\mu A_\nu\end{aligned}. \tag{2.2}$$

Note that when electrodynamics is "turned off" by setting $A_\mu$ and / or by setting $\phi=0$ this reduces to $\eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b}=\eta_{\mu\nu}$ which is solved by the tetrad being a unit matrix, $\varepsilon_{\mu a}=\delta_{\mu a}$. Subtracting $\eta_{\mu\nu}$ from each side of (2.2) we now need to solve:

$$\eta_{a\nu}\varepsilon'^{\,a}_{\mu}+\eta_{\mu b}\varepsilon'^{\,b}_{\nu}+\eta_{ab}\varepsilon'^{\,a}_{\mu}\varepsilon'^{\,b}_{\nu}=\phi^2k^2A_\mu A_\nu\,. \tag{2.3}$$

The above contains sixteen (16) equations for each of $\mu=0,1,2,3$ and $\nu=0,1,2,3$. But, this is symmetric in $\mu$ and $\nu$ so in fact there are only ten (10) independent equations. Given that $\mathrm{diag}\left(\eta_{ab}\right)=\left(1,-1,-1,-1\right)$, the four $\mu=\nu$ "diagonal" equations in (2.3) produce the relations:

$$\begin{aligned}&\eta_{a0}\varepsilon'^{a}_{0}+\eta_{0b}\varepsilon'^{b}_{0}+\eta_{ab}\varepsilon'^{a}_{0}\varepsilon'^{b}_{0}=2\varepsilon'^{0}_{0}+\varepsilon'^{0}_{0}\varepsilon'^{0}_{0}-\varepsilon'^{1}_{0}\varepsilon'^{1}_{0}-\varepsilon'^{2}_{0}\varepsilon'^{2}_{0}-\varepsilon'^{3}_{0}\varepsilon'^{3}_{0}=\phi^2k^2A_0A_0\\&\eta_{a1}\varepsilon'^{a}_{1}+\eta_{1b}\varepsilon'^{b}_{1}+\eta_{ab}\varepsilon'^{a}_{1}\varepsilon'^{b}_{1}=-2\varepsilon'^{1}_{1}+\varepsilon'^{0}_{1}\varepsilon'^{0}_{1}-\varepsilon'^{1}_{1}\varepsilon'^{1}_{1}-\varepsilon'^{2}_{1}\varepsilon'^{2}_{1}-\varepsilon'^{3}_{1}\varepsilon'^{3}_{1}=\phi^2k^2A_1A_1\\&\eta_{a2}\varepsilon'^{a}_{2}+\eta_{2b}\varepsilon'^{b}_{2}+\eta_{ab}\varepsilon'^{a}_{2}\varepsilon'^{b}_{2}=-2\varepsilon'^{2}_{2}+\varepsilon'^{0}_{2}\varepsilon'^{0}_{2}-\varepsilon'^{1}_{2}\varepsilon'^{1}_{2}-\varepsilon'^{2}_{2}\varepsilon'^{2}_{2}-\varepsilon'^{3}_{2}\varepsilon'^{3}_{2}=\phi^2k^2A_2A_2\\&\eta_{a3}\varepsilon'^{a}_{3}+\eta_{3b}\varepsilon'^{b}_{3}+\eta_{ab}\varepsilon'^{a}_{3}\varepsilon'^{b}_{3}=-2\varepsilon'^{3}_{3}+\varepsilon'^{0}_{3}\varepsilon'^{0}_{3}-\varepsilon'^{1}_{3}\varepsilon'^{1}_{3}-\varepsilon'^{2}_{3}\varepsilon'^{2}_{3}-\varepsilon'^{3}_{3}\varepsilon'^{3}_{3}=\phi^2k^2A_3A_3\end{aligned}. \tag{2.4a}$$

Likewise, the three $\mu=0$, $\nu=1,2,3$ mixed time and space relations in (2.3) are:

$$\begin{aligned}
&\eta_{a1}\varepsilon_0'^{a}+\eta_{0b}\varepsilon_1'^{b}+\eta_{ab}\varepsilon_0'^{a}\varepsilon_1'^{b}=-\varepsilon_0'^{1}+\varepsilon_1'^{0}+\varepsilon_0'^{0}\varepsilon_1'^{0}-\varepsilon_0'^{1}\varepsilon_1'^{1}-\varepsilon_0'^{2}\varepsilon_1'^{2}-\varepsilon_0'^{3}\varepsilon_1'^{3}=\phi^2k^2A_0A_1\\
&\eta_{a2}\varepsilon_0'^{a}+\eta_{0b}\varepsilon_2'^{b}+\eta_{ab}\varepsilon_0'^{a}\varepsilon_2'^{b}=-\varepsilon_0'^{2}+\varepsilon_2'^{0}+\varepsilon_0'^{0}\varepsilon_2'^{0}-\varepsilon_0'^{1}\varepsilon_2'^{1}-\varepsilon_0'^{2}\varepsilon_2'^{2}-\varepsilon_0'^{3}\varepsilon_2'^{3}=\phi^2k^2A_0A_2\ .\\
&\eta_{a3}\varepsilon_0'^{a}+\eta_{0b}\varepsilon_3'^{b}+\eta_{ab}\varepsilon_0'^{a}\varepsilon_3'^{b}=-\varepsilon_0'^{3}+\varepsilon_3'^{0}+\varepsilon_0'^{0}\varepsilon_3'^{0}-\varepsilon_0'^{1}\varepsilon_3'^{1}-\varepsilon_0'^{2}\varepsilon_3'^{2}-\varepsilon_0'^{3}\varepsilon_3'^{3}=\phi^2k^2A_0A_3
\end{aligned}\tag{2.4b}$$

Finally, the pure-space relations in (2.3) are:

$$\begin{aligned}
&\eta_{a2}\varepsilon_1'^{a}+\eta_{1b}\varepsilon_2'^{b}+\eta_{ab}\varepsilon_1'^{a}\varepsilon_2'^{b}=-\varepsilon_1'^{2}-\varepsilon_2'^{1}+\varepsilon_1'^{0}\varepsilon_2'^{0}-\varepsilon_1'^{1}\varepsilon_2'^{1}-\varepsilon_1'^{2}\varepsilon_2'^{2}-\varepsilon_1'^{3}\varepsilon_2'^{3}=\phi^2k^2A_1A_2\\
&\eta_{a3}\varepsilon_2'^{a}+\eta_{2b}\varepsilon_3'^{b}+\eta_{ab}\varepsilon_2'^{a}\varepsilon_3'^{b}=-\varepsilon_2'^{3}-\varepsilon_3'^{2}+\varepsilon_2'^{0}\varepsilon_3'^{0}-\varepsilon_2'^{1}\varepsilon_3'^{1}-\varepsilon_2{}^{2}\varepsilon_3'^{2}-\varepsilon_2'^{3}\varepsilon_3'^{3}=\phi^2k^2A_2A_3\ .\\
&\eta_{a1}\varepsilon_3'^{a}+\eta_{3b}\varepsilon_1'^{b}+\eta_{ab}\varepsilon_3'^{a}\varepsilon_1'^{b}=-\varepsilon_3'^{1}-\varepsilon_1'^{3}+\varepsilon_3'^{0}\varepsilon_1'^{0}-\varepsilon_3'^{1}\varepsilon_1'^{1}-\varepsilon_3'^{2}\varepsilon_1'^{2}-\varepsilon_3'^{3}\varepsilon_1'^{3}=\phi^2k^2A_3A_1
\end{aligned}\tag{2.4c}$$

Now, we notice that the right-hand side of all ten of (2.4) have nonlinear second-order products $\phi^2k^2A_\mu A_\nu$ of field terms, while on the left of each there is a mix of linear first-order and nonlinear second-order expressions containing the $\varepsilon_\mu'^{a}$. Our goal at the moment, therefore, is to eliminate all of the first order expressions from the left-hand sides of (2.4) to create a structural match whereby a sum of second order terms on the left is equal to a second order term on the right.

In (14.3a) the linear appearances are of $\varepsilon_0'^{0}$, $\varepsilon_1'^{1}$, $\varepsilon_2'^{2}$ and $\varepsilon_3'^{3}$ respectively. Noting that the complete tetrad $\varepsilon_\mu{}^{a}=\delta_\mu{}^{a}+\varepsilon_\mu'^{a}$ and that $\varepsilon_\mu{}^{a}=\delta_\mu{}^{a}$ when electrodynamics is turned off, we first require that $\varepsilon_\mu{}^{a}=\delta_\mu{}^{a}$ for the four $\mu=a$ diagonal components, and therefore, that $\varepsilon_0'^{0}=\varepsilon_1'^{1}=\varepsilon_2'^{2}=\varepsilon_3'^{3}=0$. As a result, the fields in $\phi^2k^2A_\mu A_\nu$ will all appear in off-diagonal components of the tetrad. With this, (2.4a) reduce to:

$$\begin{aligned}
&-\varepsilon_0'^{1}\varepsilon_0'^{1}-\varepsilon_0'^{2}\varepsilon_0'^{2}-\varepsilon_0'^{3}\varepsilon_0'^{3}=\phi^2k^2A_0A_0\\
&\varepsilon_1'^{0}\varepsilon_1'^{0}-\varepsilon_1'^{2}\varepsilon_1'^{2}-\varepsilon_1'^{3}\varepsilon_1'^{3}=\phi^2k^2A_1A_1\\
&\varepsilon_2'^{0}\varepsilon_2'^{0}-\varepsilon_2'^{1}\varepsilon_2'^{1}-\varepsilon_2'^{3}\varepsilon_2'^{3}=\phi^2k^2A_2A_2\\
&\varepsilon_3'^{0}\varepsilon_3'^{0}-\varepsilon_3'^{1}\varepsilon_3'^{1}-\varepsilon_3'^{2}\varepsilon_3'^{2}=\phi^2k^2A_3A_3
\end{aligned}\ .\tag{2.5a}$$

In (2.4b) we achieve structural match using $\varepsilon_1'^{1}=\varepsilon_2'^{2}=\varepsilon_3'^{3}=0$ from above, and also by setting $\varepsilon_0'^{1}=\varepsilon_1'^{0}$, $\varepsilon_0'^{2}=\varepsilon_2'^{0}$, $\varepsilon_0'^{3}=\varepsilon_3'^{0}$, which is symmetric under $0\leftrightarrow a=1,2,3$ interchange. Therefore:

$$\begin{aligned}
&-\varepsilon_0'^{2}\varepsilon_1'^{2}-\varepsilon_0'^{3}\varepsilon_1'^{3}=\phi^2k^2A_0A_1\\
&-\varepsilon_0'^{1}\varepsilon_2'^{1}-\varepsilon_0'^{3}\varepsilon_2'^{3}=\phi^2k^2A_0A_2\ .\\
&-\varepsilon_0'^{1}\varepsilon_3'^{1}-\varepsilon_0'^{2}\varepsilon_3'^{2}=\phi^2k^2A_0A_3
\end{aligned}\tag{2.5b}$$

In (2.4c) we use $\varepsilon_1'^{1}=\varepsilon_2'^{2}=\varepsilon_3'^{3}=0$ from above and also set $\varepsilon_1'^{2}=-\varepsilon_2'^{1}$, $\varepsilon_2'^{3}=-\varepsilon_3'^{2}$, $\varepsilon_3'^{1}=-\varepsilon_1'^{3}$ which are antisymmetric under interchange of different space indexes. Therefore, we now have:

$$\begin{aligned}
&\varepsilon_1'^{0}\varepsilon_2'^{0}-\varepsilon_1'^{3}\varepsilon_2'^{3}=\phi^2k^2A_1A_2\\
&\varepsilon_2'^{0}\varepsilon_3'^{0}-\varepsilon_2'^{1}\varepsilon_3'^{1}=\phi^2k^2A_2A_3\ .\\
&\varepsilon_3'^{0}\varepsilon_1'^{0}-\varepsilon_3'^{2}\varepsilon_1'^{2}=\phi^2k^2A_3A_1
\end{aligned} \tag{2.5c}$$

In all of (2.5), we now only have matching-structure second-order terms on both sides.

For the next step, closely studying the space indexes in all of (2.5) above, we now make an educated guess at an assignment for the fields in $\phi^2k^2A_iA_j$. Specifically, also using the symmetric-interchange $\varepsilon_0'^{1}=\varepsilon_1'^{0}$, $\varepsilon_0'^{2}=\varepsilon_2'^{0}$, $\varepsilon_0'^{3}=\varepsilon_3'^{0}$ from earlier, we now guess an assignment:

$$\varepsilon_0'^{1}=\varepsilon_1'^{0}=\phi kA_1;\quad \varepsilon_0'^{2}=\varepsilon_2'^{0}=\phi kA_2;\quad \varepsilon_0'^{3}=\varepsilon_3'^{0}=\phi kA_3\ . \tag{2.6}$$

Because all space-indexed expressions in (2.5) contain second-order products of the above, it is possible to have also tried using a minus sign in all of (2.5) whereby $\varepsilon_0'^{1}=\varepsilon_1'^{0}=-\phi kA_1$, $\varepsilon_0'^{2}=\varepsilon_2'^{0}=-\phi kA_2$ and $\varepsilon_0'^{3}=\varepsilon_3'^{0}=-\phi kA_3$. But absent motivation to the contrary, we employ a plus sign which is implicit in the above. Substituting (2.6) into all of (2.5) and reducing now yields:

$$\begin{aligned}
&-A_1A_1-A_2A_2-A_3A_3=A_0A_0\\
&-\varepsilon_1'^{2}\varepsilon_1'^{2}-\varepsilon_1'^{3}\varepsilon_1'^{3}=0\\
&-\varepsilon_2'^{1}\varepsilon_2'^{1}-\varepsilon_2'^{3}\varepsilon_2'^{3}=0\\
&-\varepsilon_3'^{1}\varepsilon_3'^{1}-\varepsilon_3'^{2}\varepsilon_3'^{2}=0
\end{aligned}\ , \tag{2.7a}$$

$$\begin{aligned}
&-\phi kA_2\varepsilon_1'^{2}-\phi kA_3\varepsilon_1'^{3}=\phi^2k^2A_0A_1\\
&-\phi kA_1\varepsilon_2'^{1}-\phi kA_3\varepsilon_2'^{3}=\phi^2k^2A_0A_2\ ,\\
&-\phi kA_1\varepsilon_3'^{1}-\phi kA_2\varepsilon_3'^{2}=\phi^2k^2A_0A_3
\end{aligned} \tag{2.7b}$$

$$-\varepsilon_1'^{3}\varepsilon_2'^{3}=-\varepsilon_2'^{1}\varepsilon_3'^{1}=-\varepsilon_3'^{2}\varepsilon_1'^{2}=0\ . \tag{2.7c}$$

Now, one way to satisfy the earlier relations $\varepsilon_1'^{2}=-\varepsilon_2'^{1}$, $\varepsilon_2'^{3}=-\varepsilon_3'^{2}$, $\varepsilon_3'^{1}=-\varepsilon_1'^{3}$ used in (2.5c) as well as to satisfy (2.7c), is to set all of the pure-space components:

$$\varepsilon_1'^{2}=\varepsilon_2'^{1}=\varepsilon_2'^{3}=\varepsilon_3'^{2}=\varepsilon_3'^{1}=\varepsilon_1'^{3}=0\ . \tag{2.8}$$

This disposes of (2.7c) and last three relations in (2.7a), leaving only the two constraints:

$$-A_1A_1-A_2A_2-A_3A_3=A_0A_0\ , \tag{2.9a}$$

$$0=\phi^2k^2A_0A_1=\phi^2k^2A_0A_2=\phi^2k^2A_0A_3\ . \tag{2.9b}$$

These above relations (2.9) are extremely important. In (2.9b), if *any one* of $A_1$, $A_2$ or $A_3$ is *not* equal to zero, then we *must* have $A_0 = 0$. So, we take as a given that at least one of $A_1$, $A_2$ or $A_3$ is non-zero, whereby (2.9a) and (2.9b) together become:

$$A_0 = 0; \quad A_1A_1 + A_2A_2 + A_3A_3 = 0, \tag{2.10}$$

*These two constraints have removed two redundant degrees of freedom from the gauge field* $A_\mu$, <u>*in a generally-covariant manner*</u>. Moreover, for the latter constraint in $A_1A_1 + A_2A_2 + A_3A_3 = 0$ to be satisfied, it is necessary that at least one of the space components of $A_j$ be *imaginary*. For example, if $A_3 = 0$, then one way to solve the entirety of (2.10) is to have:

$$A_\mu = A\varepsilon_\mu \exp\left(-iq_\sigma x^\sigma / \hbar\right), \tag{2.11a}$$

with a polarization vector

$$\varepsilon_{R,L\mu}\left(\hat{z}\right) \equiv \begin{pmatrix} 0 & \pm 1 & +i & 0 \end{pmatrix} / \sqrt{2}, \tag{2.11b}$$

where *A* has dimensions of charge / energy to provide dimensional balance given the dimensionless $\varepsilon_{R,L\mu}$. But the foregoing is instantly-recognizable as the gauge potential $A_\mu = A_{\gamma\mu}$ for an individual photon (denoted with $\gamma$) with two helicity states propagating along the *z* axis, having an energy-momentum vector

$$cq^\mu\left(\hat{z}\right) = \begin{pmatrix} E & 0 & 0 & cq_z \end{pmatrix} = \begin{pmatrix} h\nu & 0 & 0 & h\nu \end{pmatrix}. \tag{2.11c}$$

This satisfies $q_\mu q^\mu = m_\gamma{}^2 c^2 = 0$, which makes this a massless, luminous field quantum. Additionally, we see from all of (2.11) that $A_\mu q^\mu = 0$, and $A_j q^j = 0$ as is also true for a photon. The latter $A_j q^j = 0$ is the so-called Coulomb gauge which is ordinarily imposed as a non-covariant gauge condition. But here, it has emerged in an entirely covariant fashion.

In short, what we have ascertained in (2.10) and (2.11) is that if the spacetime components $G_{\mu\nu} = g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$ of the Kaluza-Klein metric tensor with $g_{\mu\nu} = \eta_{\mu\nu}$ are to produce a set of $\Gamma_\mu$ satisfying the Dirac anticommutator relation $\frac{1}{2}\left\{\Gamma_\mu, \Gamma_\nu\right\} \equiv G_{\mu\nu}$, *the gauge symmetry of* $A_\mu$ *must be broken to correspond with that of the photon,* $A_\mu = A_{\gamma\mu}$. *The very act of deconstructing* $G_{\mu\nu}$ *into square root Dirac operators covariantly removes two degrees of freedom from the gauge field and forces it to become a photon field quantum.* Moreover, (2.11a) implies that $i\hbar\partial_\alpha A_\mu = q_\alpha A_\mu$ while (2.11c) contains the energy $E = h\nu$ of a single photon. So, *starting with an entirely-classical* $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$ *and merely requiring the formation of a set of* $\Gamma_\mu$ *transforming* *covariantly in spacetime with the anticommutator $\tfrac{1}{2}\{\Gamma_\mu,\Gamma_\nu\}\equiv G_{\mu\nu}$, we covariantly end up with some of the core relations of quantum mechanics.*

*Even outside of the context of Kaluza-Klein theory, entirely in four-dimensional spacetime, the foregoing calculation solves the long-perplexing problem of how to covariantly eliminate the redundancy inherent in using a four-component Lorentz vector $A_\mu$ to describe a classical electromagnetic wave or a quantum photon field with only two transverse degrees of physical freedom*: If we posit a metric tensor given by $G_{\mu\nu}=g_{\mu\nu}+\phi^2k^2A_\mu A_\nu$, and if we require the existence of a set of Dirac operators $\Gamma_\mu$ transforming as a covariant vector in spacetime and connected to the metric tensor such that $\tfrac{1}{2}\{\Gamma_\mu,\Gamma_\nu\}\equiv G_{\mu\nu}$, then we are given *no choice* but to have $A_\mu=A_{\gamma\mu}$ be the quantum field of a photon with two degrees of freedom covariantly-removed and only two degrees of freedom remaining.

Moreover, we have also deduced all of the components of the tetrad $\varepsilon_\mu{}^a=\delta_\mu{}^a+\varepsilon'_\mu{}^a$. Pulling together all of $\varepsilon'^0_0=\varepsilon'^1_1=\varepsilon'^2_2=\varepsilon'^3_3=0$ together with (2.6) and (2.8), and setting $A_\mu=A_{\gamma\mu}$ to incorporate the pivotal finding in (2.10), (2.11) that the gauge-field must be covariantly fixed to the gauge field of a photon (again, $\gamma$ is a subscript, *not a spacetime index*), this tetrad is:

$$\varepsilon_\mu{}^a=\delta_\mu{}^a+\varepsilon'_\mu{}^a=\begin{pmatrix}1 & \phi kA_{\gamma1} & \phi kA_{\gamma2} & \phi kA_{\gamma3}\\ \phi kA_{\gamma1} & 1 & 0 & 0\\ \phi kA_{\gamma2} & 0 & 1 & 0\\ \phi kA_{\gamma3} & 0 & 0 & 1\end{pmatrix}. \tag{2.12}$$

Finally, because $\varepsilon_{\mu a}\gamma^a=\varepsilon_\mu{}^\alpha\gamma_\alpha\equiv\Gamma_\mu$, we may use (2.12) to deduce that the Dirac operators:

$$\begin{aligned}
\Gamma_0&=\varepsilon_0{}^\alpha\gamma_\alpha=\varepsilon_0{}^0\gamma_0+\varepsilon_0{}^1\gamma_1+\varepsilon_0{}^2\gamma_2+\varepsilon_0{}^3\gamma_3=\gamma_0+\phi kA_{\gamma j}\gamma_j\\
\Gamma_1&=\varepsilon_1{}^\alpha\gamma_\alpha=\varepsilon_1{}^0\gamma_0+\varepsilon_1{}^1\gamma_1=\gamma_1+\phi kA_{\gamma1}\gamma_0\\
\Gamma_2&=\varepsilon_2{}^\alpha\gamma_\alpha=\varepsilon_2{}^0\gamma_0+\varepsilon_2{}^2\gamma_2=\gamma_2+\phi kA_{\gamma2}\gamma_0\\
\Gamma_3&=\varepsilon_3{}^\alpha\gamma_\alpha=\varepsilon_3{}^0\gamma_0+\varepsilon_3{}^3\gamma_3=\gamma_3+\phi kA_{\gamma3}\gamma_0
\end{aligned}, \tag{2.13}$$

which consolidate into a set of $\Gamma_\mu$ transforming as a four-vector in spacetime, namely:

$$\Gamma_\mu=\begin{pmatrix}\gamma_0+\phi kA_{\gamma j}\gamma_j & \gamma_j+\phi kA_{\gamma j}\gamma_0\end{pmatrix}. \tag{2.14}$$

It is a useful exercise to confirm that (2.14) above, inserted into (2.1), will produce $G_{\mu\nu}=\eta_{\mu\nu}+\phi^2k^2A_{\gamma\mu}A_{\gamma\nu}$, which may then be generalized from $\eta_{\mu\nu}\mapsto g_{\mu\nu}$ in the usual way by applying the minimal coupling principle. As a result, we return to the Kaluza-Klein metric tensors in (1.1), but apply the foregoing to now rewrite these as:

$$G_{\mathrm{MN}} = \begin{pmatrix} g_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu} & \phi^2 k A_{\gamma\mu} \\ \phi^2 k A_{\gamma\nu} & \phi^2 \end{pmatrix}; \quad G^{\mathrm{MN}} = \begin{pmatrix} g^{\mu\nu} & -A_{\gamma}{}^{\mu} \\ -A_{\gamma}{}^{\nu} & g_{\alpha\beta} A_{\gamma}{}^{\alpha} A_{\gamma}{}^{\beta} + 1/\phi^2 \end{pmatrix}. \tag{2.15}$$

The only change we have made is to replace $A_\mu \mapsto A_{\gamma\mu}$, which is to represent the remarkable result that *even in four spacetime dimensions alone*, it is not possible to deconstruct $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$ into a set of Dirac $\Gamma_\mu$ defined using (2.1) without fixing the gauge field $A_\mu$ to that of a photon $A_{\gamma\mu}$. Now, we extend this general covariance to the fifth dimension.

## 3. Derivation of the "Dirac-Kaluza-Klein" (DKK) Five-Dimensional Metric Tensor

To ensure general covariance at the Dirac level in five-dimensions, it is necessary to first extend (2.1) into all five dimensions. For this we use the lower-indexed (1.3), namely:

$$\tfrac{1}{2}\{\Gamma_{\mathrm{M}}, \Gamma_{\mathrm{N}}\} = \tfrac{1}{2}\{\Gamma_{\mathrm{M}}\Gamma_{\mathrm{N}} + \Gamma_{\mathrm{N}}\Gamma_{\mathrm{M}}\} \equiv G_{\mathrm{MN}}. \tag{3.1}$$

As just shown, the spacetime components of (3.1) with $g_{\mu\nu} = \eta_{\mu\nu}$ and using (2.14) will already reproduce $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$ in (2.15). Now we turn to the fifth-dimensional components.

We first find it helpful to separate the time and space components of $G_{\mathrm{MN}}$ in (2.15) and so rewrite this as:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} g_{00} + \phi^2 k^2 A_{\gamma 0} A_{\gamma 0} & g_{0k} + \phi^2 k^2 A_{\gamma 0} A_{\gamma k} & \phi^2 k A_{\gamma 0} \\ g_{j0} + \phi^2 k^2 A_{\gamma j} A_{\gamma 0} & g_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ \phi^2 k A_{\gamma 0} & \phi^2 k A_{\gamma k} & \phi^2 \end{pmatrix}. \tag{3.2}$$

We know of course that $A_{\gamma 0} = 0$, which is the constraint that first arose from (2.10). So, if we again work with $g_{\mu\nu} = \eta_{\mu\nu}$ and set $A_{\gamma 0} = 0$, the above simplifies to:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ 0 & \phi^2 k A_{\gamma k} & \phi^2 \end{pmatrix}. \tag{3.3}$$

Next, let us *define* a $\Gamma_5$ to go along with the remaining $\Gamma_\mu$ in (2.14) in such a way as to *require* that the symmetric components $G_{j5} = G_{5j} = \phi^2 k A_{\gamma j}$ in (3.3) remain fully intact without any change. This is important, because these components in particular are largely responsible for the Kaluza "miracles" which reproduce Maxwell's equations together with the Lorentz Force motion and the Maxwell Stress-Energy Tensor. At the same time, because $A_{\gamma 0}=0$ as uncovered at (2.10), we can always maintain covariance between the space components $G_{j5}=G_{5j}=\phi^2 kA_{\gamma j}$ and the time components $G_{05}=G_{50}$ in the manner of (1.1) by adding $\phi^2 kA_{\gamma 0}=0$ to anything else we deduce for $G_{05}=G_{50}$, so we lay the foundation for the Kaluza miracles to remain intact. We impose this requirement though (3.1) by writing the $\Gamma_5$ *definition* as:

$$\tfrac{1}{2}\left\{\Gamma_j,\Gamma_5\right\}=\tfrac{1}{2}\left\{\Gamma_j\Gamma_5+\Gamma_5\Gamma_j\right\}\equiv G_{j5}=G_{j5}=\phi^2 kA_{\gamma j}\,. \tag{3.4}$$

Using $\Gamma_j=\gamma_j+\phi kA_{\gamma j}\gamma_0$ from (2.14) and adding in a zero, the above now becomes:

$$0+\phi^2 kA_{\gamma j}\equiv\tfrac{1}{2}\left\{\Gamma_j\Gamma_5+\Gamma_5\Gamma_j\right\}=\tfrac{1}{2}\left\{\gamma_j,\Gamma_5\right\}+\tfrac{1}{2}\phi kA_{\gamma j}\left\{\gamma_0,\Gamma_5\right\}, \tag{3.5}$$

which reduces down to a pair of anticommutation constraints on $\Gamma_5$, namely:

$$\begin{aligned}0&=\tfrac{1}{2}\left\{\gamma_j,\Gamma_5\right\}\\ \phi&=\tfrac{1}{2}\left\{\gamma_0,\Gamma_5\right\}\end{aligned}\,. \tag{3.6}$$

Now let's examine possible options for $\Gamma_5$.

Given that $\Gamma_0=\gamma_0+\phi kA_{\gamma j}\gamma_j$ and $\Gamma_j=\gamma_j+\phi kA_{\gamma j}\gamma_0$ in (2.14), we anticipate the general form for $\Gamma_5$ to be $\Gamma_5\equiv\gamma_X+Y$ in which we define two unknowns to be determined using (3.6). First, *X* is one of the indexes 0, 1, 2, 3 or 5 of a Dirac matrix. Second, *Y* is a complete unknown which we anticipate will also contain a Dirac matrix as do the operators in (2.14). Using $\Gamma_5\equiv\gamma_X+Y$ in (3.6) we first deduce:

$$\begin{aligned}0&=\tfrac{1}{2}\left\{\gamma_j\Gamma_5+\Gamma_5\gamma_j\right\}=\tfrac{1}{2}\left\{\gamma_j\gamma_X+\gamma_jY+\gamma_X\gamma_j+Y\gamma_j\right\}=\tfrac{1}{2}\left\{\gamma_j,\gamma_X\right\}+\tfrac{1}{2}\left\{\gamma_j,Y\right\}\\ 0+\phi&=\tfrac{1}{2}\left\{\gamma_0\Gamma_5+\Gamma_5\gamma_0\right\}=\tfrac{1}{2}\left\{\gamma_0\gamma_X+\gamma_0Y+\gamma_X\gamma_0+Y\gamma_0\right\}=\tfrac{1}{2}\left\{\gamma_0,\gamma_X\right\}+\tfrac{1}{2}\left\{\gamma_0,Y\right\}\end{aligned}\,. \tag{3.7}$$

From the top line, so long as $\gamma_X\neq -Y$ which means so long as $\Gamma_5\neq 0$, we must have both the anticommutators $\left\{\gamma_j,\gamma_X\right\}=0$ and $\left\{\gamma_j,Y\right\}=0$. The former $\left\{\gamma_j,\gamma_X\right\}=0$ excludes *X* being a space index 1, 2 or 3 leaving only $\gamma_X=\gamma_0$ or $\gamma_X=\gamma_5$. The latter $\left\{\gamma_j,Y\right\}=0$ makes clear that whatever Dirac operator is part of *Y* must likewise be either $\gamma_0$ or $\gamma_5$. From the bottom line, however, we must also have the anticommutators $\left\{\gamma_0,\gamma_X\right\}=0$ and $\tfrac{1}{2}\left\{\gamma_0,Y\right\}=\phi$. The former means that the only remaining choice is $\gamma_X = \gamma_5$, while given $\gamma_0\gamma_0 = 1$ and $\{\gamma_0, \gamma_5\} = 0$ the latter means that $Y = \phi\gamma_0$. Therefore, we conclude that $\Gamma_5 = \gamma_5 + \phi\gamma_0$. Thus, including this in (2.14) now gives:

$$\Gamma_{\mathrm{M}} = \begin{pmatrix} \gamma_0 + \phi k A_{\gamma k}\gamma_k & \gamma_j + \phi k A_{\gamma j}\gamma_0 & \gamma_5 + \phi\gamma_0 \end{pmatrix}. \tag{3.8}$$

With this final operator $\Gamma_5 \equiv \gamma_5 + \phi\gamma_0$, we can use all of (3.8) above in (3.1) to precisely reproduce $G_{j5} = \phi^2 k A_{\gamma j}$ and $G_{5k} = \phi^2 k A_{\gamma k}$ in (3.3), as well as $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$ given $A_{\gamma 0} = 0$. This leaves the remaining components $G_{05} = G_{50}$ and $G_{55}$ to which we now turn.

If we use $\Gamma_0 = \gamma_0 + \phi k A_{\gamma j}\gamma_j$ and $\Gamma_5 = \gamma_5 + \phi\gamma_0$ in (3.1) to ensure that these remaining components are also fully covariant over all five dimensions, then we determine that:

$$\begin{aligned} G_{05} = G_{50} &= \tfrac{1}{2}\{\Gamma_0\Gamma_5 + \Gamma_5\Gamma_0\} = \tfrac{1}{2}\{(\gamma_0 + \phi k A_{\gamma j}\gamma_j)(\gamma_5 + \phi\gamma_0) + (\gamma_5 + \phi\gamma_0)(\gamma_0 + \phi k A_{\gamma j}\gamma_j)\} \\ &= \phi\gamma_0\gamma_0 + \tfrac{1}{2}\{\gamma_0, \gamma_5\} + \tfrac{1}{2}\phi k A_{\gamma j}\{\gamma_j, \gamma_5\} + \tfrac{1}{2}\phi^2 k A_{\gamma j}\{\gamma_j, \gamma_0\} = \phi \end{aligned}, \tag{3.9}$$

$$G_{55} = \Gamma_5\Gamma_5 = (\gamma_5 + \phi\gamma_0)(\gamma_5 + \phi\gamma_0) = \gamma_5\gamma_5 + \phi^2\gamma_0\gamma_0 + \phi\{\gamma_5\gamma_0 + \gamma_0\gamma_5\} = 1 + \phi^2. \tag{3.10}$$

These two components are now different from those in (3.3). However, in view of this Dirac operator deconstruction these are required to be different to ensure that the metric tensor is completely generally-covariant across all five dimensions, just as we were required at (2.15) to set $A_j = A_{\gamma j}$ at (2.12) to ensure even basic covariance in four spacetime dimensions.

Consequently, changing (3.3) to incorporate (3.9) and (3.10), we now have:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \phi \\ 0 & \eta_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ \phi & \phi^2 k A_{\gamma k} & 1 + \phi^2 \end{pmatrix}. \tag{3.11}$$

This metric tensor is fully covariant across all five dimensions, and because it is rooted in the Dirac operators (3.8), we expect that this can be made fully compatible with Dirac's theory of the multitude of fermions observed in the natural world, as we shall examine further in section 5. Moreover, in the context of Kaluza-Klein theory, Dirac's Quantum Theory of the Electron [13] has also forced us to set $A_j = A_{\gamma j}$ in the metric tensor, and thereby also served up a *quantum theory of the photon*. Because of its origins in requiring Kaluza-Klein theory to be compatible with Dirac theory, we shall refer to the above as the "Dirac-Kaluza-Klein" (DKK) metric tensor, and shall give the same name to the overall theory based on this.

Importantly, when electrodynamics is turned off by setting $A_{\gamma j} = 0$ and $\phi = 0$ the signature of (3.11) becomes $\mathrm{diag}(G_{\mathrm{MN}}) = (+1, -1, -1, -1, +1)$ with a determinant $|G_{\mathrm{MN}}| = -1$, versus

$|G_{\mathrm{MN}}|=0$ in (1.1) as reviewed earlier. This means that the inverse obtained via $G^{\mathrm{MA}}G_{\mathrm{AN}}=\delta^{\mathrm{M}}{}_{\mathrm{N}}$ will be non-singular as opposed to that in (1.1), and that there is no reliance whatsoever on having $\phi\neq 0$ in order to avoid singularity. This in turn frees $G_{55}$ from the energy requirements of $\phi$ which cause the fifth dimension in (1.1) to have a spacelike signature. And in fact, we see that as a result of this signature, *the fifth dimension in (3.11) is a second timelike, not fourth spacelike, dimension*. In turn, because (3.10) shows that $G_{55}=1+\phi^2=\gamma_5\gamma_5+\phi^2$ obtains its signature from $\gamma_5\gamma_5=1$, it now becomes possible to fully associate the Kaluza-Klein fifth dimension with the $\gamma_5$ of Dirac theory. This is not possible when a theory based on (1.1) causes $G_{55}$ to be spacelike even though $\gamma_5\gamma_5=1$ is timelike, because of this conflict between timelike and spacelike signatures. Moreover, having only $G_{55}=\phi^2$ causes $G_{55}$ to shrink or expand or even zero out entirely, based on the magnitude of $\phi$. In (3.11), there is no such problem. We shall review the physics consequences of all these matters more deeply in section 9 following other development. At the moment, we wish to consolidate (3.11) into the 2x2 matrix format akin to (1.1), which consolidates all spacetime components into a single expression with manifest four-dimensional covariance.

In general, as already hinted, it will sometimes simplify calculation to set $A_{\gamma 0}=0$ simply because this puts some zeros in the equations we are working with; while at other times it will be better to explicitly include $A_{\gamma 0}$ knowing this is zero in order to take advantage of the consolidations enabled by general covariance. To consolidate (3.11) to 2x2 format, we do the latter, by restoring the zeroed $A_{\gamma 0}=0$ to the spacetime components of (3.11) and consolidating them to $G_{\mu\nu}=\eta_{\mu\nu}+\phi^2k^2A_{\gamma\mu}A_{\gamma\nu}$. This is exactly what is in the Kaluza-Klein metric tensor (1.1) when $g_{\mu\nu}=\eta_{\mu\nu}$, but for the fact that the gauge symmetry has been broken to force $A_\mu=A_{\gamma\mu}$. But we also know that $G_{05}=G_{50}$ and $G_{j5}=G_{5j}$ have been constructed at (3.9) and (3.4) to form a four-vector in spacetime. Therefore, referring to these components in (3.11), we now *define* a new covariant (lower-indexed) four-vector:

$$\Phi_\mu\equiv\begin{pmatrix}\phi & \phi^2kA_{\gamma j}\end{pmatrix}. \tag{3.12}$$

Moreover, $G_{55}=\gamma_5\gamma_5+\phi^2\gamma_0\gamma_0$ in (3.10) teaches that the underlying timelike signature (and the metric non-singularity) is rooted in $\gamma_5\gamma_5=1$, and via $\phi^2\gamma_0\gamma_0=\phi^2$ that the square of the scalar field is rooted in $\gamma_0\gamma_0=1$ which has two time indexes. So, we may now formally assign $\eta_{55}=1$ to the fifth component of the Minkowski metric signature, and we may assign $\phi^2=\Phi_0\Phi_0$ to the fields in $G_{\mu\nu}$ and $G_{55}$. With all of this, and using minimal coupling to generalize $\eta_{\mathrm{MN}}\mapsto g_{\mathrm{MN}}$ which also means accounting for non-zero $g_{\mu 5}$, $g_{5\nu}$, (3.11) may now be compacted via (3.12) to the 2x2 form:

$$\boxed{G_{\mathrm{MN}}=\begin{pmatrix}G_{\mu\nu} & G_{\mu 5}\\ G_{5\nu} & G_{55}\end{pmatrix}=\begin{pmatrix}g_{\mu\nu}+\Phi_0\Phi_0k^2A_{\gamma\mu}A_{\gamma\nu} & g_{\mu 5}+\Phi_\mu\\ g_{5\nu}+\Phi_\nu & g_{55}+\Phi_0\Phi_0\end{pmatrix}}. \tag{3.13}$$

This is the Dirac-Kaluza-Klein metric tensor which will form the basis for all continued development from here, and it should be closely contrasted with (1.1). The next step is to calculate the inverse $G^{\mathrm{MN}}$ of (3.13) above.

## 4. Calculation of the Inverse Dirac-Kaluza-Klein Metric Tensor

As already mentioned, the modified Kaluza-Klein metric tensor (3.13) has a *non-singular* inverse $G^{\mathrm{MN}}$ specified in the usual way by $G^{\mathrm{MA}}G_{\mathrm{AN}}=\delta^{\mathrm{M}}{}_{\mathrm{N}}$. We already know this because when all electromagnetic fields are turned off and $g_{\mathrm{MN}}=\eta_{\mathrm{MN}}$, we have a determinant $\left|G_{\mathrm{MN}}\right|=-1$ which is one of the litmus tests that can be used to demonstrate non-singularity. But because this inverse is essential to being able to calculate connections, equations of motion, and the Einstein field equation and related energy tensors, the next important step – which is entirely mathematical – is to explicitly calculate the inverse of (3.13). We shall now do so.

Calculating the inverse of a 5x5 matrix is a very cumbersome task if one employs a brute force approach. But we can take great advantage of the fact that the tangent space Minkowski tensor $\mathrm{diag}\left(\eta_{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,+1\right)$ has two timelike and three spacelike dimensions when we set $A_{\gamma j}=0$ and $\phi=0$ to turn off the electrodynamic fields, by using the analytic blockwise inversion method detailed, e.g., in [14]. Specifically, we split the 5x5 matrix into 2x2 and 3x3 matrices along the "diagonal", and into 2x3 and 3x2 matrices off the "diagonal." It is best to work from (3.11) which does not show the time component $A_{\gamma 0}=0$ because this is equal to zero for a photon, and which employs $g_{\mu\nu}=\eta_{\mu\nu}$. We expand this to show the entire 5x5 matrix, and we move the rows and columns so the ordering of the indexes is not $\mathrm{M}=0,1,2,3,5$, but rather is $\mathrm{M}=0,5,1,2,3$. With all of this, (3.11) may be rewritten as:

$$G_{\mathrm{MN}}=\begin{pmatrix} G_{00} & G_{05} & G_{01} & G_{02} & G_{03} \\ G_{50} & G_{55} & G_{51} & G_{52} & G_{53} \\ G_{10} & G_{15} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{25} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{35} & G_{31} & G_{32} & G_{33} \end{pmatrix}=\begin{pmatrix} 1 & \phi & 0 & 0 & 0 \\ \phi & 1+\phi^2 & \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 1} & -1+\phi^2k^2A_{\gamma 1}A_{\gamma 1} & \phi^2k^2A_{\gamma 1}A_{\gamma 2} & \phi^2k^2A_{\gamma 1}A_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 2} & \phi^2k^2A_{\gamma 2}A_{\gamma 1} & -1+\phi^2k^2A_{\gamma 2}A_{\gamma 2} & \phi^2k^2A_{\gamma 2}A_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 3} & \phi^2k^2A_{\gamma 3}A_{\gamma 1} & \phi^2k^2A_{\gamma 3}A_{\gamma 2} & -1+\phi^2k^2A_{\gamma 3}A_{\gamma 3} \end{pmatrix}. \quad (4.1)$$

Then, we find the inverse using the blockwise inversion relation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \quad (4.2)$$

with the matrix block assignments:

$$\mathbf{A}=\begin{pmatrix}1 & \phi\\ \phi & 1+\phi^2\end{pmatrix};\quad \mathbf{B}=\begin{pmatrix}0 & 0 & 0\\ \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3}\end{pmatrix};$$
$$\mathbf{C}=\begin{pmatrix}0 & \phi^2 kA_{\gamma 1}\\ 0 & \phi^2 kA_{\gamma 2}\\ 0 & \phi^2 kA_{\gamma 3}\end{pmatrix};\quad \mathbf{D}=\begin{pmatrix}-1+\phi^2 k^2 A_{\gamma 1}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 3}\\ \phi^2 k^2 A_{\gamma 2}A_{\gamma 1} & -1+\phi^2 k^2 A_{\gamma 2}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 2}A_{\gamma 3}\\ \phi^2 k^2 A_{\gamma 3}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 3}A_{\gamma 2} & -1+\phi^2 k^2 A_{\gamma 3}A_{\gamma 3}\end{pmatrix}. \tag{4.3}$$

The two inverses we must calculate are $\mathbf{A}^{-1}$ and $\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}$. The former is a 2x2 matrix easily inverted, see, e.g. [15]. Its determinant $|\mathbf{A}|=1+\phi^2-\phi^2=1$, so its inverse is:

$$\mathbf{A}^{-1}=\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}. \tag{4.4}$$

Next, we need to calculate $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$, then invert this. We first calculate:

$$\begin{aligned}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}&=-\begin{pmatrix}0 & \phi^2 kA_{\gamma 1}\\ 0 & \phi^2 kA_{\gamma 2}\\ 0 & \phi^2 kA_{\gamma 3}\end{pmatrix}\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3}\end{pmatrix}\\ &=-\begin{pmatrix}0 & \phi^2 kA_{\gamma 1}\\ 0 & \phi^2 kA_{\gamma 2}\\ 0 & \phi^2 kA_{\gamma 3}\end{pmatrix}\begin{pmatrix}-\phi^3 kA_{\gamma 1} & -\phi^3 kA_{\gamma 2} & -\phi^3 kA_{\gamma 3}\\ \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3}\end{pmatrix}=-\begin{pmatrix}\phi^4 k^2 A_{\gamma 1}A_{\gamma 1} & \phi^4 k^2 A_{\gamma 1}A_{\gamma 2} & \phi^4 k^2 A_{\gamma 1}A_{\gamma 3}\\ \phi^4 k^2 A_{\gamma 2}A_{\gamma 1} & \phi^4 k^2 A_{\gamma 2}A_{\gamma 2} & \phi^4 k^2 A_{\gamma 2}A_{\gamma 3}\\ \phi^4 k^2 A_{\gamma 3}A_{\gamma 1} & \phi^4 k^2 A_{\gamma 3}A_{\gamma 2} & \phi^4 k^2 A_{\gamma 3}A_{\gamma 3}\end{pmatrix}\end{aligned}. \tag{4.5}$$

Therefore:

$$\begin{aligned}\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}&=\begin{pmatrix}-1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 3}\\ \left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 1} & -1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 3}\\ \left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 2} & -1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 3}\end{pmatrix}.\\ &=\eta_{jk}+\left(\phi^2-\phi^4\right)k^2 A_{\gamma j}A_{\gamma k}\end{aligned} \tag{4.6}$$

We can easily invert this using the skeletal mathematical relation $(1+x)(1-x)=1-x^2$. Specifically, using the result in (4.6) we may write:

$$\begin{aligned}&\left(\eta_{jk}+\left(\phi^2-\phi^4\right)k^2 A_{\gamma j}A_{\gamma k}\right)\left(\eta_{kl}-\left(\phi^2-\phi^4\right)k^2 A_{\gamma k}A_{\gamma l}\right)\\ &=\eta_{jk}\eta_{kl}+\left(\phi^2-\phi^4\right)k^2\left(\eta_{kl}A_{\gamma j}A_{\gamma k}-\eta_{jk}A_{\gamma k}A_{\gamma l}\right)-\left(\phi^2-\phi^4\right)^2 k^4 A_{\gamma j}A_{\gamma k}A_{\gamma k}A_{\gamma l}=\delta_{jl}\end{aligned}. \tag{4.7}$$

The $A_{\gamma j}A_{\gamma k}A_{\gamma k}A_{\gamma l}$ term zeros out because $A_{\gamma k}A_{\gamma k}=0$ for the photon field. Sampling the diagonal $j=l=1$ term, $\eta_{k1}A_{\gamma 1}A_{\gamma k}-\eta_{1k}A_{\gamma k}A_{\gamma 1}=-A_{\gamma 1}A_{\gamma 1}+A_{\gamma 1}A_{\gamma 1}=0$. Sampling the off-diagonal $j=1$, $l=2$ term, $\eta_{k1}A_{\gamma 2}A_{\gamma k}-\eta_{2k}A_{\gamma k}A_{\gamma 1}=-A_{\gamma 2}A_{\gamma 1}+A_{\gamma 2}A_{\gamma 1}=0$. By rotational symmetry, all other terms zero as well. And of course, $\eta_{jk}\eta_{kl}=\delta_{jl}$. So (4.7) taken with (4.6) informs us that:

$$\begin{aligned}&\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}=\eta_{jk}-\left(\phi^2-\phi^4\right)k^2A_{\gamma j}A_{\gamma k}\\&=\begin{pmatrix}-1-\left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 3}\\ -\left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 3}\\ -\left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 2} & -1-\left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 3}\end{pmatrix}.\end{aligned}\tag{4.8}$$

We now have all the inverses we need; the balance of the calculation is matrix multiplication.

From the lower-left block in (4.2) we use $\mathbf{C}$ in (4.3), with (4.4) and (4.8), to calculate:

$$\begin{aligned}&-\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}\mathbf{CA}^{-1}\\&=\begin{pmatrix}1+\left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2A_{\gamma 1}A_{\gamma 3}\\ \left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 1} & 1+\left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2A_{\gamma 2}A_{\gamma 3}\\ \left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 2} & 1+\left(\phi^2-\phi^4\right)k^2A_{\gamma 3}A_{\gamma 3}\end{pmatrix}\begin{pmatrix}0 & \phi^2kA_{\gamma 1}\\ 0 & \phi^2kA_{\gamma 2}\\ 0 & \phi^2kA_{\gamma 3}\end{pmatrix}\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix},\\&=\begin{pmatrix}-\phi^3kA_{\gamma 1}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma 1}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma 1}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma 1}A_{\gamma k}A_{\gamma k}\\ -\phi^3kA_{\gamma 2}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma 2}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma 2}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma 2}A_{\gamma k}A_{\gamma k}\\ -\phi^3kA_{\gamma 3}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma 3}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma 3}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma 3}A_{\gamma k}A_{\gamma k}\end{pmatrix}=\begin{pmatrix}-\phi^3kA_{\gamma 1} & \phi^2kA_{\gamma 1}\\ -\phi^3kA_{\gamma 2} & \phi^2kA_{\gamma 2}\\ -\phi^3kA_{\gamma 3} & \phi^2kA_{\gamma 3}\end{pmatrix}\end{aligned}\tag{4.9}$$

again using $A_{\gamma k}A_{\gamma k}=0$. We can likewise calculate $-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}$ in the upper-right block in (4.2), but it is easier and entirely equivalent to simply use the transposition symmetry $G_{\mathrm{MN}}=G_{\mathrm{NM}}$ of the metric tensor and the result in (4.9) to deduce:

$$-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1}=\begin{pmatrix}-\phi^3kA_{\gamma 1} & -\phi^3kA_{\gamma 2} & -\phi^3kA_{\gamma 3}\\ \phi^2kA_{\gamma 1} & \phi^2kA_{\gamma 2} & \phi^2kA_{\gamma 3}\end{pmatrix},\tag{4.10}$$

For the upper left block in (4.2) we use $\mathbf{B}$ in (4.3), with (4.4) and (4.9) to calculate:

$$\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1}$$
$$=\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}+\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3}\end{pmatrix}\begin{pmatrix}\phi^3 kA_{\gamma 1} & -\phi^2 kA_{\gamma 1}\\ \phi^3 kA_{\gamma 2} & -\phi^2 kA_{\gamma 2}\\ \phi^3 kA_{\gamma 3} & -\phi^2 kA_{\gamma 3}\end{pmatrix}, \tag{4.11}$$
$$=\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}+\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \phi^5 k^2 A_{\gamma k}A_{\gamma k} & -\phi^4 k^2 A_{\gamma k}A_{\gamma k}\end{pmatrix}=\begin{pmatrix}1+\phi^2 & -\phi\\ -\phi & 1\end{pmatrix}$$

again using $A_{\gamma k}A_{\gamma k}=0$. And (4.8) already contains the complete lower-right block in (4.2).

So, we now reassemble (4.8) through (4.11) into (4.2) to obtain the complete inverse:

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}^{-1}=\begin{pmatrix}1+\phi^2 & -\phi & -\phi^3 kA_{\gamma 1} & -\phi^3 kA_{\gamma 2} & -\phi^3 kA_{\gamma 3}\\ -\phi & 1 & \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3}\\ -\phi^3 kA_{\gamma 1} & \phi^2 kA_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 3}\\ -\phi^3 kA_{\gamma 2} & \phi^2 kA_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 3}\\ -\phi^3 kA_{\gamma 3} & \phi^2 kA_{\gamma 3} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 2} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 3}\end{pmatrix} \tag{4.12}$$

Then we reorder rows and columns back to the $\mathrm{M}=0,1,2,3,5$ sequence and connect this to the contravariant (inverse) metric tensor $G^{\mathrm{MN}}$ to write:

$$G^{\mathrm{MN}}=\begin{pmatrix}1+\phi^2 & -\phi^3 kA_{\gamma 1} & -\phi^3 kA_{\gamma 2} & -\phi^3 kA_{\gamma 3} & -\phi\\ -\phi^3 kA_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 3} & \phi^2 kA_{\gamma 1}\\ -\phi^3 kA_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 3} & \phi^2 kA_{\gamma 2}\\ -\phi^3 kA_{\gamma 3} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 2} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 3} & \phi^2 kA_{\gamma 3}\\ -\phi & \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3} & 1\end{pmatrix}. \tag{4.13}$$

In a vitally-important contrast to the usual Kaluza-Klein $G^{\mathrm{MN}}$ in (1.1), this is manifestly *not singular*. This reverts to $\mathrm{diag}\left(G^{\mathrm{MN}}\right)=\mathrm{diag}\left(\eta^{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,+1\right)$ when $A_{\gamma\mu}=0$ and $\phi=0$ which is exactly the same signature as $G_{\mathrm{MN}}$ in (3.11). Then we consolidate to the 3x3 form:

$$G^{\mathrm{MN}}=\begin{pmatrix}G^{00} & G^{0k} & G^{05}\\ G^{j0} & G^{jk} & G^{j5}\\ G^{50} & G^{5k} & G^{55}\end{pmatrix}=\begin{pmatrix}1+\phi^2 & -\phi^3 kA_{\gamma k} & -\phi\\ -\phi^3 kA_{\gamma j} & \eta^{jk}-\left(\phi^2-\phi^4\right)k^2 A_{\gamma j}A_{\gamma k} & \phi^2 kA_{\gamma j}\\ -\phi & \phi^2 kA_{\gamma k} & 1\end{pmatrix}. \tag{4.14}$$

Now, the photon gauge vectors $A_{\gamma j}$ in (4.14) still have lower indexes, and with good reason: We cannot simply raise these indexes of components *inside the metric tensor* at will as we might for any other tensor. Rather, we must use the metric tensor (4.14) itself to raise and lower indexes, by calculating $A_\gamma{}^{\mathrm{M}} = G^{\mathrm{MN}} A_{\gamma\mathrm{N}}$. Nonetheless, it would be desirable to rewrite the components of (4.14) with all upper indexes, which will simplify downstream calculations. Given that $A_{\gamma 0} = 0$ for the photon and taking $A_{\gamma 5} = 0$, and raising indexes for $A_\gamma{}^0$ and $A_\gamma{}^5$ while sampling $A_\gamma{}^1$ and once again employing $A_{\gamma k} A_{\gamma k} = 0$, we may calculate:

$$\begin{aligned}
A_\gamma{}^0 &= G^{0\mathrm{N}} A_{\gamma\mathrm{N}} = G^{01} A_{\gamma 1} + G^{02} A_{\gamma 2} + G^{03} A_{\gamma 3} = -\phi^3 k A_{\gamma k} A_{\gamma k} = 0 \\
A_\gamma{}^1 &= G^{1\mathrm{N}} A_{\gamma\mathrm{N}} = G^{11} A_{\gamma 1} + G^{12} A_{\gamma 2} + G^{13} A_{\gamma 3} = -A_{\gamma 1} - \left(\phi^2 + \phi^4\right) k^2 A_{\gamma 1} A_{\gamma k} A_{\gamma k} = -A_{\gamma 1} \ , \\
A_\gamma{}^5 &= G^{5\mathrm{N}} A_{\gamma\mathrm{N}} = G^{51} A_{\gamma 1} + G^{52} A_{\gamma 2} + G^{53} A_{\gamma 3} = -\phi^2 k A_{\gamma k} A_{\gamma k} = 0
\end{aligned} \tag{4.15}$$

The middle result applies by rotational symmetry to other space indexes, so that:

$$A_\gamma{}^\mu = G^{\mu\nu} A_{\gamma\nu} = \eta^{\mu\nu} A_{\gamma\nu} \mapsto A_\gamma{}^\mu = g^{\mu\nu} A_{\gamma\nu} \, , \tag{4.16}$$

which is the usual way of raising indexes in flat spacetime, generalized to $g^{\mu\nu}$ with minimal coupling. As a result, with $g^{\mu\nu} = \eta^{\mu\nu}$ we may raise the index in (3.12) to obtain:

$$\Phi^\mu = \begin{pmatrix} \phi & \phi^2 k A_\gamma{}^j \end{pmatrix} = \begin{pmatrix} \phi & -\phi^2 k A_{\gamma j} \end{pmatrix} . \tag{4.17}$$

We then use (4.17) to write (4.14) as:

$$G^{\mathrm{MN}} = \begin{pmatrix} G^{00} & G^{0k} & G^{05} \\ G^{j0} & G^{jk} & G^{j5} \\ G^{50} & G^{5k} & G^{55} \end{pmatrix} = \begin{pmatrix} 1+\phi^2 & -\phi^3 k A_{\gamma k} & -\Phi^0 \\ -\phi^3 k A_{\gamma j} & \eta^{jk} - \left(\phi^2 - \phi^4\right) k^2 A_{\gamma j} A_{\gamma k} & -\Phi^j \\ -\Phi^0 & -\Phi^k & 1 \end{pmatrix} . \tag{4.18}$$

Now we focus on the middle term, expanded to $\eta^{jk} - \phi^2 k^2 A_{\gamma j} A_{\gamma k} + \phi^4 k^2 A_{\gamma j} A_{\gamma k}$. Working from (4.17) we now calculate:

$$\Phi^0 \Phi^0 = \phi^2; \quad \Phi^0 \Phi^k = -\phi^3 k A_{\gamma k}; \quad \Phi^j \Phi^0 = -\phi^3 k A_{\gamma j}; \quad \Phi^j \Phi^k = \phi^4 k^2 A_{\gamma j} A_{\gamma k} . \tag{4.19}$$

So, we use (4.19) in (4.18), and raise the indexes using $A_{\gamma j} A_{\gamma k} = A_\gamma{}^j A_\gamma{}^k$ from (14.16), to write:

$$G^{\mathrm{MN}} = \begin{pmatrix} G^{00} & G^{0k} & G^{05} \\ G^{j0} & G^{jk} & G^{j5} \\ G^{50} & G^{5k} & G^{55} \end{pmatrix} = \begin{pmatrix} 1+\Phi^0 \Phi^0 & \Phi^0 \Phi^k & -\Phi^0 \\ \Phi^j \Phi^0 & \eta^{jk} - \phi^2 k^2 A_\gamma{}^j A_\gamma{}^k + \Phi^j \Phi^k & -\Phi^j \\ -\Phi^0 & -\Phi^k & 1 \end{pmatrix} . \tag{4.20}$$

Then, again taking advantage of the fact that $A_{\gamma 0}=0$, while using $1=\eta_{00}=\eta^{00}$ and $1=\eta_{55}=\eta^{55}$ we may consolidate this into the 2x2 format:

$$G^{\mathrm{MN}}=\begin{pmatrix} G^{\mu\nu} & G^{\mu 5} \\ G^{5\nu} & G^{55} \end{pmatrix}=\begin{pmatrix} \eta^{\mu\nu}-\Phi^0\Phi^0 k^2 A_\gamma{}^\mu A_\gamma{}^\nu+\Phi^\mu\Phi^\nu & -\Phi^\mu \\ -\Phi^\nu & \eta^{55} \end{pmatrix}. \tag{4.21}$$

This is the inverse of (3.13) with $g_{\mu\nu}=\eta_{\mu\nu}$, and it is a good exercise to check and confirm that in fact, $G^{\mathrm{MA}}G_{\mathrm{AN}}=\delta^{\mathrm{M}}{}_{\mathrm{N}}$ .

The final step is to apply minimal coupling to generalize $\eta^{\mathrm{MN}}\mapsto g^{\mathrm{MN}}$, with possible non-zero $g_{\mu 5}$, $g_{5\nu}$, $g^{\mu 5}$ and $g^{5\nu}$. With this last step, (4.21) now becomes:

$$\boxed{G^{\mathrm{MN}}=\begin{pmatrix} G^{\mu\nu} & G^{\mu 5} \\ G^{5\nu} & G^{55} \end{pmatrix}=\begin{pmatrix} g^{\mu\nu}-\Phi^0\Phi^0 k^2 A_\gamma{}^\mu A_\gamma{}^\nu+\Phi^\mu\Phi^\nu & g^{\mu 5}-\Phi^\mu \\ g^{5\nu}-\Phi^\nu & g^{55} \end{pmatrix}}. \tag{4.22}$$

The above along with (3.13) are the direct counterparts to the Kaluza-Klein metric tensors (1.1). This inverse, in contrast to that of (1.1), is manifestly non-singular.

Finally, we commented after (2.6) that it would have been possible to choose minus rather than plus signs in the tetrad / field assignments. We make a note that had we done so, this would have carried through to a sign flip in all the $\varepsilon_k{}^0$ and $\varepsilon_0{}^k$ tetrad components in (2.12), it would have changed (2.14) to $\Gamma_\mu=\begin{pmatrix}\gamma_0-\phi k A_{\gamma j}\gamma_j & \gamma_j-\phi k A_{\gamma j}\gamma_0\end{pmatrix}$, and it would have changed (3.8) to include $\Gamma_5=\gamma_5-\phi\gamma_0$. Finally, for the metric tensors (4.22), all would be exactly the same, except that we would have had $G_{\mu 5}=G_{5\mu}=g_{\mu 5}-\Phi_\mu$ and $G^{\mu 5}=G^{5\mu}=g^{\mu 5}+\Phi^\mu$, with the vectors in (3.12) and (4.17) instead given by $\Phi_\mu=\begin{pmatrix}\phi & -\phi^2 k A_{\gamma j}\end{pmatrix}$ and $\Phi^\mu=\begin{pmatrix}\phi & -\phi^2 k A_\gamma{}^j\end{pmatrix}$. We note this because in a related preprint by the author at [16], this latter sign choice was required at [14.5] in a similar circumstance to ensure limiting-case solutions identical to those of Dirac's equation, as reviewed following [19.13] therein. Whether a similar choice may be required here cannot be known for certain without calculating detailed correspondences with Dirac theory based on the $\Gamma_{\mathrm{M}}$ in (3.8). In the next section, we will lay out the Dirac theory based on the Kaluza-Klein metric tensors having now been made generally-covariant in five dimensions.

## 5. The Dirac Equation with Five-Dimensional General Covariance

Now that we have obtained a Dirac-Kaluza-Klein metric tensor $G_{\mathrm{MN}}$ in (3.13) and its non-singular inverse $G^{\mathrm{MN}}$ in (4.22) which are fully covariant across all five dimensions and which are connected to a set of Dirac operators $\Gamma_{\mathrm{M}}$ deduced in (3.8) through the anticommutators (3.1), there are several additional calculations we shall perform which lay the foundation for deeper development. The first calculation, which vastly simplifies downstream calculation and provides the basis for a Dirac-type quantum theory of the electron and the photon based on Kaluza-Klein, is to obtain the contravariant (upper indexed) operators $\Gamma^{\mathrm{M}} = G^{\mathrm{MN}}\Gamma_{\mathrm{N}}$ in two component form which consolidates the four spacetime operators $\Gamma^{\mu}$ into a single four-covariant expression, then to do the same for the original $\Gamma_{\mathrm{M}}$ in (3.8).

As just noted, we may raise the indexes in the $\Gamma_{\mathrm{M}}$ of (3.8) by calculating $\Gamma^{\mathrm{M}} = G^{\mathrm{MN}}\Gamma_{\mathrm{N}}$. It is easiest to work from (3.8) together with the 3x3 form (4.20), then afterward consolidate to 2x2 form. So, we first calculate each of $\Gamma^0$, $\Gamma^j$ and $\Gamma^5$ as such:

$$\begin{aligned}\Gamma^0 &= G^{0\mathrm{N}}\Gamma_{\mathrm{N}} = G^{00}\Gamma_0 + G^{0k}\Gamma_k + G^{05}\Gamma_5 \\ &= \left(1+\Phi^0\Phi^0\right)\left(\gamma_0 + \phi k A_{\gamma j}\gamma_j\right) + \Phi^0\Phi^k\left(\gamma_k + \phi k A_{\gamma k}\gamma_0\right) - \Phi^0\left(\gamma_5 + \phi\gamma_0\right), \\ &= \gamma^0 + \Phi^0 k A_\gamma{}^k \gamma^k + k A_\gamma{}^0 \Phi^0 \gamma^0 - \Phi^0 \gamma^5\end{aligned} \tag{5.1a}$$

$$\begin{aligned}\Gamma^j &= G^{j\mathrm{N}}\Gamma_{\mathrm{N}} = G^{j0}\Gamma_0 + G^{jk}\Gamma_k + G^{j5}\Gamma_5 \\ &= \Phi^j\Phi^0\left(\gamma_0 + \phi k A_{\gamma k}\gamma_k\right) + \left(\eta^{jk} - \phi^2 k^2 A_\gamma{}^j A_\gamma{}^k + \Phi^j\Phi^k\right)\left(\gamma_k + \phi k A_{\gamma k}\gamma_0\right) - \Phi^j\left(\gamma_5 + \phi\gamma_0\right), \\ &= \gamma^j + \Phi^j k A_\gamma{}^k \gamma^k + k A_\gamma{}^j \Phi^0 \gamma^0 - \Phi^j \gamma^5\end{aligned} \tag{5.1b}$$

$$\begin{aligned}\Gamma^5 &= G^{5\mathrm{N}}\Gamma_{\mathrm{N}} = G^{50}\Gamma_0 + G^{5k}\Gamma_k + G^{55}\Gamma_5 \\ &= -\Phi^0\left(\gamma_0 + \phi k A_{\gamma j}\gamma_j\right) - \Phi^k\left(\gamma_k + \phi k A_{\gamma k}\gamma_0\right) + \left(\gamma_5 + \phi\gamma_0\right) = \gamma^5\end{aligned} . \tag{5.1c}$$

To reduce the above, we have employed $\Phi^\mu = \begin{pmatrix}\phi & \phi^2 k A_\gamma{}^j\end{pmatrix}$ from (4.17) which implies that $\Phi^k A_{\gamma k} = 0$ via $A_k A_k = 0$ from (2.10). We have also used $A_\gamma{}^j = \eta^{jk} A_{\gamma k} = -A_{\gamma j}$ from (14.16), and the basic Dirac identities $\gamma^0 = \gamma_0$, $\gamma^k = \eta^{jk}\gamma_k = -\gamma_k$ and $\gamma^5 = \gamma_5$. We also include a term $k A_\gamma{}^0 \Phi^0 \gamma^0 = 0$ in (5.1a) to highlight the four-dimensional spacetime covariance with (5.1b), notwithstanding that this term is a zero because the gauge symmetry has been broken to that of a photon. Making use of this, we consolidate all of (5.1) above into the two-part:

$$\boxed{\Gamma^{\mathrm{M}} = \begin{pmatrix}\gamma^\mu + \Phi^\mu k A_\gamma{}^k \gamma^k + k A_\gamma{}^\mu \Phi^0 \gamma^0 - \Phi^\mu \gamma^5 & \gamma^5\end{pmatrix}}. \tag{5.2}$$

As a final step to consolidate the Dirac matrices, we use the 2x2 consolidation of the metric tensor $G_{\mathrm{MN}}$ in (3.13), with $g_{\mu\nu} = \eta_{\mu\nu}$, to lower the indexes in (5.2) and obtain a two-part $\Gamma_{\mathrm{M}} = G_{\mathrm{MN}}\Gamma^{\mathrm{N}}$. Doing so we calculate:

$$\begin{aligned}\Gamma_\mu &= G_{\mu\mathrm{N}}\Gamma^{\mathrm{N}} = G_{\mu\nu}\Gamma^\nu + G_{\mu 5}\Gamma^5 \\ &= \left(\eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}\right)\left(\gamma^\nu + \Phi^\nu k A_\gamma{}^k \gamma^k + k A_\gamma{}^\nu \Phi^0 \gamma^0 - \Phi^\nu \gamma^5\right) + \Phi_\mu \gamma^5, \\ &= \gamma_\mu + \Phi_\mu k A_{\gamma k}\gamma_k - \Phi_0\Phi_0 k^2 A_{\gamma\mu} A_{\gamma k}\gamma_k + k A_{\gamma\mu}\Phi_0\gamma_0\end{aligned} \tag{5.3a}$$

$$\begin{aligned}\Gamma_5 &= G_{5\mathrm{N}}\Gamma^{\mathrm{N}} = G_{5\nu}\Gamma^{\nu} + G_{55}\Gamma^5 \\ &= \Phi_\nu\left(\gamma^\nu + \Phi^\nu k A_\gamma{}^k \gamma^k + k A_\gamma{}^\nu \Phi^0\gamma^0 - \Phi^\nu\gamma^5\right) + \left(1+\Phi_0\Phi_0\right)\gamma^5 . \\ &= \gamma_5 + \Phi_0\gamma_0\end{aligned} \tag{5.3b}$$

Above, we use the same reductions employed in (5.1), as well as $A_{\gamma\nu}A_\gamma{}^\nu = 0$, $A_{\gamma\nu}\Phi^\nu = 0$ and $\Phi_\nu\Phi^\nu = \phi^2$. We then consolidate this into the two-part:

$$\boxed{\Gamma_{\mathrm{M}} = \begin{pmatrix}\gamma_\mu + \left(\Phi_\mu - \Phi_0\Phi_0 k A_{\gamma\mu}\right) k A_{\gamma k}\gamma_k + k A_{\gamma\mu}\Phi_0\gamma_0 & \gamma_5 + \Phi_0\gamma_0\end{pmatrix}}. \tag{5.4}$$

Making use of $\Phi_\mu \equiv \begin{pmatrix}\phi & \phi^2 k A_{\gamma j}\end{pmatrix}$ in (3.12), again mindful that $A_{\gamma\mu} = 0$, and noting that $\Phi_\mu - \Phi_0\Phi_0 k A_{\gamma\mu} = \Phi_0 = \phi$ for the $\mu = 0$ time component and $\Phi_\mu - \Phi_0\Phi_0 k A_{\gamma\mu} = \Phi_k - \phi^2 k A_{\gamma k} = 0$ for the $\mu = k$ space components, it is a good exercise to confirm that (5.4) does reduce precisely to $\Gamma_{\mathrm{M}} = \begin{pmatrix}\gamma_0 + \phi k A_{\gamma k}\gamma_k & \gamma_j + \phi k A_{\gamma j}\gamma_0 & \gamma_5 + \phi\gamma_0\end{pmatrix}$ obtained in (3.8). Using (5.2) and (5.4) and reducing with $\Phi^\mu = \begin{pmatrix}\phi & \phi^2 k A_\gamma{}^j\end{pmatrix}$, $\gamma^k\gamma_0 = -\gamma_0\gamma^k$, $A_{\gamma j}\gamma_j A_\gamma{}^k\gamma^k = 0$, $\Phi_\mu A_\gamma{}^\mu = 0$ and $A_{\gamma\mu}A_\gamma{}^\mu = 0$, it is also a good exercise to confirm that:

$$\Gamma_{\mathrm{M}}\Gamma^{\mathrm{M}} = \gamma_{\mathrm{M}}\gamma^{\mathrm{M}} = 5 . \tag{5.5}$$

And, it is a good exercise to confirm that (5.4) and (5.2) used in (1.3), see also (3.1), respectively reproduce the covariant and contravariant metric tensors (3.13) and (4.22).

Finally, having the upper-indexed (5.2) enables us to extend the Dirac equation governing fermion behavior into all five of the Kaluza-Klein dimensions, in the form of:

$$\boxed{\left(i\hbar c\Gamma^{\mathrm{M}}\partial_{\mathrm{M}} - mc^2\right)\Psi = 0}. \tag{5.6}$$

If we then define a five-dimensional energy-momentum vector $cp^{\mathrm{M}} = \begin{pmatrix}cp^\mu & cp^5\end{pmatrix}$ containing the usual four-dimensional $cp^\mu = \begin{pmatrix}E & c\mathbf{p}\end{pmatrix}$, and given that (3.13) and (4.22) provide the means to lower and raise indexes at will, we may further define the wavefunction $\Psi \equiv U_0\left(p^\Sigma\right)\exp\left(-ip_\Sigma x^\Sigma/\hbar\right)$ to include a Fourier kernel $\exp\left(-ip_\Sigma x^\Sigma/\hbar\right)$ over all five dimensions $x^\Sigma = \begin{pmatrix}ct^0 & \mathbf{x} & ct^5\end{pmatrix}$. These coordinates now include a *timelike* $x^5 = ct^5$ which is heretofore distinguished from the ordinary time dimension $x^0 = ct^0$ because as earlier reviewed, (3.13) has the tangent-space signature $\mathrm{diag}\left(G_{\mathrm{MN}}\right) = \left(+1,-1,-1,-1,+1\right)$. And $U_0\left(p^\Sigma\right)$ is a Dirac spinor which is now a function of all five components of $p^\Sigma$ but independent of the coordinates $x^\Sigma$. In other words, $\partial_{\mathrm{M}} U_0\left(p^\Sigma\right)=0$, which is why we include the 0 subscript. With all of this, we can convert (5.6) from configuration space to momentum space in the usual way, to obtain:

$$\boxed{\left(\Gamma^{\mathrm{M}} cP_{\mathrm{M}} - mc^2\right)U_0\left(p^\Sigma\right)=0}. \tag{5.7}$$

It is important to note that it is *not possible* to obtain the Dirac-type equations (5.6) and (5.7) from the usual Kaluza-Klein metric tensor and inverse (1.1), precisely because this metric tensor is *not generally-covariant* across all five dimensions. And in fact, as we first deduced at (2.10), the Kaluza-Klein (1.1) are not even truly-covariant in the four spacetime dimensions alone unless we set the gauge field $A_\mu \mapsto A_{\gamma\mu}$ to that of a photon with only two transverse degrees of freedom. Of course, we do not at this juncture know precisely how to understand the fifth component $cp^5$ of the energy momentum or the second time dimension $x^5 = ct^5$. But it is the detailed development and study of the Dirac-Kaluza-Klein (DKK) equations (5.6) and (5.7) which may provide one set of avenues for understanding precisely how the energy $cp^5$ and the time $t^5$ are manifest in the natural world.

## 6. The Dirac-Kaluza-Klein Metric Tensor Determinant and Inverse Determinant

It is also helpful to calculate the metric tensor determinants. These are needed in a variety of settings, for example, to calculate the five-dimensional Einstein-Hilbert action, see e.g. [17], which expressly contains the determinant as part of the volume element $\sqrt{-g}\,\mathrm{d}^4x$ in four dimensions and which we anticipate will appear as $\sqrt{-G}\,\mathrm{d}^5x$ in five dimensions. As we shall later elaborate in section 10, the Einstein-Hilbert action provides what is perhaps the most direct path for understanding the fifth dimension as a "matter" dimension along the lines long-advocated by the 5D Space-Time-Matter Consortium [18]. Moreover, the Einstein-Hilbert action, from which the Einstein equation is also derived as reviewed in [17], is also essential for calculating quantum mechanical path integrals which would effectively provide a quantum field theory of gravitation in five-dimensions. For all these reasons, it is helpful to have obtained this determinant.

To calculate the determinant, we employ the block calculation method reviewed, e.g., at [19]. Specifically, for an invertible matrix which we have shown $G_{\mathrm{MN}}$ to be via $G^{\mathrm{MN}}$ in (4.22), the determinant is calculated with:

$$\left|G_{\mathrm{MN}}\right| = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \left|\mathbf{A}\right|\left|\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right|, \tag{6.1}$$

using the exact same blocks specified in (4.3) to calculate (4.2). Keep in mind that the blocks in (4.3) are based on having used what we now understand to be the tangent Minkowski-space $g_{\mathrm{MN}} = \eta_{\mathrm{MN}}$. As we found following (4.3), $\left|\mathbf{A}\right| = 1+\phi^2-\phi^2 = 1$, so (6.1) simplifies to

$\left|G_{\mathrm{MN}}\right| = \left|\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right|$. Moreover, we already found $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ in (4.6). So, all that we need do is calculate the determinant of this 3x3 matrix, and we will have obtained $\left|G_{\mathrm{MN}}\right|$.

From (4.6) which we denote as the matrix $m_{ij} \equiv \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$, we write out the full determinant, substitute (4.6), then reduce to obtain:

$$\begin{aligned}\left|m_{ij}\right| &= m_{11}m_{22}m_{33} + m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} - m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32} \\ &= -1 + \left(\phi^2 - \phi^4\right)k^2\left(A_{\gamma 1}A_{\gamma 1} + A_{\gamma 2}A_{\gamma 2} + A_{\gamma 3}A_{\gamma 3}\right) = -1\end{aligned}. \tag{6.2}$$

Most of the terms cancel identically because of the equal number of + and – signs in the top line of (6.2). The only remaining term besides –1 itself, contains $A_{\gamma j}A_{\gamma j} = 0$, which is zero because of (2.10) which removed two degrees of freedom from the gauge field and turned it into $A_\mu = A_{\gamma\mu}$ for a massless, luminous photon. So, we conclude, neatly, that $\left|\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right| = -1$, and because $\left|\mathbf{A}\right| = 1$, that $\left|G_{\mathrm{MN}}\right| = -1 = \left|\eta_{\mathrm{MN}}\right|$. Moreover, because $\left|M^{-1}\right| = \left|M\right|^{-1}$ for any square matrix, we likewise conclude that $\left|G^{\mathrm{MN}}\right| = -1 = \left|\eta^{\mathrm{MN}}\right|$. Then, because the blocks in (4.3) are based on having used $g_{\mathrm{MN}} = \eta_{\mathrm{MN}}$, we may employ minimal coupling to generalize from $\eta_{\mathrm{MN}} \mapsto g_{\mathrm{MN}}$, so that the complete five-dimensional determinant and its inverse are:

$$\boxed{G \equiv \left|G_{\mathrm{MN}}\right| = \left|g_{\mathrm{MN}}\right| \equiv g; \quad G^{-1} \equiv \left|G^{\mathrm{MN}}\right| = \left|g^{\mathrm{MN}}\right| \equiv g^{-1}}. \tag{6.3}$$

In the above, the massless, luminous $A_\mu = A_{\gamma\mu}$ and the scalar field $\phi$ wash entirely out of the determinant, leaving the determinants entirely dependent upon $g_{\mathrm{MN}}$ which accounts for all curvatures *other than* those produced by $A_{\gamma\mu}$ and $\phi$.

For the determinant of the four-dimensional spacetime components $G_{\mu\nu}$ alone, we employ the exact same calculation used in (6.1), but now we split $G_{\mu\nu}$ into a 1x1 time "block" with $\mathbf{A} = \left|\mathbf{A}\right| = 1$, a 3x3 space block with the same $\mathbf{D} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu}$, and the 1x3 and 3x1 blocks $\mathbf{B} = 0$ and $\mathbf{C} = 0$. So (6.1) becomes $\left|G_{\mu\nu}\right| = \left|\mathbf{A}\right|\left|\mathbf{D}\right| = \left|\mathbf{D}\right|$. We next note that $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ in (4.6) differs from $\mathbf{D}$ in (4.3) merely by the term $-\phi^4 k^2 A_{\gamma\mu}A_{\gamma\nu}$, which tells us that the calculation of $\left|\mathbf{D}\right|$ will produce the exact same result as (6.2) leading to $\left|G_{\mu\nu}\right| = -1 = \left|\eta_{\mu\nu}\right|$, with the inverse following suit. Consequently, after generalizing $\eta_{\mu\nu} \mapsto g_{\mu\nu}$ via minimal coupling, we find that for the four dimensions of spacetime alone:

$$\left|G_{\mu\nu}\right| = \left|g_{\mu\nu}\right|; \quad \left|G^{\mu\nu}\right| = \left|g^{\mu\nu}\right|. \tag{6.4}$$

Here too, the massless, luminous $A_\mu = A_{\gamma\mu}$ with two degrees of freedom and the scalar $\phi$ are washed out entirely. Note, comparing (6.3) and (6.4), that we have reserved the notational definitions $G \equiv |G_{\mathrm{MN}}|$ and $g = |g_{\mathrm{MN}}|$ for the *five-dimensional* determinants. In four dimensions, we simply use the spacetime indexes to designate that (6.4) represents the four-dimensional spacetime subset of the five-dimensional metric tensor determinant and inverse.

## 7. The Dirac-Kaluza-Klein Lorentz Force Motion

Kaluza-Klein theory which will celebrate its centennial next year, has commanded attention for the past century for the very simple reason that despite all of its difficulties (most of which as will be reviewed in section 9 arise directly or indirectly from the degeneracy of the metric tensor (1.1) and its lack of five-dimensional covariance at the Dirac level) because it successfully explains Maxwell's equations, the Lorentz Force motion and the Maxwell stress-energy tensor on an entirely geometrodynamic foundation. This successful geometrodynamic representation of Maxwell's electrodynamics – popularly known as the "Kaluza miracle" – arises particularly from the components $G_{\mu 5} = G_{5\mu} = \phi^2 k A_\mu$ of the metric tensor (1.1), because the electromagnetic field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is among the objects which appear in the five-dimensional Christoffel connections $\tilde{\Gamma}^{\mathrm{M}}_{\mathrm{AB}}$ (particularly in $\tilde{\Gamma}^{\mu}_{\alpha 5}$ as we shall now detail), and because these $F^{\mu\nu}$ then make their way into the geodesic equation of motion in a form that can be readily connected to the Lorentz Force motion, and because they also enter the Einstein field equation in a form that can be likewise connected to the Maxwell stress-energy tensor. Therefore, it is important to be assured that in the process of remediating the various difficulties of Kaluza-Klein's metric tensor (1.1), the 5-covariant metric tensor (3.13) does not sacrifice any of the Kaluza miracle in the process.

In (3.13), $G_{\mu 5} = G_{5\mu} = \phi^2 k A_\mu$ from (1.1) which are responsible for the Kaluza miracle are replaced by $G_{\mu 5} = G_{5\mu} = g_{\mu 5} + \Phi_\mu$. For a flat Minkowski tangent space $g_{\mathrm{MN}} = \eta_{\mathrm{MN}}$ these reduce to $G_{\mu 5} = G_{5\mu} = \Phi_\mu$. At (3.4) we *required* $G_{j5} = G_{5j} = \phi^2 k A_{\gamma j}$ to precisely match $G_{\mathrm{MN}}$ from the Kaluza-Klein metric (1.1), maintaining the same spacetime covariance as $G_{\mu 5} = G_{5\mu}$ in (1.1) because $\phi^2 k A_{\gamma 0} = 0$, to keep the "miracle" intact. So, for a five-dimensional metric defined by:

$$c^2 d\mathrm{T}^2 \equiv G_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} \tag{7.1}$$

the equation of motion obtained by minimizing the geodesic variation is:

$$\frac{d^2 x^{\mathrm{M}}}{c^2 d\mathrm{T}^2} = -\tilde{\Gamma}^{\mathrm{M}}_{\mathrm{AB}} \frac{dx^{\mathrm{A}}}{cd\mathrm{T}} \frac{dx^{\mathrm{B}}}{cd\mathrm{T}} = -\tilde{\Gamma}^{\mathrm{M}}_{\alpha\beta} \frac{dx^{\alpha}}{cd\mathrm{T}} \frac{dx^{\beta}}{cd\mathrm{T}} - 2\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5} \frac{dx^{\alpha}}{cd\mathrm{T}} \frac{dx^{5}}{cd\mathrm{T}} - \tilde{\Gamma}^{\mathrm{M}}_{55} \frac{dx^{5}}{cd\mathrm{T}} \frac{dx^{5}}{cd\mathrm{T}} \tag{7.2}$$

just as in Kaluza-Klein theory, with connections of the "first" and "second" kinds specified by:

$$\begin{aligned} &\tilde{\Gamma}_{\Sigma\,\mathrm{AB}} = \tfrac{1}{2}\left(\partial_{\mathrm{B}} G_{\Sigma \mathrm{A}} + \partial_{\mathrm{A}} G_{\mathrm{B}\Sigma} - \partial_{\Sigma} G_{\mathrm{AB}}\right); \\ &\tilde{\Gamma}^{\mathrm{M}}_{\mathrm{AB}} = \tfrac{1}{2} G^{\mathrm{M}\Sigma}\left(\partial_{\mathrm{B}} G_{\Sigma \mathrm{A}} + \partial_{\mathrm{A}} G_{\mathrm{B}\Sigma} - \partial_{\Sigma} G_{\mathrm{AB}}\right) = G^{\mathrm{M}\Sigma} \tilde{\Gamma}_{\Sigma\,\mathrm{AB}}, \end{aligned} \tag{7.3}$$

likewise, just as in Kaluza-Klein theory. One may multiply (7.2) through by $d\mathrm{T}^2 / d\tau^2$ to obtain:

$$\frac{d^2x^{\mathrm{M}}}{c^2d\tau^2} = -\tilde{\Gamma}^{\mathrm{M}}_{\mathrm{AB}}\frac{dx^{\mathrm{A}}}{cd\tau}\frac{dx^{\mathrm{B}}}{cd\tau} = -\tilde{\Gamma}^{\mathrm{M}}_{\alpha\beta}\frac{dx^{\alpha}}{cd\tau}\frac{dx^{\beta}}{cd\tau} - 2\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}\frac{dx^{\alpha}}{cd\tau}\frac{dx^{5}}{cd\tau} - \tilde{\Gamma}^{\mathrm{M}}_{55}\frac{dx^{5}}{cd\tau}\frac{dx^{5}}{cd\tau} \tag{7.4}$$

which is the equation of motion with regard to the ordinary invariant spacetime metric line element $d\tau$, in which this four-dimensional proper time is defined by:

$$c^2d\tau^2 \equiv G_{\mu\nu}dx^{\mu}dx^{\nu} = g_{\mu\nu}dx^{\mu}dx^{\nu} + \phi^2k^2A_{\gamma\mu}A_{\gamma\nu}dx^{\mu}dx^{\nu}. \tag{7.5}$$

The space acceleration with regard to proper time $\tau$ is then given by $d^2x^j / d\tau^2$ with $\mathrm{M} = j = 1,2,3$ in (7.4). And if we then multiply this through by $d\tau^2 / dt^{0\ 2}$ (mindful again that we now need to distinguish $dt^0$ from the second time dimension $dt^5$), we obtain the space acceleration $d^2x^j / dt^{0\ 2}$ with regard to the ordinary time coordinate.

The above (7.1) through (7.5) are exactly the same as their counterparts in Kaluza-Klein theory, and they are exactly the same as what is used in the General Theory of Relativity in four spacetime dimensions alone, aside from minor notational changes intended to distinguish four- from five-dimensional objects. The only difference is that Kaluza-Klein theory uses the metric tensor (1.1) which has a spacelike fifth dimension, while the present DKK theory uses the metric tensor (3.13) which as a timelike fifth dimension. But the main reasons we are reviewing the equation of five-dimensional motion (7.4) is to be assured that the Kaluza miracle is not compromised by using the different metric tensor (3.13) rather than the usual (1.1).

As noted above, the connections $\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}$ are the particular ones responsible for the Kaluza-Klein representation of electrodynamics, whereby $\tilde{\Gamma}^{\mu}_{\alpha 5}$ governs accelerations in the four spacetime dimensions and $\tilde{\Gamma}^{5}_{\alpha 5}$ governs the fifth-dimensional acceleration. So, let's examine $\tilde{\Gamma}^{\mu}_{\alpha 5}$ more closely. Using (3.13) and (4.22) in (7.3) along with the symmetric $G_{\mathrm{MN}} = G_{\mathrm{NM}}$ we obtain:

$$\begin{aligned}
&\tilde{\Gamma}^{\mu}_{\alpha 5} = \tfrac{1}{2}G^{\mu\Sigma}\left(\partial_5 G_{\Sigma\alpha} + \partial_\alpha G_{5\Sigma} - \partial_\Sigma G_{\alpha 5}\right) \\
&= \tfrac{1}{2}G^{\mu\sigma}\left(\partial_5 G_{\sigma\alpha} + \partial_\alpha G_{5\sigma} - \partial_\sigma G_{\alpha 5}\right) + \tfrac{1}{2}G^{\mu 5}\partial_\alpha G_{55} \\
&= \tfrac{1}{2}\left(g^{\mu\sigma} - \Phi^0\Phi^0k^2A_\gamma{}^{\mu}A_\gamma{}^{\sigma} + \Phi^\mu\Phi^\sigma\right)\left(\partial_5\left(g_{\sigma\alpha} + \Phi_0\Phi_0k^2A_{\gamma\sigma}A_{\gamma\alpha}\right) + \partial_\alpha\left(g_{5\sigma} + \Phi_\sigma\right) - \partial_\sigma\left(g_{\alpha 5} + \Phi_\alpha\right)\right) \\
&+ \tfrac{1}{2}\left(g^{\mu 5} - \Phi^\mu\right)\partial_\alpha\left(g_{55} + \Phi_0\Phi_0\right)
\end{aligned}. \tag{7.6}$$

For a flat tangent space $G_{\mathrm{MN}} = \eta_{\mathrm{MN}}$ with $\mathrm{diag}\left(\eta_{\mathrm{MN}}\right) = \left(+1,-1,-1,-1,+1\right)$ thus $\partial_\alpha G_{\mathrm{MN}} = 0$ this simplifies to:

$$\tilde{\Gamma}^{\mu}_{\alpha 5} = \tfrac{1}{2}\left(\eta^{\mu\sigma} - \Phi^0\Phi^0k^2A_\gamma{}^{\mu}A_\gamma{}^{\sigma} + \Phi^\mu\Phi^\sigma\right)\left(\partial_5\left(\Phi_0\Phi_0k^2A_{\gamma\sigma}A_{\gamma\alpha}\right) + \partial_\alpha\Phi_\sigma - \partial_\sigma\Phi_\alpha\right) - \tfrac{1}{2}\Phi^\mu\partial_\alpha\left(\Phi_0\Phi_0\right). \tag{7.7}$$

What is of special interest in (7.7) is the antisymmetric tensor term $\partial_\alpha \Phi_\sigma - \partial_\sigma \Phi_\alpha$, because this is responsible for an electromagnetic field strength $F_{\gamma\mu\nu} = \partial_\mu A_{\gamma\nu} - \partial_\nu A_{\gamma\mu}$. To see this, we rewrite (3.12) as:

$$\Phi_\mu = \begin{pmatrix} \phi + \phi^2 k A_{\gamma 0} & \phi^2 k A_{\gamma j} \end{pmatrix}, \tag{7.8}$$

again taking advantage of $A_{\gamma 0} = 0$ to display the spacetime covariance of $A_{\gamma\mu}$. We then calculate the antisymmetric tensor in (7.7) in two separate bivector parts, as follows:

$$\begin{aligned} \partial_0 \Phi_k - \partial_k \Phi_0 &= \partial_0 \left( \phi^2 k A_{\gamma k} \right) - \partial_k \left( \phi + \phi^2 k A_{\gamma 0} \right) \\ &= \phi^2 k \left( \partial_0 A_{\gamma k} - \partial_k A_{\gamma 0} \right) + 2\phi k \left( A_{\gamma k} \partial_0 - A_{\gamma 0} \partial_k \right) \phi - \partial_k \phi , \\ &= \phi^2 k F_{\gamma 0 k} - 2\phi k \left( A_{\gamma 0} \partial_k - A_{\gamma k} \partial_0 \right) \phi - \partial_k \phi \end{aligned} \tag{7.9a}$$

$$\begin{aligned} \partial_j \Phi_k - \partial_k \Phi_j &= \partial_j \left( \phi^2 k A_{\gamma k} \right) - \partial_k \left( \phi^2 k A_{\gamma j} \right) \\ &= \phi^2 k \left( \partial_j A_{\gamma k} - \partial_k A_{\gamma j} \right) + 2\phi k \left( A_{\gamma k} \partial_j - A_{\gamma j} \partial_k \right) \phi . \\ &= \phi^2 k F_{\gamma jk} - 2\phi k \left( A_{\gamma j} \partial_k - A_{\gamma k} \partial_j \right) \phi \end{aligned} \tag{7.9b}$$

We see the emergence of the field strength tensor $F_{\gamma\mu\nu} = \partial_\mu A_{\gamma\nu} - \partial_\nu A_{\gamma\mu}$ in its usual Kaluza-Klein form $\phi^2 k F_{\gamma\mu\nu}$, modified to indicate that this arises from taking $F_\gamma^{\ \mu\nu}$ for a photon $A_\gamma^{\ \nu}$, which is a point to which we shall return momentarily. The only term which bars immediately merging both of (7.9) in a generally-covariant manner is the gradient $-\partial_k \phi$ in the 0*k* components of (7.9a). For this, noting that with reversed indexes $\partial_j \Phi_0 - \partial_0 \Phi_j$ (7.9a) will produce a gradient $+\partial_j \phi$ in the *j*0 components, we define a four-component $I_\mu \equiv \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix}$ and use this to form:

$$\begin{pmatrix} 0 & -\partial_k \phi \\ \partial_j \phi & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_j \phi \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & \partial_k \phi \end{pmatrix} = \partial_\mu \phi I_\nu - I_\mu \partial_\nu \phi = -\left( I_\mu \partial_\nu - I_\nu \partial_\mu \right) \phi . \tag{7.10}$$

We then use this to covariantly combine both of (7.9) into:

$$\begin{aligned} \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu &= \phi^2 k F_{\gamma\mu\nu} - 2\phi k \left( A_{\gamma\mu} \partial_\nu - A_{\gamma\nu} \partial_\mu \right) \phi - \left( I_\mu \partial_\nu - I_\nu \partial_\mu \right) \phi \\ &= \phi^2 k F_{\gamma\mu\nu} - \left( \left( I_\mu + 2\phi k A_{\gamma\mu} \right) \partial_\nu - \left( I_\nu + 2\phi k A_{\gamma\nu} \right) \partial_\mu \right) \phi \end{aligned} \tag{7.11}$$

The newly-appearing vector $I_\mu + 2\phi k A_{\gamma\mu} = \begin{pmatrix} 1 & 2\phi k A_{\gamma j} \end{pmatrix}$ which we represent by now removing $A_{\gamma 0} = 0$, is itself of interest, because the breaking of the gauge symmetry in section 2 caused $A_{\gamma 0} = 0$ to come out of the photon gauge vector which only has two transverse degrees of freedom.

But in this new vector $\begin{pmatrix}1 & 2\phi kA_{\gamma j}\end{pmatrix}$, the removed $A_{\gamma 0}=0$ is naturally replaced by the number 1, which is then included along with the remaining photon components $A_{\gamma j}$ multiplied by $2\phi k$. Again, the very small constant *k* which Kaluza-Klein theory fixes to (1.2) has dimensions of charge/energy, $\phi$ is taken to be dimensionless, and so $2\phi kA_{\gamma j}$ is dimensionless as well. Compare also $\Phi_\mu = \begin{pmatrix}\phi & \phi^2 kA_{\gamma j}\end{pmatrix}$, then observe that $\Phi_\mu + \phi^2 kA_{\gamma\mu} = \phi\left(I_\mu + 2\phi kA_{\gamma\mu}\right)$.

Most importantly, we now see in (7.11) that the field strength $F_{\gamma\mu\nu}$ which is needed for the Lorentz Force motion and the Maxwell tensor, does indeed emerge inside of $\tilde{\Gamma}^{\mu}_{\alpha 5}$ as seen in (7.7) just as it does from the usual Kaluza-Klein metric tensor (1.1), with the identical coefficients. But there is one wrinkle: $F_\gamma^{\ \mu\nu}$ is the field strength of a *single photon*, not a general classical $F^{\mu\nu}$ sourced by a material current density $J^\nu = \begin{pmatrix}\rho & \mathbf{J}\end{pmatrix}$ with a gauge potential $A^\mu = \begin{pmatrix}\phi & \mathbf{A}\end{pmatrix}$ which can always be Lorentz-transformed into a rest frame with $A^\mu = \begin{pmatrix}\phi_0 & \mathbf{0}\end{pmatrix}$ with $\phi_0$ being the proper potential (note: this is a different $\phi$ from the Kaluza-Klein $\phi$). In contrast, the photon $A_\gamma^{\ \mu}$ in (2.11) can never be placed at rest because the photon is a luminous, massless field quantum.

However, this can be surmounted using gauge symmetry, while making note of Heaviside's intuitions half a century before gauge theory which led him to formulate Maxwell's original theory without what would later be understood as a gauge potential. Specifically, even though the gauge symmetry is broken for $A_\gamma^{\ \mu}$ and it is therefore impossible to *Lorentz* transform the luminous $A_\gamma^{\ \mu}$ into a classical potential $A^\mu = \begin{pmatrix}\phi & \mathbf{A}\end{pmatrix}$ which can be placed at rest, or even to gauge transform $A_\gamma^{\ \mu} \to A^\mu$ from a luminous to a material potential because its gauge has already been fixed, the same impossibility *does not apply to gauge transformations of* $F_\gamma^{\ \mu\nu} = \partial^\mu A_\gamma^{\ \nu} - \partial^\nu A_\gamma^{\ \mu}$ obtained from this $A_\gamma^{\ \mu}$. This is because $F_{\gamma\mu\nu} = \partial_\mu A_{\gamma\nu} - \partial_\nu A_{\gamma\mu}$ is an antisymmetric tensor which, as is well-known, is *invariant under gauge transformations* $qA_\mu \to qA'_\mu \equiv qA_\mu + \hbar c\partial_\mu \Lambda$, where *q* is an electric charge and $\Lambda(t,\mathbf{x})$ is an unobservable scalar gauge parameter. To review, if we gauge transform some $qF_{\mu\nu} = q\partial_{;[\mu}A_{\nu]} \to qF'_{\mu\nu} = q\partial_{;[\mu}A_{\nu]} + \hbar c\left[\partial_{;\mu},\partial_{;\nu}\right]\Lambda = qF_{\mu\nu}$, the gauge transformation washes out because the commutator $\left[\partial_{;\mu},\partial_{;\nu}\right]\Lambda = 0$ even in curved spacetime. This is because the covariant derivative of a scalar is the same as its ordinary derivative, so that the covariant derivative $\partial_{;\mu}\partial_{;\nu}\Lambda = \partial_{;\mu}\partial_\nu\Lambda = \partial_\mu\partial_\nu\Lambda - \Gamma^\sigma_{\mu\nu}\partial_\sigma\Lambda$, with a similar expression under $\mu \leftrightarrow \nu$ interchange, and because $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ is symmetric under such interchange.

So even though we cannot Lorentz transform $A_\gamma^{\ \mu}$ into $A^\mu$, and even though the gauge of $A_\gamma^{\ \mu}$ is fixed so we cannot even gauge transform $A_\gamma^{\ \mu}$ into $A^\mu$, we may perform a gauge transformation $F_{\gamma\mu\nu} \to F_{\mu\nu}$ precisely because the field strength (which was central to Heaviside's formulation of Maxwell in terms of its bivectors **E** and **B**) is invariant with respect to the gauge that was fixed to the photon in (2.11) as a result of (2.10). Another way of saying this is that $F_{\gamma\mu\nu}=\partial_\mu A_{\gamma\nu}-\partial_\nu A_{\gamma\mu}$ for a photon has the exact same form as $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ for a materially-sourced potential which can be placed at rest, and that $F_{\gamma\mu\nu}$ enters into Maxwell's equations in exactly the same form as $F_{\mu\nu}$. The difference is that $F_{\gamma\mu\nu}$ emerges in source-free electrodynamics where the source current $J^\nu=0$ while $F_{\mu\nu}$ emerges when there is a non-zero $J^\nu\neq 0$.

So irrespective of this $A^\mu=A_\gamma{}^\mu$ symmetry breaking which arose from (2.10) to ensure Dirac-level covariance of the Kaluza-Klein metric tensor, the luminous photon fields $F_{\gamma\mu\nu}$ emerging in (7.7) via (7.11) can always be gauge-transformed using $F_\gamma{}^{\mu\nu}\to F^{\mu\nu}$ into the classical field strength of a classical materially-sourced potential $A^\mu=\begin{pmatrix}\phi & \mathbf{A}\end{pmatrix}$. Moreover, once we gauge transform $F_\gamma{}^{\mu\nu}\to F^{\mu\nu}$, the classical field strength $F^{\mu\nu}$ will contain innumerably-large numbers of photons mediating electromagnetic interactions, and so will entirely swamp out the individual $A_\gamma{}^\mu$ which represent individual photons. This transformation of $F_\gamma{}^{\mu\nu}\to F^{\mu\nu}$ by taking advantage of gauge symmetry, following by drowning out the impacts of individual photons as against classical fields, is exactly what the author did in Sections 21 and 23 of [16] to obtain the empirically-observed lepton magnetic moments at [23.5] and [23.6] of that same paper.

So, we now substitute (7.11) with a gauge-transformed $F_{\gamma\mu\nu}\to F_{\mu\nu}$ into (7.7), to find that:

$$\begin{aligned}\tilde{\Gamma}^\mu_{\alpha 5}=&\tfrac{1}{2}\left(\eta^{\mu\sigma}-\Phi^0\Phi^0k^2A_\gamma{}^\mu A_\gamma{}^\sigma+\Phi^\mu\Phi^\sigma\right)\phi^2kF_{\alpha\sigma}\\&+\tfrac{1}{2}\left(\eta^{\mu\sigma}-\Phi^0\Phi^0k^2A_\gamma{}^\mu A_\gamma{}^\sigma+\Phi^\mu\Phi^\sigma\right)\partial_5\left(\Phi_0\Phi_0k^2A_{\gamma\sigma}A_{\gamma\alpha}\right)\\&-\tfrac{1}{2}\left(\eta^{\mu\sigma}-\Phi^0\Phi^0k^2A_\gamma{}^\mu A_\gamma{}^\sigma+\Phi^\mu\Phi^\sigma\right)\left(\left(I_\alpha+2\phi kA_{\gamma\alpha}\right)\partial_\sigma-\left(I_\sigma+2\phi kA_{\gamma\sigma}\right)\partial_\alpha\right)\phi\\&-\tfrac{1}{2}\Phi^\mu\partial_\alpha\left(\Phi_0\Phi_0\right)\end{aligned}\quad . \tag{7.12}$$

From here, further mathematical reductions are possible. First, we noted earlier that $i\hbar\partial_\alpha A_{\gamma\mu}=q_\alpha A_{\gamma\mu}$ for the photon field in (2.11), which we extend to five dimensions as $i\hbar\partial_{\mathrm{A}}A_{\gamma\mu}=q_{\mathrm{A}}A_{\gamma\mu}$ by appending a fifth dimension in the Fourier kernel in (2.11a) just as we did for the fermion wavefunction following (5.6). Thus, we find $i\hbar A_\gamma{}^\sigma\partial_5A_{\gamma\sigma}=A_\gamma{}^\sigma q_5A_{\gamma\sigma}=0$ and so may set $A_\gamma{}^\sigma\partial_5A_{\gamma\sigma}=0$. For similar reasons, see (4.17) and recall that $A_{\gamma 0}=0$, we set $\Phi^\sigma\partial_5A_{\gamma\sigma}=0$. We also clear any remaining $A_\gamma{}^\sigma A_{\gamma\sigma}=0$ and $\Phi^\sigma A_{\gamma\sigma}=0$, and use $A_\gamma{}^\sigma I_\sigma=0$ because $A_{\gamma 0}=0$. Next, because $A_{\gamma 0}=0$, wherever there is a remaining $A_{\gamma\sigma}$ summed with an object with an upper $\sigma$ index, we set $\sigma=k=1,2,3$ to the space indexes only. We also use $\eta^{\mu\sigma}I_\sigma=\eta^{00}I_0=1$. And we substitute $\Phi^0=\Phi_0=\phi$ throughout. Again mindful that $i\hbar\partial_{\mathrm{A}}A_{\gamma\mu}=q_{\mathrm{A}}A_{\gamma\mu}$, we also use $A_\gamma{}^j=\eta^{jk}A_{\gamma k}$ from (4.16) to raise some indexes. Finally, we apply all remaining derivatives, separate out time and space components for any summed indexes still left except for in $F_{\alpha\sigma}$, and reconsolidate. The result is that strictly mathematically, (7.12) reduces to:

$$\begin{aligned}\tilde{\Gamma}^{\mu}_{\alpha 5} &= \tfrac{1}{2}\left(\eta^{\mu\sigma} - \phi^2 k^2 A_{\gamma}{}^{\mu} A_{\gamma}{}^{\sigma} + \Phi^{\mu}\Phi^{\sigma}\right)\phi^2 k F_{\alpha\sigma} \\ &\quad + \tfrac{1}{2}\left(\eta^{\mu 0} + 2\eta^{\mu k} k A_{\gamma k}\phi - \Phi^{\mu}\phi\right)\partial_{\alpha}\phi \\ &\quad - \tfrac{1}{2}\left(\eta^{\mu 0} + \Phi^{\mu}\phi\right)\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\partial_0\phi \\ &\quad - \tfrac{1}{2}\left(\eta^{\mu k} - \phi^2 k^2 A_{\gamma}{}^{\mu} A_{\gamma}{}^{k} + \Phi^{\mu}\phi^2 k A_{\gamma}{}^{k}\right)\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\partial_k\phi \\ &\quad + \phi\partial_5\phi k^2 A_{\gamma}{}^{\mu} A_{\gamma\alpha} + \tfrac{1}{2}\phi^2 k^2 \partial_5 A_{\gamma}{}^{\mu} A_{\gamma\alpha} + \tfrac{1}{2}\phi^2 k^2 A_{\gamma}{}^{\mu}\partial_5 A_{\gamma\alpha}\end{aligned} \quad . \tag{7.13}$$

Now, it is the upper $\mu$ index in $\tilde{\Gamma}^{\mu}_{\alpha 5}$ which, when used in the equation of motion (7.4), will determine the coordinate against which the acceleration is specified in relation to the proper time interval $d\tau$. So, we now separate (7.13) into its time and space components, as such:

$$\begin{aligned}\tilde{\Gamma}^{0}_{\alpha 5} &= \tfrac{1}{2}\left(\eta^{0\sigma} + \phi\Phi^{\sigma}\right)\phi^2 k F_{\alpha\sigma} \\ &\quad + \tfrac{1}{2}\left(1-\phi^2\right)\partial_{\alpha}\phi - \tfrac{1}{2}\left(1+\phi^2\right)\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\partial_0\phi - \tfrac{1}{2}\phi^3 k A_{\gamma}{}^{k}\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\partial_k\phi\end{aligned} , \tag{7.14a}$$

$$\begin{aligned}\tilde{\Gamma}^{j}_{\alpha 5} &= \tfrac{1}{2}\left(\eta^{j\sigma} - \phi^2 k^2 A_{\gamma}{}^{j} A_{\gamma}{}^{\sigma} + \phi^2 k A_{\gamma}{}^{j}\Phi^{\sigma}\right)\phi^2 k F_{\alpha\sigma} \\ &\quad + \left(1 - \tfrac{1}{2}\phi^2\right) k A_{\gamma}{}^{j}\phi\partial_{\alpha}\phi \\ &\quad - \tfrac{1}{2}\phi^2 k A_{\gamma}{}^{j}\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\phi\partial_0\phi \\ &\quad - \tfrac{1}{2}\left(\eta^{jk} - \left(\phi^2 - \phi^4\right)k^2 A_{\gamma}{}^{j} A_{\gamma}{}^{k}\right)\left(I_{\alpha} + 2\phi k A_{\gamma\alpha}\right)\partial_k\phi \\ &\quad + k^2 A_{\gamma}{}^{j} A_{\gamma\alpha}\phi\partial_5\phi + \tfrac{1}{2}\phi^2 k^2 A_{\gamma\alpha}\partial_5 A_{\gamma}{}^{j} + \tfrac{1}{2}\phi^2 k^2 A_{\gamma}{}^{j}\partial_5 A_{\gamma\alpha}\end{aligned} \quad . \tag{7.14b}$$

It is noteworthy that all terms in (7.13) containing the fifth dimensional derivative $\partial_5 = \partial / \partial x^5 = \partial / c\partial t^5$ also contain $A_{\gamma}{}^{\mu}$ and so drop out entirely from (7.14a) because $A_{\gamma}{}^{0} = 0$.

Now, as previewed prior to (7.12), $A_{\gamma\alpha}$ is the field for a single photon, which is inconsequential in physical effect compared to $F_{\alpha\sigma}$ which has now been gauge-transformed to a classical electric and magnetic field bivector consisting of innumerable photons. This is to say, if there is some interaction occurring in a classical electromagnetic field, a single photon more, or a single photon less, will be entirely undetectable for that interaction, akin to a single drop of water in an ocean. Moreover, the constant *k* is very small, so that the dimensionless $kA_{\gamma\alpha}$ will be very small in relation to the numbers $\pm 1$ contained in $\eta^{\mu\nu}$. With this in mind, we may set $A_{\gamma\alpha} \cong 0$ as an extraordinarily-close approximation to zero all terms which contain $A_{\gamma\alpha}$ in (7.14). This includes for (7.14a), only retaining $\Phi^0 = \phi$ in $\phi\Phi^{\sigma}\phi^2 k F_{\alpha\sigma} = \phi\Phi^0\phi^2 k F_{\alpha 0}$. And in (7.14b) we further use $\eta^{jk}\partial_k = -\partial_j$. So now, both of (7.14) reduce to:

$$\tilde{\Gamma}^0_{\alpha 5}=\tfrac{1}{2}\left(1+\phi^2\right)\phi^2 kF_{\alpha 0}+\tfrac{1}{2}\left(1-\phi^2\right)\partial_\alpha\phi-\tfrac{1}{2}\left(1+\phi^2\right)I_\alpha\partial_0\phi, \tag{7.15a}$$

$$\tilde{\Gamma}^j_{\alpha 5}=\tfrac{1}{2}\phi^2 kF_\alpha{}^j+\tfrac{1}{2}I_\alpha\partial_j\phi. \tag{7.15b}$$

Contrasting, we see that the former contains $F_{\alpha 0}$ while the latter contains $F_\alpha{}^j$ with a raised index. To properly compare we need to carefully raise the time index in (7.15a). To do this, we recall from after (2.11) that $i\hbar\partial_\alpha A_\mu=q_\alpha A_\mu$, $A_\gamma{}^\alpha q_\alpha=0$, and $A^j q_j=0$, which also means that $A_\gamma{}^\alpha\partial_\alpha=0$ and $A_\gamma{}^j\partial_j=0$, thus $\Phi^j\partial_j=0$ when $\partial_\alpha$ operates on $A_{\gamma\mu}$. Recall as well that $A_\gamma{}^\sigma A_{\gamma\sigma}=0$ and $\Phi^\sigma A_{\gamma\sigma}=0$. So, working from $F_{\gamma\sigma\nu}=\partial_\sigma A_{\gamma\nu}-\partial_\nu A_{\gamma\sigma}$ for an individual photon and using (4.22) with $g^{\mu\nu}=\eta^{\mu\nu}$, we first obtain, without yet fully reducing:

$$F_\gamma{}^\mu{}_\nu=G^{\mu\sigma}F_{\gamma\sigma\nu}=G^{\mu\sigma}\partial_\sigma A_{\gamma\nu}-G^{\mu\sigma}\partial_\nu A_{\gamma\sigma}=\left(\eta^{\mu\sigma}+\Phi^\mu\Phi^\sigma\right)\partial_\sigma A_{\gamma\nu}-\left(\eta^{\mu\sigma}+\Phi^\mu\Phi^\sigma\right)\partial_\nu A_{\gamma\sigma}. \tag{7.16}$$

Then, extracting the electric field bivector we obtain the field strength with a raised time index:

$$\begin{aligned}F_\gamma{}^0{}_\nu&=\left(\eta^{0\sigma}\partial_\sigma A_{\gamma\nu}+\Phi^0\Phi^\sigma\partial_\sigma A_{\gamma\nu}\right)-\left(\eta^{0\sigma}\partial_\nu A_{\gamma\sigma}+\Phi^0\Phi^\sigma\partial_\nu A_{\gamma\sigma}\right)\\&=\left(\partial_0 A_{\gamma\nu}+\Phi^0\Phi^0\partial_0 A_{\gamma\nu}\right)-\left(\partial_\nu A_{\gamma 0}+\Phi^0\Phi^0\partial_\nu A_{\gamma 0}\right)\\&=\left(1+\phi^2\right)\left(\partial_0 A_{\gamma\nu}-\partial_\nu A_{\gamma 0}\right)=\left(1+\phi^2\right)F_{\gamma 0\nu}\end{aligned}. \tag{7.17}$$

Using the gauge transformation $F_{\gamma\mu\nu}\to F_{\mu\nu}$ discussed prior to (7.12) to write this as $F_\alpha{}^0=\left(1+\phi^2\right)F_{\alpha 0}$, then using this in (7.15a), now reduces the equation pair (7.15) to:

$$\tilde{\Gamma}^0_{\alpha 5}=\tfrac{1}{2}\phi^2 kF_\alpha{}^0+\tfrac{1}{2}\left(1-\phi^2\right)\partial_\alpha\phi-\tfrac{1}{2}\left(1+\phi^2\right)I_\alpha\partial_0\phi, \tag{7.18a}$$

$$\tilde{\Gamma}^j_{\alpha 5}=\tfrac{1}{2}\phi^2 kF_\alpha{}^j+\tfrac{1}{2}I_\alpha\partial_j\phi. \tag{7.18b}$$

These clearly manifest general spacetime covariance between the $\tfrac{1}{2}\phi^2 kF_\alpha{}^0$ and $\tfrac{1}{2}\phi^2 kF_\alpha{}^j$ terms.

At this point we are ready to use the above in the equation of motion (7.4). Focusing on the motion contribution from the $\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}$ term, we first write (7.4) as:

$$\frac{d^2x^{\mathrm{M}}}{c^2d\tau^2}=-2\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}\frac{dx^\alpha}{cd\tau}\frac{dx^5}{cd\tau}+\ldots \tag{7.19}$$

with a reminder that we are focusing on this particular term out of the three terms in (7.4). We then separate this into time and space components and use (7.18) with $F_\alpha{}^\mu=-F^\mu{}_\alpha$ and $I_\alpha=\begin{pmatrix}1 & \mathbf{0}\end{pmatrix}$. Importantly, we also use the differential chain rule on the $\phi$ terms. We thus obtain:

$$\begin{aligned}\frac{d^2x^0}{c^2d\tau^2}&=-2\tilde{\Gamma}^0_{\alpha 5}\frac{dx^\alpha}{cd\tau}\frac{dx^5}{cd\tau}+...=-\left(\phi^2kF_\alpha{}^0+\left(1-\phi^2\right)\partial_\alpha\phi-\left(1+\phi^2\right)I_\alpha\partial_0\phi\right)\frac{dx^\alpha}{cd\tau}\frac{dx^5}{cd\tau}+...\\&=\phi^2k\frac{dx^5}{cd\tau}F^0{}_\alpha\frac{dx^\alpha}{cd\tau}+2\frac{dx^5}{cd\tau}\phi^2\frac{d\phi}{cd\tau}+...\end{aligned} \tag{7.20a}$$

$$\begin{aligned}\frac{d^2x^j}{c^2d\tau^2}&=-2\tilde{\Gamma}^j_{\alpha 5}\frac{dx^\alpha}{cd\tau}\frac{dx^5}{cd\tau}+...=-\left(\phi^2kF_\alpha{}^j+I_\alpha\partial_j\phi\right)\frac{dx^\alpha}{cd\tau}\frac{dx^5}{cd\tau}\\&=\phi^2k\frac{dx^5}{cd\tau}F^j{}_\alpha\frac{dx^\alpha}{cd\tau}-\frac{dx^5}{cd\tau}\frac{dx^0}{dx^j}\frac{d\phi}{cd\tau}+...\end{aligned} \tag{7.20b}$$

In both of the above, for the scalar we find a *derivative along the curve*, $d\phi / cd\tau$. Note further that in (7.20b) this is multiplied by the inverse of $dx^j / dx^0 = v^j / c$ where $v^j = dx^j / dt^0$ is an ordinary space velocity with reference to the ordinary time $t^0$ (versus the fifth-dimensional $t^5$). In contrast, in (7.20a) the objects covariant with this velocity term simply turned into the number 1 via the chain rule. Given its context, we understand $v^j$ to be the space velocity of the scalar $\phi$.

This raises an important question and gives us our first piece of solid information about the *physical nature* of the Kaluza-Klein scalar $\phi$: Without the $d\phi / cd\tau$ term (7.20) consolidate into $d^2x^\mu / c^2d\tau^2 = \left(\phi^2kdx^5 / cd\tau\right)F^\mu{}_\alpha dx^\alpha cd\tau$ following which we can make the usual "Kaluza miracle" association with the Lorentz Force law. However, with this term, if $\phi$ is a *material* field or particle which can be Lorentz transformed to a rest frame with $v^j = 0$, then we have a problem, because the latter term in (7.20b) will become infinite, causing the space acceleration to likewise become infinite. The only way to avoid this problem, is to understand the scalar $\phi$ as a *luminous* entity which travels at the speed of light and which can never be Lorentz transformed to a rest frame, just like the photon. More to the point in terms of scientific method: we know from observation that the Lorentz force does not become infinite nor does it exhibit any observable deviations from the form $d^2x^\mu / c^2d\tau^2 = \left(\phi^2kdx^5 / cd\tau\right)F^\mu{}_\alpha dx^\alpha cd\tau$. Therefore, we use this observational evidence in view of (7.20b) to deduce that $\phi$ must be luminous.

To implement this luminosity, we first write the four-dimensional spacetime metric for a luminous particle such as the photon, and now also the scalar $\phi$, using mixed indexes, as $0 = d\tau^2 = dx^0dx_0 + dx^jdx_j$. This easily is rewritten as $dx^0dx_0 = -dx^jdx_j$ and then again as:

$$\frac{dx^0}{dx^j}=-\frac{dx_j}{dx_0}. \tag{7.21}$$

This is the term of interest in (7.20b). Now, we want to raise indexes on the right side of (7.21) but must do so with (3.13). Using $\Phi_0 = \phi$ and $g_{\mu\nu} = \eta_{\mu\nu}$ as well as $A_{\gamma 0} = 0$ and $A_\gamma{}^\mu = \eta^{\mu\nu}A_{\gamma\nu}$ from (4.16), we find:

$$\begin{aligned} dx_0 &= G_{0\nu}dx^\nu = \left(\eta_{0\nu} + \phi^2 k^2 A_{\gamma 0} A_{\gamma\nu}\right) dx^\nu = \eta_{0\nu} dx^\nu = dx^0 \\ dx_j &= G_{j\nu}dx^\nu = \left(\eta_{j\nu} + \phi^2 k^2 A_{\gamma j} A_{\gamma\nu}\right) dx^\nu = -dx^j + \phi^2 k^2 A_\gamma{}^j A_\gamma{}^k dx^k \end{aligned} . \tag{7.22}$$

Using the above in (7.21) then yields the *luminous* particle relation:

$$\frac{dx^0}{dx^j} = \frac{dx^j}{dx^0} - \phi^2 k^2 A_\gamma{}^j A_\gamma{}^k \frac{dx^k}{dx^0} = \hat{u}^j - \phi^2 k^2 A_\gamma{}^j A_\gamma{}^k \hat{u}^k . \tag{7.23}$$

Above, we also introduce a unit vector $\hat{u}^j = dx^j / dx^0$ with $\hat{u}^j \hat{u}^j = 1$ pointing in the direction of the luminous propagation of $\phi$.

Inserting (7.23) for a luminous scalar into (7.20b) then produces:

$$\frac{d^2 x^j}{c^2 d\tau^2} = \phi^2 k \frac{dx^5}{cd\tau} F^j{}_\alpha \frac{dx^\alpha}{cd\tau} - \hat{u}^j \frac{dx^5}{cd\tau}\frac{d\phi}{cd\tau} + \frac{dx^5}{cd\tau}\phi^2 k^2 A_\gamma{}^j A_\gamma{}^k \hat{u}^k \frac{d\phi}{cd\tau} \tag{7.24}$$

As we did starting at (7.15) we then set $A_{\gamma\alpha} \cong 0$ because the gauge vector for a single photon will be swamped by the innumerable photons contained in the classical field strength $F^j{}_\alpha$. As a result, using (7.24), we find that (7.20) together now become:

$$\frac{d^2 x^0}{c^2 d\tau^2} = \phi^2 k \frac{dx^5}{cd\tau} F^0{}_\alpha \frac{dx^\alpha}{cd\tau} + 2\frac{dx^5}{cd\tau}\phi^2 \frac{d\phi}{cd\tau} \tag{7.25a}$$

$$\frac{d^2 x^j}{c^2 d\tau^2} = \phi^2 k \frac{dx^5}{cd\tau} F^j{}_\alpha \frac{dx^\alpha}{cd\tau} - \hat{u}^j \frac{dx^5}{cd\tau}\frac{d\phi}{cd\tau} \tag{7.25b}$$

In (7.25b), $\phi$ has now been made luminous.

Finally, we are ready to connect this to the Lorentz Force motion, which we write as:

$$\frac{d^2 x^\mu}{c^2 d\tau^2} = \frac{q}{mc^2} F^\mu{}_\alpha \frac{dx^\alpha}{cdt} . \tag{7.26}$$

We start with the space components in (7.25b) combined with $\mu = j$ in (7.26) and use these to *define* the association:

$$\frac{d^2 x^j}{c^2 d\tau^2} = \phi^2 k \frac{dx^5}{cd\tau} F^j{}_\alpha \frac{dx^\alpha}{cd\tau} - \hat{u}^j \frac{dx^5}{cd\tau}\frac{d\phi}{cd\tau} \equiv \frac{q}{mc^2} F^j{}_\alpha \frac{dx^\alpha}{cdt} . \tag{7.27}$$

For the moment, let us ignore the term $d\phi / d\tau$ to which we shall shortly return, and focus on the term with $F^{j}{}_{\alpha}$. If this is to represent Lorentz motion insofar as the $F^{j}{}_{\alpha}$ terms, then factoring out common terms from both sides, we obtain the following relation and its inverse:

$$\boxed{\phi^2 k \frac{dx^5}{cd\tau} = \phi^2 k \frac{dt^5}{d\tau} = \frac{q}{mc^2}; \quad \frac{dx^5}{cd\tau} = \frac{dt^5}{d\tau} = \frac{q}{\phi^2 kmc^2}}. \tag{7.28}$$

This is why electric charge – and to be precise, the charge-to-mass ratio – is interpreted as "motion" through the fifth dimension. However, because of the timelike fifth dimension in the metric tensor (3.13), the charge-to-energy ratio of a charged material body is no longer interpreted as *spatial motion through an unseen fourth space dimension*. Rather, it is understood as a *rate of time flow in a second time dimension*.

Next, we substitute the above for $dx^5 / cd\tau$ in each of (7.25) and reduce to obtain:

$$\frac{d^2x^0}{c^2 d\tau^2} = \frac{q}{mc^2} F^{0}{}_{\alpha} \frac{dx^\alpha}{cd\tau} + 2\frac{q}{kmc^2}\frac{d\phi}{cd\tau} \tag{7.29a}$$

$$\frac{d^2x^j}{c^2 d\tau^2} = \frac{q}{mc^2} F^{j}{}_{\alpha} \frac{dx^\alpha}{cd\tau} - \frac{\hat{u}^j}{\phi^2}\frac{q}{kmc^2}\frac{d\phi}{cd\tau} \tag{7.29b}$$

This does indeed reproduce the Lorentz motion, except for the $d\phi / d\tau$ term in each. Now, because there is no observed deviation for the Lorentz motion, in order to minimize the physical impact of these final terms, one might suppose that the luminous $\phi$ is an extremely small field $\phi \cong 0$ with $d\phi / d\tau \cong 0$, but this is problematic for two reasons: First, if *k* turns out to be the extremely small ratio $k = \left(2/c^2\right)\sqrt{G/k_e}$ given by (1.2) as it is in Kaluza-Klein theory – and there is no reason to believe that *k* will turn out otherwise here – then the 1/*k* in both of (7.29) is an extremely large coefficient, which means that $d\phi / d\tau$ would have to be even more extraordinarily small. Second, even if $d\phi / d\tau \cong 0$ in part because we make $\phi$ extremely small, the presence of $1/\phi^2$ in (7.29b) still causes a problem, because an extremely small $\phi \to 0$ implies an extremely large $1/\phi^2 \to \infty$. Ironically, the $1/\phi^2$ which causes $G^{\mathrm{MN}} \to \infty$ in the usual Kaluza-Klein metric tensor (1.1) – which problem was solved by the non-singular (4.22) – nevertheless still persists, because of its appearance in (7.29b). And it persists in the form of a very large yet unobserved impact on the physical, observable Lorentz motion. The only apparent way to resolve this, is to *require* that $d\phi / d\tau = 0$. If that is the case, then (7.29) both condense precisely into the Lorentz Force motion.

Now, on first appearance, the thought that $d\phi / d\tau = 0$ seems to suggest that $\phi$ must be a constant field with no gradient, which as pointed out in [11] imposes unwarranted constraints on the electromagnetic field, and which also defeats the purpose of a "field" if that field has to be constant. But in (7.29), $d\phi / d\tau$ is *not* a gradient nor is it a time derivative. Rather, it is a *derivative along the curve* with curvature specified by the metric tensor (2.15), and it is related to the four-gradient $\partial_\mu \phi$ by the chain rule $d\phi / d\tau = \left(\partial\phi / \partial x^\mu\right)\left(dx^\mu / d\tau\right) = \partial_\mu \phi u^\mu$ with $u^\mu \equiv dx^\mu / d\tau$. Moreover, we have now learned at (7.20) that $\phi$ must be a luminous field, which requirement has been embedded in (7.29b). So, this derivative along the curve will be taken in *frames of reference which travel with the luminous field*, which luminous reference frames cannot ever be transformed into the rest frame – or even into a relatively-moving frame – of a *material* observer. As a result, it is indeed possible to have a zero $d\phi / d\tau$ in the luminous reference frame "along the curve" simultaneously with a non-zero gradient $\partial_\mu \phi \neq 0$ taken with reference to coordinates defined by a material observer. As we now shall elaborate, this solves the "constant field / zero gradient" problems which have long plagued Kaluza-Klein theory, and teaches a great deal of new intriguing information about the physical properties of the scalar field $\phi$.

## 8. Luminosity and Internal Second-Rank Dirac Symmetry of the Dirac-Kaluza-Klein Scalar Field

Let us take the final step of connecting (7.29) to the observed Lorentz Force motion with nothing else in the way, by formally setting the derivative along the curve for $\phi$ to zero, thus:

$$\frac{d\phi}{cd\tau} = \frac{\partial\phi}{\partial x^\mu}\frac{dx^\mu}{cd\tau} = 0 . \tag{8.1}$$

With this, both of (7.29) immediately become synonymous with the Lorentz Force motion (7.26). From the standpoint of scientific method, we can take (7.29) together with (7.26) as empirical evidence that (8.1) must be true. Now, let's explore what (8.1) – if it really is true – teaches us about the physical properties of $\phi$.

To start, let us square (8.1) and so write this as:

$$\left(\frac{d\phi}{cd\tau}\right)^2 = \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} = \partial_\mu\phi\partial_\nu\phi\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau} = 0 . \tag{8.2}$$

Next, let's write the four-dimensional spacetime metric (7.5) for a luminous particle using (3.13) with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Phi_0 = \phi$ as:

$$0 = c^2 d\tau^2 = G_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu}dx^\mu dx^\nu . \tag{8.3}$$

We already used a variant of this to obtain (7.23). Then, also appending a $\phi^2$ and using an overall minus sign which will become useful momentarily, we restructure this to:

$$-\left(\eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu}\right)\frac{dx^\mu}{cd\tau}\frac{dx^\nu}{cd\tau}\phi^2 = 0 . \tag{8.4}$$

The above (8.4) describes a luminous particle in a five-dimensional spacetime with the metric tensor (3.13). So, we can use this luminosity to supply the zero for the squared derivative along the curve in (8.2) if, comparing (8.2) and (8.4), we define the relation:

$$\partial_\mu \phi \partial_\nu \phi \equiv -\left(\eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}\right) \phi^2 / \lambdabar^2 \neq 0 , \tag{8.5}$$

where $\lambdabar \equiv \lambda / 2\pi$ is a reduced wavelength of the scalar, needed and therefore introduced to balance the $1/\text{length}^2$ dimension of $\partial_\mu \phi \partial_\nu \phi$ with the dimensionless $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$. Now, all we need to do is determine the first-order $\partial_\mu \phi$ which satisfies (8.5).

What becomes apparent on close study of (8.5) is that there is no way to isolate a first-order $\partial_\mu \phi$ unless we make use of the Dirac gamma operators in a manner very similar to what Dirac originally used in [13] to take the operator "square root" of the Klein-Gordon equation. And in fact, the operator square root we need to take to separate out a linear $\partial_\mu \phi$ from (8.5) is precisely the $\Gamma_\mu = \left(\gamma_0 + kA_{\gamma j}\gamma_j \quad \gamma_j + kA_{\gamma j}\gamma_0\right)$ we found in (2.14) which satisfy (2.1) with $g_{\mu\nu} = \eta_{\mu\nu}$, that is, which satisfy $\frac{1}{2}\left\{\Gamma_\mu, \Gamma_\nu\right\} = \eta_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$. Therefore, we may now use these $\Gamma_\mu$ to take the square root of (8.5), where we also use $-i = \sqrt{-1}$ choosing $-i$ rather than $+i$ for reasons which will momentarily become apparent, to obtain:

$$\lambdabar \partial_\mu \phi = -i \Gamma_\mu \phi . \tag{8.6}$$

Now, just as the photon gauge field (2.11a) contains a Fourier kernel $\exp\left(-iq_\sigma x^\sigma / \hbar\right)$ where $q^\mu$ is the photon energy-momentum, and the fermion wavefunction used in (5.6) contains a Fourier kernel $\exp\left(-ip_\Sigma x^\Sigma / \hbar\right)$ with a fermion five-momentum $p^{\mathrm{M}}$ (and we anticipate (5.6) will be used to inform us regarding $p^5$), let us specify a Fourier kernel $\exp\left(-is_\Sigma x^\Sigma / \hbar\right)$ with a five-dimensional $s^{\mathrm{M}}$ which we regard as the five-momentum of the luminous scalar $\phi$. Moreover, because $\phi$ is dimensionless and so too is $\exp\left(-is_\Sigma x^\Sigma / \hbar\right)$, let us simply define:

$$\phi \equiv \exp\left(-is_\Sigma x^\Sigma / \hbar\right) \tag{8.7}$$

to be a Fourier kernel in five dimensions, for which $\partial_\mu \phi = -is_\mu \phi / \hbar$. Our goal is to now learn as much as we can about $s^\Sigma$ and especially its spacetime components $s^\mu$.

Substituting $\partial_\mu \phi = -is_\mu \phi / \hbar$ based on (8.7) into (8.6) we first obtain:

$$-i \lambdabar s_\mu \phi / \hbar = -i \Gamma_\mu \phi . \tag{8.8}$$

Then stripping off all $\phi$, and following some algebraic rearrangement including multiplying through by $c$, then using $\hbar c / ƛ = \hbar\omega = hf$ for the energy magnitude of the scalar, we arrive at:

$$\begin{aligned} cs_\mu &= \Gamma_\mu \hbar c / ƛ = \hbar\omega\Gamma_\mu = hf\,\Gamma_\mu = \hbar\omega\left(\gamma_0 + kA_{\gamma j}\gamma_j \quad \gamma_j + kA_{\gamma j}\gamma_0\right) \\ &= \hbar\omega\left(\begin{pmatrix} I & k\mathbf{A}_\gamma\cdot\boldsymbol{\sigma} \\ -k\mathbf{A}_\gamma\cdot\boldsymbol{\sigma} & -I \end{pmatrix} \quad \begin{pmatrix} Ik\mathbf{A}_\gamma & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & -Ik\mathbf{A}_\gamma \end{pmatrix}\right) \end{aligned} , \tag{8.9}$$

using the Dirac representation. So, we now see that the reason we used $-i = \sqrt{-1}$ at (8.6) was to ensure that the component $cs_0 = hf\left(\gamma_0 + kA_{\gamma j}\gamma_j\right)$ for the energy of the scalar is positive for the upper components of $\mathrm{diag}(\gamma_0) = (+I, -I)$ in the Dirac representation.

Then, if $cs_\mu = \hbar\omega\Gamma_\mu$ in four dimensions, the natural five-dimensional covariant extension of (8.9) is $cs_\mathrm{M} = \hbar\omega\Gamma_\mathrm{M}$, and with (8.9) so-extended, the scalar wavefunction in (8.7) becomes:

$$\boxed{\phi_{AB} \equiv \exp\left(-i\frac{\omega}{c}\Gamma_\Sigma x^\Sigma\right)_{AB} = \cos\left(\frac{\omega}{c}\Gamma_\Sigma x^\Sigma\right)_{AB} - i\sin\left(\frac{\omega}{c}\Gamma_\Sigma x^\Sigma\right)_{AB}} . \tag{8.10}$$

Because the magnitude of the energy (8.9) for the scalar is $E = hf$ just as it is for a single photon, we now must interpret $\phi$ as an individual scalar particle quantum, just as at (2.11) we were required to regard $A_\mu = A_{\gamma\mu}$ and an individual photon quantum.

In (8.10) above, to emphasize a very significant new finding, we have made explicit the Dirac spinor indexes $A = 1,2,3,4$ and $B = 1,2,3,4$. Specifically, what we now learn from (8.10), and from (8.9) written with Dirac indexes explicit as $cs_{\mathrm{M}AB} = \hbar\omega\Gamma_{\mathrm{M}AB}$, is that if $d\phi / cd\tau = 0$ in (8.1) is to be true, which enables (7.29) to consolidate precisely into the Lorentz Force law with no other problematic terms, then the energy momentum $cs_{\mathrm{M}AB} = \hbar\omega\Gamma_{\mathrm{M}AB}$ scalar field must in fact be a second rank 4x4 Dirac object with implied Dirac indexes, and a total of 4x4x5=80 components when this is taken over the five space dimensions. Moreover, when calculated out using the McLaurin series $\exp(-i\theta) = 1 - i\theta - \theta^2/2! + i\theta^3/3! + \theta^4/4! + \ldots$ including taking square, cubes, etc. of $\Gamma_\Sigma$, we realize that the Kaluza-Klein scalar in physical reality, must not only be luminous, but must also be a 4x4=16 component second rank Dirac-based object $\phi_{AB}$ with indexes explicit.

This solves the Kaluza-Klein problem of how to make the scalar field "constant" to remove what are otherwise some very large terms, while not unduly constraining the electromagnetic fields: The gradient can be non-zero, while the derivative along the curve can be zero, so long as the scalar is a luminous particle which also has a second rank Dirac structure. In turn, if we then return to the metric tensor $G_\mathrm{MN}$ in the form of, say, (3.11), we find that this too must also have implied Dirac indexes, that is, $G_\mathrm{MN} = G_{\mathrm{MN}AB}$ owing to the structure (8.10) of the scalar fields which sit in its fifth dimensional components. So (8.10) gives a second rank Dirac structure to the metric tensor, alongside of its already second-rank, five dimensional spacetime structure.

And so, the Kaluza-Klein fifth dimension, taken together with using Dirac theory to enforce general covariance across all five dimensions, has turned a metric tensor (1.1) with an entirely classical character, into a quantum field theory metric tensor with luminous photons and luminous scalar field quanta. If this is all in accord with physical reality, this means that nature actually has three spin types of massless, luminous field quanta: spin-2 gravitons, spin-1 photons and gluons, and spin-0 scalars with an internal second rank Dirac-tensor symmetry. This also means that the Kaluza-Klein scalar is not the same scalar as the Higgs, because the latter is massive and material. Finally, one must a least consider the prospect that these scalars (8.10) contribute additional energy content to the universe which may not have previously been accounted for.

## 9. How the Dirac-Kaluza-Klein Metric Tensor Resolves the Challenges faced by Kaluza-Klein Theory without Diminishing the Kaluza "Miracle," and Grounds the Now-Timelike Fifth Dimension in Manifestly-Observed Physical Reality

Now let's review the physics implications of everything that has been developed here so far. As has been previously pointed out, in the circumstance where all electrodynamic interactions are turned off by setting $A_{\gamma j}=0$ and what is now $\Phi_{\mu}=0$, then (3.13) reduces when $g_{\mu\nu}=\eta_{\mu\nu}$ to $\mathrm{diag}\left(G_{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,+1\right)$ with $\left|G_{\mathrm{MN}}\right|=-1$. And we saw at (6.3) that this result does not change at all, even when $A_{\gamma j}\neq 0$ and $\Phi_{\mu}\neq 0$. But in the same situation the usual Kaluza-Klein metric tensor (1.1) reduces to $\mathrm{diag}\left(G_{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,0\right)$ with a determinant $\left|G_{\mathrm{MN}}\right|=0$. This of course means the Kaluza-Klein metric tensor is not-invertible and therefore becomes singular when electrodynamic interactions are turned off. Again, this may be seen directly from the fact that when we set $A_{\gamma j}=0$ and $\phi=0$, in (1.1) we get $G^{55}=g_{\alpha\beta}A^{\alpha}A^{\beta}+1/\phi^{2}=0+\infty$. This degeneracy leads to a number of interrelated ills which have hobbled Kaluza-Klein as a viable theory of the natural world for a year shy of a century.

First, the scalar field $\phi$ carries a much heavier burden than it should, because Kaluza-Klein theory relies upon this field being non-zero to ensure that the five-dimensional spacetime geometry is non-singular. This imposes constraints upon $\phi$ which would not exist if it was not doing "double duty" as both a scalar field and as a structural element required to maintain the non-degeneracy of Minkowski spacetime extended to five dimensions.

Second, this makes it next-to-impossible to account for the fifth dimension in the observed physical world. After all, the space and time of real physical experience have a flat spacetime signature $\mathrm{diag}\left(\eta_{\mu\nu}\right)=\left(+1,-1,-1,-1\right)$ which is structurally sound even in the absence of any fields whatsoever. But what is one to make of a signature which, when $g_{\mu\nu}=\eta_{\mu\nu}$ and $A_{\gamma k}=0$, is given by $\mathrm{diag}\left(\eta_{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,\phi^{2}\right)$ with $\left|\eta_{\mathrm{MN}}\right|=-\phi^{2}$? How is one to explain the physicality of a

$G_{55} = \phi^2$ in the Minkowski signature which is based upon a field, rather than being either a timelike +1 or a spacelike –1 Pythagorean metric component? The Minkowski signature defines the *flat tangent* spacetime at each event, absent curvature. How can a tangent space which by definition should not be curved, be dependent upon a field $\phi$ which if it has even the slightest modicum of energy will cause curvature? This is an internal logical contradiction of the Kaluza-Klein metric tensor (1.1) that had persisted for a full century, and it leads to such hard-to-justify oddities as a fifth dimensional metric component $G_{55} = \phi^2$ and determinant $|\eta_{\mathrm{MN}}| = -\phi^2$ which dilates or contracts (hence the sometime-used name "dilaton") in accordance with the behavior of $\phi^2$.

Third, the DKK metric tensor (3.13) is obtained by *requiring* that it be possible to deconstruct the Kaluza-Klein metric tensor into a set of Dirac matrices obeying (3.1), with the symmetry of full five-dimensional general covariance. What we have found is that it is *not possible* to have 5-dimensional general covariance if $G_{05} = G_{50} = 0$ and $G_{55} = \phi^2$ as in (1.1). Rather, general 5-dimensional covariance *requires* that $G_{05} = G_{50} = \phi$ and $G_{55} = 1 + \phi^2$ in (3.13). Further, even to have spacetime covariance in *four dimensions alone*, we are *required* to gauge the electromagnetic potential to that of the photon. Without these changes to the metric tensor components, it is simply not possible to make Kaluza-Klein theory compatible with Dirac theory and to have 5-dimensional general covariance. This means that there is no consistent way of using the usual (1.1) to account for the fermions which are at the heart of observed matter in the material universe. Such an omission – even without any of its other known ills – most-assuredly renders the KK metric (1.1) "unphysical."

Finally, there is the century-old demand which remains unmet to this date: "show me the fifth dimension!" There is no observational evidence at all to support the fifth dimension, at least in the form specified by (1.1), or in the efforts undertaken to date to remedy these problems.

But the metric tensors (3.13) and (4.22) lead to a whole other picture. First, by definition, a 5-covariant Dirac equation (5.6) can be formed, so there is no problem of incompatibility with Dirac theory. Thus, all aspects of fermion physics may be fully accounted for. Second, it should be obvious to anyone familiar with the $\gamma_\mu$ and $\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3$ that one may easily use an anticommutator $\eta_{\mathrm{MN}} \equiv \frac{1}{2}\{\gamma_{\mathrm{M}}, \gamma_{\mathrm{N}}\}$ to form a five-dimensional Minkowski tensor with $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1, -1, -1, -1, +1)$, which has a Minkowski signature with two timelike and three spacelike dimensions. But it is not at all obvious how one might proceed to regard $\gamma_5$ as the generator of *a truly-physical fifth dimension* which is on an absolute par with the generators $\gamma_\mu$ of the four truly-physical dimensions which are time and space. This is true, notwithstanding the clear observational evidence that $\gamma_5$ has a multitude of observable physical impacts. The reality of $\gamma_5$ is most notable in the elementary fermions that contain the factor $\frac{1}{2}(1 \pm \gamma_5)$ for right- and left-chirality; in the one particle and interaction namely neutrinos acting weakly that are always left-chiral; and in the many observed pseudo-scalar mesons ( $J^{PC} = 0^{-+}$ ) and pseudo-vector mesons ( $J^{PC} = 1^{++}$ and $J^{PC} = 1^{+-}$ ) laid out in [20], all of which require the use of $\gamma_5$ to underpin their theoretical origins. So $\gamma_5$ is real and physical, as would therefore be any fifth dimension which can be *properly-connected* with $\gamma_5$.

But the immediate problem as pointed out in toward the end of [11], is that because $G_{55} = \phi^2$ in the Kaluza-Klein metric tensor (1.1), if we require electromagnetic energy densities to be positive, the fifth-dimension must have a *spacelike* signature. And this directly contradicts making $\gamma_5$ the generator of the fifth dimension because $\gamma_5\gamma_5 = 1$ produces a *timelike* signature. So, as physically-real and pervasive as are the observable consequences of the $\gamma_5$ matrix, the Kaluza-Klein metric tensor (1.1) does *not* furnish a theoretical basis for associating $\gamma_5$ with a fifth dimension, at the very least because of this timelike-versus-spacelike contradiction. This is yet another problem stemming from having $\phi$ carry the burden of maintaining the fifth-dimensional signature and the fundamental Pythagorean character of the Minkowski tangent space.

So, to summarize, on the one hand, Kaluza-Klein theory has a fifth physical dimension on a par with space and time, but it has been impossible to connect that dimension with actual observations in the material, physical universe, or to make credible sense of the dilation and contraction of that dimension based on the behavior of a scalar field. On the other hand, Dirac theory has an eminently-physical $\gamma_5$ with pervasive observational manifestations on an equal footing with $\gamma_\mu$, but it has been impossible to connect this $\gamma_5$ with a true physical fifth dimension (or at least, with the Kaluza-Klein metric tensor (1.1) in five dimensions). At minimum this is because the metric tensor signatures conflict. *Kaluza-Klein has a fifth-dimension unable to connect to physical reality, while Dirac theory has a physically-real $\gamma_5$ unable to connect to a fifth dimension.* And the origin of this disconnect on both hands, is that the Kaluza-Klein metric tensor (1.1) cannot be deconstructed into Dirac-type matrices while maintaining five-dimensional general covariance according to (3.1). *To maintain general covariance and achieve a Dirac-type square root operator deconstruction of the metric tensor, <u>(1.1) must be replaced by (3.13) and (4.22)</u>.*

Once we use (3.13) and (4.22) all these problems evaporate. Kaluza-Klein theory becomes fully capable of describing fermions as shown in (5.6). With $G_{55} = 1 + \phi^2$ the metric signature is decoupled from the energy requirements for $\phi$, and with $|G_{\mathrm{MN}}| = |g_{\mathrm{MN}}|$ from (6.3) the metric tensor determinant is entirely independent of both $A_{\gamma\mu}$ and $\phi$. Most importantly, when $A_{\gamma j} = 0$ and $\phi = 0$ and $g_{\mathrm{MN}} = \eta_{\mathrm{MN}}$, because $\mathrm{diag}(G_{\mathrm{MN}}) = (+1, -1, -1, -1, +1) = \mathrm{diag}\left(\frac{1}{2}\{\gamma_{\mathrm{M}}, \gamma_{\mathrm{N}}\}\right) = \mathrm{diag}(\eta_{\mathrm{MN}})$, and because of this decoupling of $\phi$ from the metric signature, we now have a timelike $\eta_{55} = \gamma_5\gamma_5 = +1$ which is directly generated by $\gamma_5$. As a consequence, the fifth dimension of Kaluza-Klein theory which has heretofore been disconnected from physical reality, can now be identified with a true physical dimension that has $\gamma_5$ as its generator, just as $\gamma_0$ is the generator of a truly-physical time dimension and $\gamma_j$ are the generators of a truly-physical space dimensions. And again, $\gamma_5$ has a wealth of empirical evidence to support its reality.

Further, with a tangent space $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$ we now have two timelike and three spacelike dimensions, with matching tangent-space signatures between Dirac theory and the Dirac-Kaluza-Klein theory. With the fifth-dimension now being timelike not spacelike, the notion of "curling up" the fifth dimension into a tiny "cylinder" comes off the table completely, while the Feynman-Wheeler concept of "many-fingered time" returns to the table, providing a possible avenue to study future probabilities which congeal into past certainties as the arrow of time progresses forward with entropic increases. And because $\gamma_5$ is connected to a multitude of confirmed observational phenomena in the physical universe, the physical reality of the fifth dimension in the metric tensors (3.13) and (4.22) is now supported by every single observation ever made of the reality of $\gamma_5$ in particle physics, regardless of any other epistemological interpretations one may also arrive at for this fifth dimension.

Moreover, although the field equations obtained from (3.13) and (4.22) rather than (1.1) will change somewhat because now $G_{05} = G_{50} = \phi$ and $G_{55} = 1 + \phi^2$ and the gauge fields are fixed to the photon $A_\mu = A_{\gamma\mu}$ with only two degrees of freedom, there is no reason to suspect that the many good benefits of Kaluza-Klein theory will be sacrificed because of these changes which eliminate the foregoing problems. Indeed, we have already seen in sections 7 and 8 how the Lorentz force motion is faithfully reproduced. Rather, we simply expect some extra terms (and so expect some additional phenomenology) to emerge in the equations of motion and the field equations because of these modifications. But the Kaluza-Klein benefits having of Maxwell's equations, the Lorentz Force motion and the Maxwell-stress energy embedded, should remain fully intact when using (3.13) and (4.22) in lieu of (1.1), as illustrated in sections 7 and 8.

Finally, given all of the foregoing, beyond the manifold observed impacts of $\gamma_5$ in particle physics, there is every reason to believe that using the five-dimensional Einstein equation with the DKK metric tensors will fully enable us to understand this fifth dimension, at bottom, as a *matter dimension*, along the lines long-advocated by the 5D Space-Time-Matter Consortium [18]. This will be further examined in the final section to follow, and may thereby bring us ever-closer to uncovering the truly-geometrodynamic theoretical foundation at the heart of all of nature.

## 10. Conclusion – Pathways for Continued Exploration: The Einstein Equation, the "Matter Dimension," Quantum Field Path Integration, Epistemology of a Second Time Dimension, and All-Interaction Unification

Starting at (7.6) we obtained the connection $\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}$ in order to study the $\tilde{\Gamma}^{\mathrm{M}}_{\alpha 5}$ term in the equation of motion (7.4), because this is the term which provides the Lorentz Force motion which becomes (7.29) once $\phi$ is understood to be a luminous field with $d\phi / d\tau = 0$ as in (8.1). The reason this was developed in detail here, is to demonstrate that the DKK metric tensors (3.13) and (4.22) in lieu of the usual (1.1) of Kaluza-Klein do not in any way forego the Kaluza miracle, at least as regards the Lorentz Force equation of electrodynamic motion. But there are a number of further steps which can and should be taken to further develop the downstream implications of using the DKK metric tensors (3.13) and (4.22) in lieu of the usual (1.1) of Kaluza-Klein.

First, it is necessary to calculate all of the other connections $\tilde{\Gamma}^{\mathrm{M}}_{\mathrm{AB}}$ using (7.3) and the metric tensors (3.13) and (4.22) similarly to what was done in section 7, then to fully develop the remaining terms in the equations of motion (7.2), (7.4) which have not yet been elaborated here, and also to obtain the five-dimensional Riemann and Ricci tensors, and the Ricci scalar:

$$\begin{aligned}
&\hat{R}^{\mathrm{A}}_{\mathrm{BMN}} = \partial_{\mathrm{M}}\Gamma^{\mathrm{A}}{}_{\mathrm{NB}} - \partial_{\mathrm{N}}\Gamma^{\mathrm{A}}{}_{\mathrm{MB}} + \Gamma^{\mathrm{A}}{}_{\mathrm{M\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{NB}} - \Gamma^{\mathrm{A}}{}_{\mathrm{N\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{MB}} \\
&\hat{R}_{\mathrm{BM}} = \hat{R}^{\mathrm{T}}_{\mathrm{BMT}} = \partial_{\mathrm{M}}\Gamma^{\mathrm{T}}{}_{\mathrm{TB}} - \partial_{\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\mathrm{MB}} + \Gamma^{\mathrm{T}}{}_{\mathrm{M\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{TB}} - \Gamma^{\mathrm{T}}{}_{\mathrm{T\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{MB}} \\
&\hat{R} = \hat{R}^{\Sigma}{}_{\Sigma} = G^{\mathrm{BM}}\hat{R}_{\mathrm{BM}} = G^{\mathrm{BM}}\partial_{\mathrm{M}}\Gamma^{\mathrm{T}}{}_{\mathrm{TB}} - G^{\mathrm{BM}}\partial_{\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\mathrm{MB}} + G^{\mathrm{BM}}\Gamma^{\mathrm{T}}{}_{\mathrm{M\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{TB}} - G^{\mathrm{BM}}\Gamma^{\mathrm{T}}{}_{\mathrm{T\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{MB}}
\end{aligned} \quad . \tag{10.1}$$

Once these are obtained, these may then be placed into a fifth-dimensional Einstein field equation:

$$-\mathrm{K}\hat{T}_{\mathrm{MN}} = \hat{R}_{\mathrm{MN}} - \tfrac{1}{2}G_{\mathrm{MN}}\hat{R} \tag{10.2}$$

with a suitably-dimensioned constant $\mathrm{K}$ related to the usual $\kappa$ to be discussed momentarily. This provides the basis for studying the field dynamics and energy tensors of the DKK geometry.

The development already presented here, should make plain that the Kaluza miracle will also be undiminished when the DKK metric tensors (3.13) and (4.22) are used in (10.2) in lieu of the usual Kaluza-Klein (1.1). Because $\tilde{\Gamma}^{\mu}_{\alpha 5}$ which we write as $\tilde{\Gamma}^{\mu}_{\alpha 5} = \tfrac{1}{2}\eta^{\mu\sigma}\phi^2 kF_{\alpha\sigma} + \ldots$ contains the electromagnetic field strength as first established at (7.13), we may be comfortable that the terms needed in the Maxwell tensor will be embedded in the (10.1) terms housed originally in $\Gamma^{\mathrm{A}}{}_{\mathrm{M\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{NB}} - \Gamma^{\mathrm{A}}{}_{\mathrm{N\Sigma}}\Gamma^{\Sigma}{}_{\mathrm{MB}}$. Moreover, because the electromagnetic source current density $\mu_0 J^{\mu} = \partial_{\sigma}F^{\sigma\mu}$, we may also be comfortable that Maxwell's source equation will be embedded in the terms housed originally in $\partial_{\mathrm{M}}\Gamma^{\mathrm{A}}{}_{\mathrm{NB}} - \partial_{\mathrm{N}}\Gamma^{\mathrm{A}}{}_{\mathrm{MB}}$. Moreover, because $\left(\hat{R}^{\mathrm{MN}} - \tfrac{1}{2}G^{\mathrm{MN}}\hat{R}\right)_{;\mathrm{M}} = 0$ which via (10.2) ensures a locally-conserved energy $\hat{T}^{\mathrm{MN}}{}_{;\mathrm{M}} = 0$ is contracted from the second Bianchi identity $\hat{R}^{\mathrm{A}}_{\mathrm{BMN;P}} + \hat{R}^{\mathrm{A}}_{\mathrm{BNP;M}} + \hat{R}^{\mathrm{A}}_{\mathrm{BPM;N}} = 0$, we may also be comfortable that Maxwell's magnetic charge equation $\partial_{;\alpha}F_{\mu\nu} + \partial_{;\mu}F_{\nu\alpha} + \partial_{;\nu}F_{\alpha\mu} = 0$ will likewise be embedded. In short, we may be comfortable based on what has already been developed here, that the Kaluza miracle will remain intact once field equations are calculated. But we should expect some additional terms and information emerging from the field equation which do not appear when we use the usual (1.1).

Second, the Ricci scalar $\hat{R}$ is especially important because of the role it plays in the Einstein-Hilbert Action. This action provides a very direct understanding of the view that the fifth dimension is a matter dimension [18], and because this action can be used to calculate five-dimensional gravitational path integrals which may be of assistance in better understanding the nature of the second time dimension $t^5$. Let us briefly preview these development paths.

The Einstein-Hilbert action reviewed for example in [21], in four dimensions, is given by:

$$S = \int \left( (1/2\kappa) R + \mathcal{L}_{\mathrm{M}} \right) \sqrt{-g}\, \mathrm{d}^4 x \ . \tag{10.3}$$

The derivation of the four-dimensional (10.2) from this is well-known, where $R = R^{\sigma}{}_{\sigma}$. So, in five dimensions, we immediately expect that (10.2) will emerge from extending (10.3) to:

$$\hat{S} = \int \left( (1/2\mathrm{K}) \hat{R} + \hat{\mathcal{L}}_{\mathrm{M}} \right) \sqrt{-G} \mathrm{d}^5 x = \int \left( (1/2\mathrm{K}) \hat{R}^{\sigma}{}_{\sigma} + (1/2\mathrm{K}) \hat{R}^{5}{}_{5} + \hat{\mathcal{L}}_{\mathrm{M}} \right) \sqrt{-G} \mathrm{d}^5 x \ , \tag{10.4}$$

using $\hat{R} = \hat{R}^{\sigma}{}_{\sigma} + \hat{R}^{5}{}_{5}$ from (10.1) and the $G$ already obtained in (6.3), and where $\mathrm{K} \equiv \lambda\kappa$ contains some suitable length $\lambda$ to balance the extra space dimensionality in $\mathrm{d}^5 x$ versus $\mathrm{d}^4 x$. In Kaluza-Klein theory based on (1.1) $\lambda$ is normally the radius of the compactified fourth space dimension and is very small. Here, because there is a second time dimension, this should become associated with some equally-suitable period of time=length/*c*, but it may not necessarily be small if it is associated, for example, with the reduced wavelength $\lambda\!\!\!^{-} = c / \omega$ of the scalar deduced in (8.10), and if that wavelength is fairly large which is likely because these scalars $\phi$ have not exactly overwhelmed the detectors in anybody's particle accelerators or cosmological observatories.

However, the energy tensor $T^{\mu\nu}$ in four dimensions is placed into the Einstein equation by hand. This is why Einstein characterized the $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ side of his field equation as "marble" and the $-\kappa T^{\mu\nu}$ side as "wood." And this $T^{\mu\nu}$ is defined from the Lagrangian density of matter by:

$$T_{\mu\nu} \equiv -2 \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_{\mathrm{M}} \ . \tag{10.5}$$

Therefore, in the Five-Dimensional Space-Time-Matter view of [18], and referring to (10.4), the "wood" of $\hat{\mathcal{L}}_{\mathrm{M}}$ is discarded entirely, and rather, we associate

$$\hat{\mathcal{L}}_{\mathrm{M}} \equiv (1/2\mathrm{K}) \hat{R}^{5}{}_{5} \tag{10.6}$$

with the matter Lagrangian density. As a result, this is now also made of "marble."

Then (10.4) may be simplified to the 5-dimensional "vacuum" equation (see [22] at 428 and 429):

$$\hat{S} = \int (1/2\mathrm{K}) \hat{R} \sqrt{-G} \mathrm{d}^5 x \,, \tag{10.7}$$

and the field equation (10.2) derived from varying (10.7) becomes the vacuum equation:

$$0 = \hat{R}_{\mathrm{MN}} - \tfrac{1}{2} G_{\mathrm{MN}} \hat{R} \,. \tag{10.8}$$

And we anticipate that the variation itself will produce the usual relation:

$$\hat{R}_{\mathrm{MN}}=\frac{\delta\hat{R}}{\delta g^{\mathrm{MN}}}=\frac{\delta\hat{R}^{\sigma}{}_{\sigma}}{\delta g^{\mathrm{MN}}}+\frac{\delta\hat{R}^{5}{}_{5}}{\delta g^{\mathrm{MN}}} \tag{10.9}$$

for the Ricci tensor, but now in five dimensions.

So, in view of (10.5) and (10.6), what we ordinarily think of as the energy tensor – which is now made of entirely geometric "marble," – is contained in those *components of (10.8)* which, also in view of (10.9) and $\hat{R}=\hat{R}^{\sigma}{}_{\sigma}+\hat{R}^{5}{}_{5}$, and given the zero of the vacuum in (10.8), are in:

$$-\mathrm{K}\hat{T}_{\mathrm{MN}}=\frac{\delta\hat{R}^{5}{}_{5}}{\delta G^{\mathrm{MN}}}-\frac{1}{2}G_{\mathrm{MN}}\hat{R}^{5}{}_{5}=\hat{R}_{\mathrm{MN}}-\frac{1}{2}G_{\mathrm{MN}}\hat{R}-\frac{\delta\hat{R}^{\sigma}{}_{\sigma}}{\delta g^{\mathrm{MN}}}+\frac{1}{2}G_{\mathrm{MN}}\hat{R}^{\sigma}{}_{\sigma}=-\frac{\delta\hat{R}^{\sigma}{}_{\sigma}}{\delta g^{\mathrm{MN}}}+\frac{1}{2}G_{\mathrm{MN}}\hat{R}^{\sigma}{}_{\sigma}. \tag{10.10}$$

In four dimensions, the salient part of the above now becomes (note sign flip):

$$\mathrm{K}\hat{T}_{\mu\nu}=\frac{\delta\hat{R}^{\sigma}{}_{\sigma}}{\delta g^{\mu\nu}}-\frac{1}{2}G_{\mu\nu}\hat{R}^{\sigma}{}_{\sigma}. \tag{10.11}$$

We then look for geometrically-rooted energy tensors that emerge in (10.8) and (10.11) using (10.1) which contain field configurations which up to multiplicative coefficients, resemble the Maxwell tensor, the tensors for dust, perfect fluids, and the like, which is all part of the Kaluza miracle. And because $T^{00}$ is an energy-density, and because the integral of this over a three-dimensional space volume is an energy which divided by $c^2$ is a mass, from this view we see how the fifth dimension really is responsible for creating matter out of geometric "marble" rather than hand-introduced "wood."

In a similar regard, one of the most important outstanding problems in particle physics, is how to introduce fermion rest masses theoretically rather than by hand, and hopefully thereby explain why the fermions have the observed masses that they do. Here, just as the five spacetime dimensions introduce a "marble" energy tensor (10.11), we may anticipate that when the five-dimensional Dirac equation (5.6) is fully developed, there will appear amidst its Lagrangian density terms a fermion rest energy term $m'c^2\overline{\Psi}\Psi$ in which the $mc^2$ in (5.6) is occupied, not by a hand-added "wood" mass, but by some energy-dimensioned scalar number which emerges entirely from the five dimensional geometry. In this event, just as we discarded $\hat{\mathcal{L}}_{\mathrm{M}}$ in (10.4) and replaced it with $(1/2\mathrm{K})\hat{R}^{5}{}_{5}$ at (10.6) to arrive at (10.7) and (10.8), we would discard the $mc^2$ in (5.6), change (5.6) to $i\hbar c\Gamma^{\mathrm{M}}\partial_{\mathrm{M}}\Psi=0$ without any hand-added "wood" mass, and in its place use the $m'c^2$ emergent from the geometry in the $m'c^2\overline{\Psi}\Psi$ terms.

Third, the action $\hat{S}=\int(1/2\kappa)\hat{R}\sqrt{-G}\mathrm{d}^5x$, like any action, is directly used in the quantum field path integral, which using (10.7) is:

$$Z=\int DG_{\mathrm{MN}}\exp\left(i\hat{S}/\hbar\right)=\int DG_{\mathrm{MN}}\exp\left((i/\hbar)\int(1/2\mathrm{K})\hat{R}\sqrt{-G}\mathrm{d}^5x\right). \tag{10.12}$$

Here, the only field over which the integration needs to take place is $G_{\mathrm{MN}}$, because this contains not only the usual $g_{\mu\nu}$, but also the photon $A_{\gamma\mu}$ and the scalar $\phi$. But aside from the direct value of (10.12) in finally quantizing gravity, one of the deeply-interesting epistemological issues raised by path integration, relates to the meaning of the fifth time dimension – not only as the matter dimension just reviewed – but also as an actual second dimension of time.

For example, Feynman's original formulation of path integration considers the multiple paths that an individual field quantum might take to get from a source point A to a detection point B, *in a given time*. And starting with Feynman-Stueckelberg it became understood that negative energy particles traversing forward in time may be interpreted as positive energy antiparticles moving backward through time. But with a second time dimension $t^5$, the path integral must now take into account *all of the possible paths through time* that the particle may have taken, which are no longer just forward and backward, but also *sideways* through what is now a *time plane*. Now, the time $t^0$ that we actually observe may well become associated with *the actual path taken through time* from amidst multiple time travel possibilities each with their own probability amplitudes, and $t^5$ may become associated with alternative paths not taken. If one has a deterministic view of nature, then of course the only reality rests with events which did occur, while events which may have occurred but did not have no meaning. But if one has a non-deterministic view of nature, then having a second time dimension to account for all the paths through time *which were not taken* makes eminent sense, and certainly makes much more intuitive and experiential sense than curling up a space dimension into a tiny loop. And if path integral calculations should end up providing a scientific foundation for the physical reality of time paths which could have occurred but never did, this could deeply affect human viewpoints of life and nature. So, while the thoughts just stated are highly preliminary, one would anticipate that a detailed analysis of path integration when there is a second time dimension may help us gain further insight into the physical nature of the fifth dimension as a time dimension, in addition to how this dimension may be utilized to turn the energy tensor from "wood" into "marble."

Finally, Kaluza-Klein theory only unifies gravitation and electromagnetism. As noted in the introduction, weak and strong interactions, and electroweak unification, were barely a glimmer a century ago when Kaluza first passed his new theory along to Einstein in 1919. This raises the question whether Kaluza-Klein theory "repaired" to be compatible with Dirac theory using the DKK metric tensor (3.13) and its inverse (4.22) might also provide the foundation for all-interaction unification to include the weak and strong interactions in addition to gravitation and electromagnetism.

In ordinary four-dimensional gravitational theory, the metric tensor only contains gravitational fields $g_{\mu\nu}$. The addition of a Kaluza Klein fifth dimension adds a spin one vector gauge potential $A_\mu$ as well as a spin 0 scaler $\phi$ to the metric tensor as seen in (1.1). The former becomes the luminous $A_{\gamma\mu}$ of (2.11) and the latter becomes the luminous $\phi_{AB}$ of (8.10) for the DKK metric tensor (3.13) and inverse (4.22). So, it may be thought that if adding an extra dimension can unify gravitation with electromagnetism, adding additional dimensions beyond the fifth might bring in the other interactions as well. This has been one of the motivations for string theory in higher dimensions, which are then compactified down to the observed four space dimensions. But these higher-dimensional theories invariably regard the extra dimensions to be *spacelike* dimensions curled up into tiny loops just like the spacelike fifth dimension in Kaluza Klein. And as we have shown here, the spacelike character of this fifth dimension is needed to compensate for the singularity of the metric tensor when $\phi \to 0$ which is one of the most serious KK problems repaired by DKK. Specifically, when Kaluza-Klein is repaired by being made compatible with Dirac theory, the fifth dimension instead becomes a second *timelike* rather than a fourth spacelike dimension. So, if the curled-up spacelike dimension is actually a flaw in the original Kaluza-Klein theory because it is based on a metric degeneracy which can be and is cured by enforcing compatibility with Dirac theory over all five dimensions, it appears to make little sense to replicate this flaw into additional spacelike dimensions.

Perhaps the more fruitful path is to recognize, as is well-established, that weak and strong interactions are very similar to electromagnetic interactions insofar as all three are all mediated by spin-1 bosons in contrast to gravitation which is mediated by spin-2 gravitons. The only salient difference among the three spin-1 mediated interactions is that weak and strong interactions employ SU(2) and SU(3) Yang-Mills [23] internal symmetry gauge groups in which the gauge fields are non-commuting and may gain an extra degree of freedom and thus a rest mass by symmetry breaking, versus the commuting U(1) group of electromagnetism. Moreover, Yang-Mills theories have been extraordinarily successful describing observed particle and interaction phenomenology. So, it would appear more likely than not that once we have a U(1) gauge field with only the two photon degrees of freedom integrated into the metric tensor in five dimensions as is the case for the DKK metric tensors (3.13) and inverse (4.22), it is unnecessary to add any additional dimensions in order to pick up the phenomenology of weak and strong interactions. Rather, one simply generalizes abelian electromagnetic gauge theory to non-abelian Yang-Mills gauge theory in the usual way, all within the context of the DKK metric tensors (3.13) and inverse (4.22) and the geodesic equation of motion and Einstein equation machinery that goes along with them. Then the trick is to pick the right gauge group, the right particle representations, and the right method of symmetry breaking.

So from this line of approach, it seems as though one would first regard the U(1) gauge fields $A_{\gamma\mu}$ which are already part of the five dimensional DKK metric tensor (3.13), as non-abelian SU(N) gauge fields $G_\mu = T^i G^i{}_\mu$ with internal symmetry established by the group generators $T^i$ which have a commutation relation $\left[T^i, T^j\right] = if^{ijk}T^k$ with group structure constants $f^{ijk}$. Prior to any symmetry breaking each gauge field would have only two degrees of freedom and so be massless and luminous just like the photon because this constraint naturally emerges from (2.10). Then, starting with the metric tensor (3.13), one would replace $A_{\gamma\mu} \mapsto G_{\gamma\mu} = T^i G^i{}_{\gamma\mu}$ everywhere this field appears (with $\gamma$ now understood to denote, not a photon, but another luminous field quantum), then re-symmetrize the metric tensor by replacing $G_{\gamma\mu}G_{\gamma\nu} \mapsto \tfrac{1}{2}\left\{G_{\gamma\mu}, G_{\gamma\nu}\right\}$ because these fields are now non-commuting. Then – at the risk of understating what is still a highly nontrivial problem – all we need do is discover the correct Yang-Mills GUT gauge group to use for these $G_{\gamma\mu}$, discover what particles are associated with various representations of this group, discover the particular way or ways in which the symmetry of this GUT group is broken and at what energy stages including how to add an extra degree of freedom to some of these $G_{\gamma\mu}$ or combinations of them to give them a mass such as is required for the weak W and Z bosons, discover the origin of the chiral asymmetries observed in nature such as those of the weak interactions, discover how the observed fermion phenomenology becomes replicated into three fermion generations, discover how to produce the observed $G \supset SU(3)_C \times SU(2)_W \times U(1)_{em}$ phenomenology observed at low energies, and discover the emergence during symmetry breaking of the observed baryons and mesons of hadronic physics, including protons and neutrons with three confined quarks. How do we do this?

There have been many GUT theories proposed since 1954 when Yang-Mills theory was first developed, and the correct choice amongst these theories is still on open question. As an example, in an earlier paper [24] the author did address these questions using a $G = SU(8)$ GUT group in which the up and down quarks with three colors each and the electron and neutrino leptons form the 8 components of an octuplet $(\nu,(u_R,d_G,d_B),e,(d_R,u_G,u_B))$ in the fundamental representation of SU(8), with $(u_R,d_G,d_B)$ having the quark content of a neutron and $(d_R,u_G,u_B)$ the quark content of a proton. Through three stages of symmetry breaking at the Planck energy, at a GUT energy, and at the Fermi vev energy, this was shown to settle into the observed $SU(3)_C \times SU(2)_W \times U(1)_{em}$ low-energy phenomenology including the condensing of the quark triplets into protons and neutrons, the replication of fermions into three generations, the chiral asymmetry of weak interactions, and the Cabibbo mixing of the left-chiral projections of those generations. As precursor to this SU(8) GUT group, in [25] and [26], based on [27], it was shown that the nuclear binding energies of fifteen distinct nuclides, namely $^2$H, $^3$H, $^3$He, $^4$He, $^6$Li, $^7$Li, $^7$Be, $^8$Be, $^{10}$B, $^9$Be, $^{10}$Be, $^{11}$B, $^{11}$C, $^{12}$C and $^{14}$N, are genomic "fingerprints" which can be used to establish "current quark" masses for the up and down quarks to better than 1 part in $10^5$ and in some cases $10^6$ *for all fifteen nuclides*, entirely independently of the renormalization scheme that one might otherwise use to characterize current quark masses. This is because one does not need to probe the nucleus at all to ascertain quark masses, but merely needs to decode the mass defects, alternatively nuclide weights, which are well-known with great precision and are independent of observational methodology. Then, in [7.6] of [28], the quark masses so-established by decoding the fingerprints of the light nucleon mass defects, in turn, were used to retrodict the observed masses of the proton and neutron as a function of only these up and down quark masses and the Fermi vev and a determinant of the CKM mixing matrix, within all experimental errors for all of these input and output parameters, *based directly on the SU(8) GUT group and particle representation and symmetry breaking cascade of [24]*. So if one were to utilize the author's example of a GUT, the $A_{\gamma\mu} \mapsto G_{\gamma\mu} = T^i G^i{}_{\gamma\mu}$ in the DKK metric (3.13) would be regarded to have an SU(8) symmetry with the foregoing octuplet in its fundamental representation. Then one would work through the same symmetry breaking cascade, but now also having available the equation of motion (7.2) and the Einstein equation (10.8) so that the motion for all interactions is strictly geodesic motion and the field dynamics and energy tensors are at bottom strictly geometrodynamic and fully gravitational.

In 2019, the scientific community will celebrate the centennial of Kaluza-Klein theory. Throughout this entire century, Kaluza-Klein theory has been hotly debated and has had its staunch supporters and its highly-critical detractors. And both are entirely justified. The miracle of geometrizing Maxwell's electrodynamics and the Lorentz motion and the Maxwell stress-energy tensors in a theory which is unified with gravitation and turns Einstein's "wood" tensor into the "marble" of geometry is tremendously attractive. But a theory which is rooted in a degenerate metric tensor with a singular inverse and a scalar field which carries the entire new dimension on its shoulders and which contains an impossible-to-observe curled up fourth space dimension, not to mention a structural incompatibility with Dirac theory and thus no ability to account for fermion phenomenology, is deeply troubling.

By using Dirac theory itself to force five-dimensional general covariance upon Kaluza-Klein theory and cure all of these troubles while retaining all the Kaluza miracles and naturally and covariantly breaking the symmetry of the gauge fields by removing two degrees of freedom and thereby turning classical fields into quantum fields, to uncover additional new knowledge about our physical universe in the process, and to possibly lay the foundation for all-interaction unification, we deeply honor the work and aspirations of our physicist forebears toward a unified geometrodynamic understanding of nature as the Kaluza-Klein centennial approaches.

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