Grounding the Fifth Dimension from Kaluza-Klein Theory in Observed Physical Reality, Using Dirac's Quantum Theory of the Electron to Ensure Five-Dimensional General Covariance

Jay R. Yablon, June 6, 2018

Abstract: We require all the components of the Kaluza-Klein metric tensor to be generally-covariant across all five dimensions by deconstructing the metric tensor into Dirac-type square root operators. This decouples the fifth dimension from the Kaluza-Klein scalar, makes this dimension timelike not spacelike, makes the metric tensor inverse non-singular, covariantly reveals the quantum fields of the photon, makes Kaluza-Klein fully compatible with Dirac theory, and roots this fifth dimension in the physical reality of the chiral, pseudo-scalar and pseudo-vector particles abundantly observed in particle physics based on Dirac's gamma-5 operator, thereby "fixing" all of the most perplexing problem in Kaluza-Klein theory. Albeit with additional new dynamics expected, all of the benefits of Kaluza-Klein theory are retained, insofar as providing a geometrodynamic foundation for Maxwell's equations, the Lorentz Force motion and the Maxwell-Stress energy tensor, and insofar as supporting the viewpoint that the fifth dimension is, at bottom, the matter dimension.

Contents

1. Introduction

About a century ago, with the 1920s approaching, much of the physics community was trying to understand the quantum reality that Planck had first uncovered almost two decades prior [1]. But with the General Theory of Relativity [2] having recently placed gravitation and the dynamical behavior of gravitating objects onto an entirely geometric and geodesic foundation (which several decades later Wheeler would dub "geometrodynamics" [3]), a few scientists were moving on to try to scale the next logical hill, which – with weak and strong interactions not yet known – was to obtain a geometrodynamic theory of electromagnetism. Besides Einstein's own work on this which continued for the rest of his life [4], the two most notable efforts were those of Hermann Weyl [5], [6] who was just starting to develop his U(1) gauge theory in four dimensions (which turned out to be a theory of "phase" invariance [7] that still retains the original moniker "gauge"), and Kaluza [8] then Klein [9], [10] who quite successfully used a fifth dimension to geometrize the Lorentz Force motion and the Maxwell Stress-Energy tensor (see, e.g., [11] and [12]). This is a very attractive aspect of Kaluza-Klein theory, and it remains so because even today, despite almost a century of efforts to do so, U(1) gauge theory has not yet successfully been able to place the Lorentz Force dynamics and the Maxwell Stress Energy on an entirely geometrodynamic foundation. And as will be appreciated by anyone who has studied this problem seriously, that it is the inequivalence of electrical mass (a.k.a. charge) and inertial mass which has been the prime hindrance to being able to do so.

Notwithstanding these Kaluza "miracles" of geometrizing the Lorentz motion and the Maxwell Stress-Energy, this fifth dimension and an associated scalar field known as the graviscalar or radion or dilaton, raised its own new challenges, many of which will be reviewed later in the present paper. These have been a legitimate hurdle to the widespread acceptance of Kaluza-Klein theory as a theory of what is observed in the natural world. It is important to keep this historical sequencing in mind, because Kaluza's work in particular predated what we now know to be modern gauge theory and so was the "first" geometrodynamic theory of electrodynamics, and of special interest in this paper, because Kaluza-Klein also preceded Dirac's seminal Quantum Theory of the Electron [13] which today is the foundation of how we understand fermion behavior.

Dirac's theory in particular, arose from taking an operator square root of the Minkowski metric tensor with $\mathrm{diag}\left(\eta^{\mu\nu}\right)=\left(+1,-1,-1,-1\right)$, by defining ("$\equiv$") a set of four operator matrices γ^{μ} according to the anticommutator relation $\tfrac{1}{2}\left\{\gamma^{\mu},\gamma^{\nu}\right\}=\tfrac{1}{2}\left\{\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}\right\}\equiv\eta^{\mu\nu}$. To generalize to curved spacetime thus to gravitation which employs the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ defined such that $g^{\mu\alpha}g_{\alpha\nu}\equiv\delta^{\mu}{}_{\nu}$, we define a set of Γ^{μ} with a parallel definition $\tfrac{1}{2}\left\{\Gamma^{\mu},\Gamma^{\nu}\right\}\equiv g^{\mu\nu}$. We simultaneously define a vierbein a.k.a. tetrad e^{μ}_{a} with both a superscripted Greek "spacetime / world" index and a subscripted Latin "local / Lorentz / Minkowski" index using the relation $e^{\mu}_{a}\gamma^{a}\equiv\Gamma^{\mu}$. Consequently, we deduce that $g^{\mu\nu}=\tfrac{1}{2}\left\{\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}\right\}e^{\mu}_{a}e^{\nu}_{b}=\eta^{ab}e^{\mu}_{a}e^{\nu}_{b}$. So just as the metric tensor $g^{\mu\nu}$ transforms in four-dimensional spacetime as a contravariant (upper-indexed) tensor, these deconstructed operators Γ^{μ} likewise transform in spacetime as a contravariant vector.

Now in Kaluza-Klein theory, the metric tensor which we denote by G_{MN} and its inverse G^{MN} obtained by $G^{\mathrm{MA}}G_{\mathrm{AN}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ are specified in five dimensions with an index $\mathrm{M} = 0,1,2,3,5$, and may be represented in the 2x2 matrix format:

$$G_{\mathrm{MN}} = \begin{pmatrix} g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu & \phi^2 k A_\mu \\ \phi^2 k A_\nu & \phi^2 \end{pmatrix}; \quad G^{\mathrm{MN}} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & g_{\alpha\beta} A^\alpha A^\beta + 1/\phi^2 \end{pmatrix}. \tag{1.1}$$

In the above, $g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$ transforms as a 4x4 tensor in spacetime. The components $G_{\mu 5} = \phi^2 k A_\mu$ and $G_{5\mathrm{N}} = \phi^2 k A_\nu$ transform as covariant (lower-indexed) vectors in spacetime. And the component $G_{55} = \phi^2$ transforms as a scalar in spacetime. If we regard ϕ to be a dimensionless scalar, then the constant *k* must have dimensions of charge/energy because the metric tensor is dimensionless and because the gauge field A_μ has dimensions of energy/charge.

It is important to note that when we turn off all electromagnetism by setting $A_\mu = 0$ and $\phi = 0$, G^{MN} in (1.1) becomes singular. This is indicated from the fact that in this situation $\mathrm{diag}(G_{\mathrm{MN}}) = (g_{00}, g_{11}, g_{22}, g_{33}, 0)$ with a determinant $|G_{\mathrm{MN}}| = 0$, and is seen directly from the fact that $G^{55} = g_{\alpha\beta} A^\alpha A^\beta + 1/\phi^2 = 0 + \infty$. Therefore, (1.1) relies upon ϕ or $g_{\alpha\beta} A^\alpha A^\beta$ being non-zero to avoid singularity, not to mention that $G_{55} = 0$ disappears entirely when $\phi = 0$ and $A_\mu = 0$.

Following identifying the Maxwell tensor in the Kaluza-Klein field equation, this constant *k* is found to be:

$$\frac{k^2}{2} \equiv \frac{2G}{c^4} 4\pi\epsilon_0 = \frac{2}{c^4}\frac{G}{k_e}, \tag{1.2}$$

where $k_e = 1/4\pi\varepsilon_0 = \mu_0 c^2 / 4\pi$ is Coulomb's constant and *G* is Newton's gravitational constant.

One might presume in view of Dirac theory, that the five-dimensional G_{MN} and G^{MN} can be likewise deconstructed into square root operators using the anticommutator relations:

$$\tfrac{1}{2}\{\Gamma_{\mathrm{M}}, \Gamma_{\mathrm{N}}\} = \tfrac{1}{2}\{\Gamma_{\mathrm{M}}\Gamma_{\mathrm{N}} + \Gamma_{\mathrm{N}}\Gamma_{\mathrm{M}}\} \equiv G_{\mathrm{MN}}; \quad \tfrac{1}{2}\{\Gamma^{\mathrm{M}}, \Gamma^{\mathrm{N}}\} = \tfrac{1}{2}\{\Gamma^{\mathrm{M}}\Gamma^{\mathrm{N}} + \Gamma^{\mathrm{N}}\Gamma^{\mathrm{M}}\} \equiv G^{\mathrm{MN}}, \tag{1.3}$$

where Γ_{M} and Γ^{M} transform as *five-dimensional vectors* in five-dimensional spacetime. This would presumably include a five-dimensional definition $\varepsilon^{\mathrm{M}}_{A}\gamma^{A} \equiv \Gamma^{\mathrm{M}}$ for a tetrad $\varepsilon^{\mathrm{M}}_{A}$, where $\mathrm{M} = 0,1,2,3,5$ is a world index and $A = 0,1,2,3,5$ is a local index, and where γ^5 is a fifth operator matrix which may or may not be associated with Dirac's $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, depending upon the detailed mathematical calculations which determine this γ^5.

However, as we shall now demonstrate, the Kaluza-Klein metric tensors in (1.1) *cannot* be deconstructed into Γ_{M} and Γ^{M} in the manner of (1.3) without modification to their $G_{05}=G_{50}$ and G_{55} components, and without imposing certain constraints on the gauge fields A^{μ} which remove two degrees of freedom and fix the gauge of these fields to that of a photon. We represent these latter constraints by $A^{\mu}=A_{\gamma}{}^{\mu}$ with the subscripted γ which is *not a spacetime index*, but which rather denotes a photon. This means that in fact, in view of Dirac theory which was developed afterwards, the Kaluza-Klein metric tensors (1.3) are really *not* generally-covariant in five dimensions. Rather, they only have a four-dimensional spacetime covariance represented in the components $G_{\mu\nu}=g_{\mu\nu}+\phi^2k^2A_{\mu}A_{\nu}$ and $G^{\mu\nu}=g^{\mu\nu}$, and in $G_{\mu 5}=\phi^2kA_{\mu}$ and $G^{\mu 5}=-A^{\mu}$, which are all patched together with fifth-dimensional components that are not generally-covariant with the components in the four spacetime dimensions.

In today's era when the General Theory of Relativity [2] is now a few years past its centenary, and where at least in classical field theory general covariance is firmly-established as a required principle for the laws of nature, it would seem essential that any theory of nature which purports to operate in five dimensions which include the four dimensions of spacetime, ought to manifest general covariance *across all five dimensions*. Accordingly, it is necessary to "fix" Kaluza-Klein theory to make certain that it adheres to such five-dimensional covariance. In so doing, many of the most nagging, century-old difficulties of Kaluza-Klein theory are immediately resolved, including those related to the scalar field in $G_{55}=\phi^2$. And of extreme importance, the Kaluza-Klein fifth dimension which has spent a century looking for direct observational grounding, may be tied directly to the clear observational physics built around the Dirac γ^5, and the multitude of observed chiral and pseudoscalar and axial vector particle states that are centered about this γ^5. All this happens without sacrificing the Kaluza "miracle" of placing electrodynamics onto a geometrodynamic footing. This is what will now be demonstrated.

2. The Kaluza-Klein Tetrad and Dirac Operators in Four Dimensional Spacetime, and the Covariant Fixing of Gauge Fields to the Photon

The first step to ensure that Kaluza-Klein theory is covariant in five dimensions using the operator deconstruction (1.3), is to obtain the four-dimensional deconstruction:

$$\tfrac{1}{2}\left\{\Gamma_{\mu},\Gamma_{\nu}\right\}=\tfrac{1}{2}\left\{\Gamma_{\mu}\Gamma_{\nu}+\Gamma_{\nu}\Gamma_{\mu}\right\}=\tfrac{1}{2}\varepsilon_{\mu a}\varepsilon_{\nu b}\left\{\gamma^a\gamma^b+\gamma^b\gamma^a\right\}=\eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b}\equiv G_{\mu\nu}=g_{\mu\nu}+\phi^2k^2A_{\mu}A_{\nu} \tag{2.1}$$

using a four-dimensional tetrad $\varepsilon_{\mu a}$ defined by $\varepsilon_{\mu a}\gamma^a\equiv\Gamma_{\mu}$, where $\mu=0,1,2,3$ is a world spacetime index raised and lowered with $G^{\mu\nu}$ and $G_{\mu\nu}$, and $a=0,1,2,3$ is a local Minkowski spacetime index raised and lowered with η^{ab} and η_{ab}. To simplify calculation, we will work in "flat" spacetime where $g_{\mu\nu}=\eta_{\mu\nu}$, and later on will generalize back from $\eta_{\mu\nu}\mapsto g_{\mu\nu}$. In the circumstance where $g_{\mu\nu}=\eta_{\mu\nu}$ and so $G_{\mu\nu}=\eta_{\mu\nu}+\phi^2k^2A_{\mu}A_{\nu}$, "flat" spacetime means a spacetime that is flat *except for* the curvature in $G_{\mu\nu}$ brought about by the electrodynamic terms $\phi^2k^2A_{\mu}A_{\nu}$.

If we further define an $\varepsilon'_{\mu a}$ according to $\delta_{\mu a} + \varepsilon'_{\mu a} \equiv \varepsilon_{\mu a}$ to represent the degree to which $\varepsilon_{\mu a}$ differs from the unit matrix $\delta_{\mu a}$, we may now write the salient portion of (2.1) as:

$$\begin{aligned} \eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b} &= \eta^{ab}\left(\delta_{\mu a} + \varepsilon'_{\mu a}\right)\left(\delta_{\nu b} + \varepsilon'_{\nu b}\right) = \eta^{ab}\delta_{\mu a}\delta_{\nu b} + \delta_{\nu b}\eta^{ab}\varepsilon'_{\mu a} + \delta_{\mu a}\eta^{ab}\varepsilon'_{\nu b} + \eta^{ab}\varepsilon'_{\mu a}\varepsilon'_{\nu b} \\ &= \eta_{\mu\nu} + \eta_{a\nu}\varepsilon'^{\ a}_{\mu} + \eta_{\mu b}\varepsilon'^{b}_{\nu} + \eta_{ab}\varepsilon'^{\ a}_{\mu}\varepsilon'^{b}_{\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu \end{aligned} . \tag{2.2}$$

Note that when electrodynamics is "turned off" by setting A_μ and / or by setting $\phi = 0$ this reduces to $\eta^{ab}\varepsilon_{\mu a}\varepsilon_{\nu b} = \eta_{\mu\nu}$ which is solved by the tetrad becoming a unit matrix, $\varepsilon_{\mu a} = \delta_{\mu a}$. Subtracting $\eta_{\mu\nu}$ from each side, we now need to solve:

$$\eta_{a\nu}\varepsilon'^{\ a}_{\mu} + \eta_{\mu b}\varepsilon'^{b}_{\nu} + \eta_{ab}\varepsilon'^{\ a}_{\mu}\varepsilon'^{b}_{\nu} = \phi^2 k^2 A_\mu A_\nu . \tag{2.3}$$

The above contains sixteen (16) equations for each of $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$. But, this is symmetric in μ and ν so in fact there are only ten (10) independent equations. Given that $\mathrm{diag}\left(\eta^{yz}\right) = (1,-1,-1,-1)$, the four $\mu = \nu$ "diagonal" equations in (2.3) produce the relations:

$$\begin{aligned} \eta_{a0}\varepsilon'^{a}_{0} + \eta_{0b}\varepsilon'^{b}_{0} + \eta_{ab}\varepsilon'^{a}_{0}\varepsilon'^{b}_{0} &= 2\varepsilon'^{0}_{0} + \varepsilon'^{0}_{0}\varepsilon'^{0}_{0} - \varepsilon'^{1}_{0}\varepsilon'^{1}_{0} - \varepsilon'^{2}_{0}\varepsilon'^{2}_{0} - \varepsilon'^{3}_{0}\varepsilon'^{3}_{0} = \phi^2 k^2 A_0 A_0 \\ \eta_{a1}\varepsilon'^{a}_{1} + \eta_{1b}\varepsilon'^{b}_{1} + \eta_{ab}\varepsilon'^{a}_{1}\varepsilon'^{b}_{1} &= -2\varepsilon'^{1}_{1} + \varepsilon'^{0}_{1}\varepsilon'^{0}_{1} - \varepsilon'^{1}_{1}\varepsilon'^{1}_{1} - \varepsilon'^{2}_{1}\varepsilon'^{2}_{1} - \varepsilon'^{3}_{1}\varepsilon'^{3}_{1} = \phi^2 k^2 A_1 A_1 \\ \eta_{a2}\varepsilon'^{a}_{2} + \eta_{2b}\varepsilon'^{b}_{2} + \eta_{ab}\varepsilon'^{a}_{2}\varepsilon'^{b}_{2} &= -2\varepsilon'^{2}_{2} + \varepsilon'^{0}_{2}\varepsilon'^{0}_{2} - \varepsilon'^{1}_{2}\varepsilon'^{1}_{2} - \varepsilon'^{2}_{2}\varepsilon'^{2}_{2} - \varepsilon'^{3}_{2}\varepsilon'^{3}_{2} = \phi^2 k^2 A_2 A_2 \\ \eta_{a3}\varepsilon'^{a}_{3} + \eta_{3b}\varepsilon'^{b}_{3} + \eta_{ab}\varepsilon'^{a}_{3}\varepsilon'^{b}_{3} &= -2\varepsilon'^{3}_{3} + \varepsilon'^{0}_{3}\varepsilon'^{0}_{3} - \varepsilon'^{1}_{3}\varepsilon'^{1}_{3} - \varepsilon'^{2}_{3}\varepsilon'^{2}_{3} - \varepsilon'^{3}_{3}\varepsilon'^{3}_{3} = \phi^2 k^2 A_3 A_3 \end{aligned} . \tag{2.4a}$$

Likewise, the three $\mu = 0$, $\nu = 1,2,3$ mixed time and space relations in (2.3) are:

$$\begin{aligned} \eta_{a1}\varepsilon'^{a}_{0} + \eta_{0b}\varepsilon'^{b}_{1} + \eta_{ab}\varepsilon'^{a}_{0}\varepsilon'^{b}_{1} &= -\varepsilon'^{1}_{0} + \varepsilon'^{0}_{1} + \varepsilon'^{0}_{0}\varepsilon'^{0}_{1} - \varepsilon'^{1}_{0}\varepsilon'^{1}_{1} - \varepsilon'^{2}_{0}\varepsilon'^{2}_{1} - \varepsilon'^{3}_{0}\varepsilon'^{3}_{1} = \phi^2 k^2 A_0 A_1 \\ \eta_{a2}\varepsilon'^{a}_{0} + \eta_{0b}\varepsilon'^{b}_{2} + \eta_{ab}\varepsilon'^{a}_{0}\varepsilon'^{b}_{2} &= -\varepsilon'^{2}_{0} + \varepsilon'^{0}_{2} + \varepsilon'^{0}_{0}\varepsilon'^{0}_{2} - \varepsilon'^{1}_{0}\varepsilon'^{1}_{2} - \varepsilon'^{2}_{0}\varepsilon'^{2}_{2} - \varepsilon'^{3}_{0}\varepsilon'^{3}_{2} = \phi^2 k^2 A_0 A_2 \\ \eta_{a3}\varepsilon'^{a}_{0} + \eta_{0b}\varepsilon'^{b}_{3} + \eta_{ab}\varepsilon'^{a}_{0}\varepsilon'^{b}_{3} &= -\varepsilon'^{3}_{0} + \varepsilon'^{0}_{3} + \varepsilon'^{0}_{0}\varepsilon'^{0}_{3} - \varepsilon'^{1}_{0}\varepsilon'^{1}_{3} - \varepsilon'^{2}_{0}\varepsilon'^{2}_{3} - \varepsilon'^{3}_{0}\varepsilon'^{3}_{3} = \phi^2 k^2 A_0 A_3 \end{aligned} . \tag{2.4b}$$

Finally, the pure-space relations are:

$$\begin{aligned} \eta_{a2}\varepsilon'^{a}_{1} + \eta_{1b}\varepsilon'^{b}_{2} + \eta_{ab}\varepsilon'^{a}_{1}\varepsilon'^{b}_{2} &= -\varepsilon'^{2}_{1} - \varepsilon'^{1}_{2} + \varepsilon'^{0}_{1}\varepsilon'^{0}_{2} - \varepsilon'^{1}_{1}\varepsilon'^{1}_{2} - \varepsilon'^{2}_{1}\varepsilon'^{2}_{2} - \varepsilon'^{3}_{1}\varepsilon'^{3}_{2} = \phi^2 k^2 A_1 A_2 \\ \eta_{a3}\varepsilon'^{a}_{2} + \eta_{2b}\varepsilon'^{b}_{3} + \eta_{ab}\varepsilon'^{a}_{2}\varepsilon'^{b}_{3} &= -\varepsilon'^{3}_{2} - \varepsilon'^{2}_{3} + \varepsilon'^{0}_{2}\varepsilon'^{0}_{3} - \varepsilon'^{1}_{2}\varepsilon'^{1}_{3} - \varepsilon_{2}{}^{2}\varepsilon'^{2}_{3} - \varepsilon'^{3}_{2}\varepsilon'^{3}_{3} = \phi^2 k^2 A_2 A_3 \\ \eta_{a1}\varepsilon'^{a}_{3} + \eta_{3b}\varepsilon'^{b}_{1} + \eta_{ab}\varepsilon'^{a}_{3}\varepsilon'^{b}_{1} &= -\varepsilon'^{1}_{3} - \varepsilon'^{3}_{1} + \varepsilon'^{0}_{3}\varepsilon'^{0}_{1} - \varepsilon'^{1}_{3}\varepsilon'^{1}_{1} - \varepsilon'^{2}_{3}\varepsilon'^{2}_{1} - \varepsilon'^{3}_{3}\varepsilon'^{3}_{1} = \phi^2 k^2 A_3 A_1 \end{aligned} . \tag{2.4c}$$

Now, the right-hand side of all ten of (2.4) have nonlinear products $\phi^2 k^2 A_\mu A_\nu$ of field terms. On the left of each there is a mix of linear and nonlinear expressions containing the $\varepsilon'^{\ a}_{\mu}$. Our goal at the moment, therefore, is to eliminate all of the linear expressions from the left-hand sides of (2.4) to create a structural match between the left and right sides.

In (14.3a) the linear appearances are of $\varepsilon_0^{\prime 0}$, $\varepsilon_1^{\prime 1}$, $\varepsilon_2^{\prime 2}$ and $\varepsilon_3^{\prime 3}$ respectively. Noting that the complete tetrad $\varepsilon_\mu{}^a = \delta_\mu{}^a + \varepsilon'_\mu{}^a$ and that $\varepsilon_\mu{}^a = \delta_\mu{}^a$ when electrodynamics is turned off, we first require that $\varepsilon_\mu{}^a = \delta_\mu{}^a$ for the four $\mu = a$ diagonal components, and therefore, that $\varepsilon_0^{\prime 0} = \varepsilon_1^{\prime 1} = \varepsilon_2^{\prime 2} = \varepsilon_3^{\prime 3} = 0$. As a result, the fields in $\phi^2 k^2 A_\mu A_\nu$ will all appear in off-diagonal components of the tetrad. With this, (2.4a) reduces to:

$$\begin{aligned}
&-\varepsilon_0^{\prime 1}\varepsilon_0^{\prime 1} - \varepsilon_0^{\prime 2}\varepsilon_0^{\prime 2} - \varepsilon_0^{\prime 3}\varepsilon_0^{\prime 3} = \phi^2 k^2 A_0 A_0 \\
&\varepsilon_1^{\prime 0}\varepsilon_1^{\prime 0} - \varepsilon_1^{\prime 2}\varepsilon_1^{\prime 2} - \varepsilon_1^{\prime 3}\varepsilon_1^{\prime 3} = \phi^2 k^2 A_1 A_1 \\
&\varepsilon_2^{\prime 0}\varepsilon_2^{\prime 0} - \varepsilon_2^{\prime 1}\varepsilon_2^{\prime 1} - \varepsilon_2^{\prime 3}\varepsilon_2^{\prime 3} = \phi^2 k^2 A_2 A_2 \\
&\varepsilon_3^{\prime 0}\varepsilon_3^{\prime 0} - \varepsilon_3^{\prime 1}\varepsilon_3^{\prime 1} - \varepsilon_3^{\prime 2}\varepsilon_3^{\prime 2} = \phi^2 k^2 A_3 A_3
\end{aligned} \quad . \tag{2.5a}$$

In (2.4b) we achieve structural match using $\varepsilon_1^{\prime 1} = \varepsilon_2^{\prime 2} = \varepsilon_3^{\prime 3} = 0$ from above, and also by setting $\varepsilon_0^{\prime 1} = \varepsilon_1^{\prime 0}$, $\varepsilon_0^{\prime 2} = \varepsilon_2^{\prime 0}$, $\varepsilon_0^{\prime 3} = \varepsilon_3^{\prime 0}$, which is symmetric under $0 \leftrightarrow a = 1,2,3$ interchange. Therefore:

$$\begin{aligned}
&-\varepsilon_0^{\prime 2}\varepsilon_1^{\prime 2} - \varepsilon_0^{\prime 3}\varepsilon_1^{\prime 3} = \phi^2 k^2 A_0 A_1 \\
&-\varepsilon_0^{\prime 1}\varepsilon_2^{\prime 1} - \varepsilon_0^{\prime 3}\varepsilon_2^{\prime 3} = \phi^2 k^2 A_0 A_2 \\
&-\varepsilon_0^{\prime 1}\varepsilon_3^{\prime 1} - \varepsilon_0^{\prime 2}\varepsilon_3^{\prime 2} = \phi^2 k^2 A_0 A_3
\end{aligned} \quad . \tag{2.5b}$$

And in (2.4c) we use $\varepsilon_1^{\prime 1} = \varepsilon_2^{\prime 2} = \varepsilon_3^{\prime 3} = 0$ from above and also set $\varepsilon_1^{\prime 2} = -\varepsilon_2^{\prime 1}$, $\varepsilon_2^{\prime 3} = -\varepsilon_3^{\prime 2}$, $\varepsilon_3^{\prime 1} = -\varepsilon_1^{\prime 3}$ which are antisymmetric under interchange of different space indexes. Therefore, we now have:

$$\begin{aligned}
&\varepsilon_1^{\prime 0}\varepsilon_2^{\prime 0} - \varepsilon_1^{\prime 3}\varepsilon_2^{\prime 3} = \phi^2 k^2 A_1 A_2 \\
&\varepsilon_2^{\prime 0}\varepsilon_3^{\prime 0} - \varepsilon_2^{\prime 1}\varepsilon_3^{\prime 1} = \phi^2 k^2 A_2 A_3 \\
&\varepsilon_3^{\prime 0}\varepsilon_1^{\prime 0} - \varepsilon_3^{\prime 2}\varepsilon_1^{\prime 2} = \phi^2 k^2 A_3 A_1
\end{aligned} \quad . \tag{2.5c}$$

In all of (2.5), we now only have matching-structure non-linear terms on both sides.

For the next step, closely studying the space indexes in all of the above, we now make an educated guess at an assignment for the fields in $\phi^2 k^2 A_\mu A_\nu$. Specifically, also using $\varepsilon_0^{\prime 1} = \varepsilon_1^{\prime 0}$, $\varepsilon_0^{\prime 2} = \varepsilon_2^{\prime 0}$, $\varepsilon_0^{\prime 3} = \varepsilon_3^{\prime 0}$ from earlier, we now guess an assignment:

$$\varepsilon_0^{\prime 1} = \varepsilon_1^{\prime 0} = \phi k A_1; \quad \varepsilon_0^{\prime 2} = \varepsilon_2^{\prime 0} = \phi k A_2; \quad \varepsilon_0^{\prime 3} = \varepsilon_3^{\prime 0} = \phi k A_3 . \tag{2.6}$$

Because all expressions in (2.5) contain nonlinear products of the above, it is possible to have also tried using a minus sign in all of the above whereby $\varepsilon_0^{\prime 1} = \varepsilon_1^{\prime 0} = -\phi k A_1$, $\varepsilon_0^{\prime 2} = \varepsilon_2^{\prime 0} = -\phi k A_2$ and

$\varepsilon_0'^3 = \varepsilon_3'^0 = -\phi k A_3$. But absent motivation to the contrary, we employ a plus sign implicit in the above. Substituting (2.6) into all of (2.5) and reducing now yields:

$$\begin{aligned} -A_1A_1 - A_2A_2 - A_3A_3 &= A_0A_0 \\ -\varepsilon_1'^2\varepsilon_1'^2 - \varepsilon_1'^3\varepsilon_1'^3 &= 0 \\ -\varepsilon_2'^1\varepsilon_2'^1 - \varepsilon_2'^3\varepsilon_2'^3 &= 0 \\ -\varepsilon_3'^1\varepsilon_3'^1 - \varepsilon_3'^2\varepsilon_3'^2 &= 0 \end{aligned} , \tag{2.7a}$$

$$\begin{aligned} -\phi kA_2\varepsilon_1'^2 - \phi kA_3\varepsilon_1'^3 &= \phi^2 k^2 A_0 A_1 \\ -\phi kA_1\varepsilon_2'^1 - \phi kA_3\varepsilon_2'^3 &= \phi^2 k^2 A_0 A_2 , \\ -\phi kA_1\varepsilon_3'^1 - \phi kA_2\varepsilon_3'^2 &= \phi^2 k^2 A_0 A_3 \end{aligned} \tag{2.7b}$$

$$-\varepsilon_1'^3\varepsilon_2'^3 = -\varepsilon_2'^1\varepsilon_3'^1 = -\varepsilon_3'^2\varepsilon_1'^2 = 0 . \tag{2.7c}$$

Now, one way to satisfy the earlier relations $\varepsilon_1'^2 = -\varepsilon_2'^1$, $\varepsilon_2'^3 = -\varepsilon_3'^2$, $\varepsilon_3'^1 = -\varepsilon_1'^3$ used in (2.5c) as well as to satisfy (2.7c), is to set all of the pure-space components:

$$\varepsilon_1'^2 = \varepsilon_2'^1 = \varepsilon_2'^3 = \varepsilon_3'^2 = \varepsilon_3'^1 = \varepsilon_1'^3 = 0 . \tag{2.8}$$

This disposes of (2.7c) and last three relations in (2.7a), leaving only the two constraints:

$$-A_1A_1 - A_2A_2 - A_3A_3 = A_0A_0 , \tag{2.9a}$$

$$0 = \phi^2 k^2 A_0 A_1 = \phi^2 k^2 A_0 A_2 = \phi^2 k^2 A_0 A_3 . \tag{2.9b}$$

These above relations (2.9) are extremely important. In (2.9b), if *any one* of A_1, A_2 or A_3 is *not* equal to zero, then we *must* have $A_0 = 0$. With this, (2.9a) and (2.9b) together become:

$$A_0 = 0; \quad A_1A_1 + A_2A_2 + A_3A_3 = 0 , \tag{2.10}$$

These two constraints have removed two degrees of freedom from the gauge field A_μ, <u>*in a generally-covariant manner*</u>. Moreover, for the latter constraint in $A_1A_1 + A_2A_2 + A_3A_3 = 0$ to be satisfied, it is necessary that at least one of the space components be *imaginary*. For example, if $A_3 = 0$, then one way to satisfy the entirety of (2.10) is to have:

$$A_\mu = A\varepsilon_\mu \exp\left(-iq_\sigma x^\sigma / \hbar\right) , \tag{2.11a}$$

with a polarization vector

$$\varepsilon_{R,L\mu}(\hat{z}) \equiv (0 \quad \pm 1 \quad +i \quad 0)/\sqrt{2}, \tag{2.11b}$$

where A has dimensions of charge / energy to provide dimensional balance given the dimensionless $\varepsilon_{R,L\mu}$. But the foregoing is instantly-recognizable as the gauge potential $A_\mu = A_{\gamma\mu}$ for an individual photon with two helicity states propagating along the z axis, having an energy-momentum vector:

$$cq^\mu(\hat{z}) = (E \quad 0 \quad 0 \quad cq_z) = (h\nu \quad 0 \quad 0 \quad h\nu) \tag{2.11c}$$

which satisfies $q_\mu q^\mu = 0$ and so makes this a massless field quantum.

In short, what we have ascertained in (2.10) and (2.11) is that if the spacetime components $G_{\mu\nu} = g_{\mu\nu} + \phi^2 k^2 A_\mu A_\nu$ of the Kaluza-Klein metric tensor with $g_{\mu\nu} = \eta_{\mu\nu}$ are to produce a set of Γ_μ satisfying the Dirac anticommutator relation $\frac{1}{2}\{\Gamma_\mu, \Gamma_\nu\} \equiv G_{\mu\nu}$, *the gauge symmetry of* A_μ *must be broken to correspond with that of the photon,* $A_\mu = A_{\gamma\mu}$. *The very act of deconstructing* $G_{\mu\nu}$ *into square root operators covariantly removes two degrees of freedom from the gauge field and forces it to become a photon field quantum.*

Also, we now have all of the components of the tetrad $\varepsilon_\mu{}^a = \delta_\mu{}^a + \varepsilon'_\mu{}^a$. Pulling together all of $\varepsilon'^0_0 = \varepsilon'^1_1 = \varepsilon'^2_2 = \varepsilon'^3_3 = 0$ together with (2.6) and (2.8), and setting $A_\mu = A_{\gamma\mu}$ to incorporate the pivotal finding in (2.10), (2.11), we have now deduced the tetrad to be:

$$\varepsilon_\mu{}^a = \delta_\mu{}^a + \varepsilon'_\mu{}^a = \begin{pmatrix} 1 & \phi k A_{\gamma 1} & \phi k A_{\gamma 2} & \phi k A_{\gamma 3} \\ \phi k A_{\gamma 1} & 1 & 0 & 0 \\ \phi k A_{\gamma 2} & 0 & 1 & 0 \\ \phi k A_{\gamma 3} & 0 & 0 & 1 \end{pmatrix} \tag{2.12}$$

Finally, because $\varepsilon_{\mu a}\gamma^a = \varepsilon_\mu{}^\alpha \gamma_\alpha \equiv \Gamma_\mu$, we may use (2.12) to deduce that the Dirac operators:

$$\begin{aligned} \Gamma_0 &= \varepsilon_0{}^\alpha \gamma_\alpha = \varepsilon_0{}^0\gamma_0 + \varepsilon_0{}^1\gamma_1 + \varepsilon_0{}^2\gamma_2 + \varepsilon_0{}^3\gamma_3 = \gamma_0 + \phi k A_{\gamma j}\gamma_j \\ \Gamma_1 &= \varepsilon_1{}^\alpha \gamma_\alpha = \varepsilon_1{}^0\gamma_0 + \varepsilon_1{}^1\gamma_1 = \gamma_1 + \phi k A_{\gamma 1}\gamma_0 \\ \Gamma_2 &= \varepsilon_2{}^\alpha \gamma_\alpha = \varepsilon_2{}^0\gamma_0 + \varepsilon_2{}^2\gamma_2 = \gamma_2 + \phi k A_{\gamma 2}\gamma_0 \\ \Gamma_3 &= \varepsilon_3{}^\alpha \gamma_\alpha = \varepsilon_3{}^0\gamma_0 + \varepsilon_3{}^3\gamma_3 = \gamma_3 + \phi k A_{\gamma 3}\gamma_0 \end{aligned}, \tag{2.13}$$

which consolidates into:

$$\Gamma_\mu = (\gamma_0 + \phi k A_{\gamma j}\gamma_j \quad \gamma_j + \phi k A_{\gamma j}\gamma_0). \tag{2.14}$$

It is a useful exercise to confirm that (2.14) above, inserted into (2.1), will produce $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$, which may then be generalized from $\eta_{\mu\nu} \mapsto g_{\mu\nu}$ in the usual way by applying the minimal coupling principle. As a result, we return to the Kaluza-Klein metric tensors in (1.1), but apply the foregoing to now rewrite these as:

$$G_{\mathrm{MN}} = \begin{pmatrix} g_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu} & \phi^2 k A_{\gamma\mu} \\ \phi^2 k A_{\gamma\nu} & \phi^2 \end{pmatrix}; \quad G^{\mathrm{MN}} = \begin{pmatrix} g^{\mu\nu} & -A_\gamma{}^\mu \\ -A_\gamma{}^\nu & g_{\alpha\beta} A_\gamma{}^\alpha A_\gamma{}^\beta + 1/\phi^2 \end{pmatrix}. \tag{2.15}$$

The only change we have made is to replace $A_\mu \mapsto A_{\gamma\mu}$, which is to recognize the remarkable result that *even in four spacetime dimensions alone*, it is not possible to deconstruct $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$ into a set of Dirac Γ_μ defined using (2.1), without fixing the gauge field A_μ to that of a photon $A_{\gamma\mu}$. Now, we extend this general covariance to the fifth dimension.

3. Derivation of the "Dirac-Kaluza-Klein" (DKK) Metric Tensor

In order to ensure general covariance at the Dirac level in five-dimensions, its is necessary that we first extend (2.1) into all five dimensions as such using the lower-indexed (1.3), namely:

$$\tfrac{1}{2}\{\Gamma_{\mathrm{M}}, \Gamma_{\mathrm{N}}\} = \tfrac{1}{2}\{\Gamma_{\mathrm{M}}\Gamma_{\mathrm{N}} + \Gamma_{\mathrm{N}}\Gamma_{\mathrm{M}}\} \equiv G_{\mathrm{MN}}. \tag{3.1}$$

The spacetime components of (3.1) with $g_{\mu\nu} = \eta_{\mu\nu}$ and using (2.14) will already reproduce $G_{\mu\nu} = \eta_{\mu\nu} + \phi^2 k^2 A_{\gamma\mu} A_{\gamma\nu}$ in (2.15). Now we turn to the fifth-dimensional components.

We first find it helpful to separate the time and space components of G_{MN} in (2.15), and so rewrite this as:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} g_{00} + \phi^2 k^2 A_{\gamma 0} A_{\gamma 0} & g_{0k} + \phi^2 k^2 A_{\gamma 0} A_{\gamma k} & \phi^2 k A_{\gamma 0} \\ g_{j0} + \phi^2 k^2 A_{\gamma j} A_{\gamma 0} & g_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ \phi^2 k A_{\gamma 0} & \phi^2 k A_{\gamma k} & \phi^2 \end{pmatrix}. \tag{3.2}$$

We know of course that $A_{\gamma 0} = 0$, which is the constraint that first arose from (2.10). And if we again work with $g_{\mu\nu} = \eta_{\mu\nu}$, then the above simplifies to:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ 0 & \phi^2 k A_{\gamma k} & \phi^2 \end{pmatrix}. \tag{3.3}$$

Next, let us *define* a Γ_5 to go along with the remaining Γ_μ in (2.14) in such a way as to *require* that $G_{j5} = \phi^2 kA_{\gamma j}$ and $G_{5k} = \phi^2 kA_{\gamma k}$ in (3.3) remain fully intact without any change. This is important, because these components in particular are largely responsible for the Kaluza "miracles" which reproduce Maxwell's equations and together with the Lorentz Force motion and the Maxwell Stress-Energy Tensor. We impose this requirement though (3.1) by writing:

$$\tfrac{1}{2}\{\Gamma_j, \Gamma_5\} = \tfrac{1}{2}\{\Gamma_j\Gamma_5 + \Gamma_5\Gamma_j\} \equiv G_{j5} = G_{j5} = \phi^2 kA_{\gamma j}\,. \tag{3.4}$$

Using $\Gamma_j = \gamma_j + \phi kA_{\gamma j}\gamma_0$ from (2.14) the above now becomes:

$$0 + \phi^2 kA_{\gamma j} \equiv \tfrac{1}{2}\{\Gamma_j\Gamma_5 + \Gamma_5\Gamma_j\} = \tfrac{1}{2}\{\gamma_j, \Gamma_5\} + \tfrac{1}{2}\phi kA_{\gamma j}\{\gamma_0, \Gamma_5\}\,, \tag{3.5}$$

which reduces down to a pair of anticommutation constraints on Γ_5, namely:

$$\begin{aligned} 0 &= \tfrac{1}{2}\{\gamma_j, \Gamma_5\} \\ \phi &= \tfrac{1}{2}\{\gamma_0, \Gamma_5\} \end{aligned}\,. \tag{3.6}$$

Now let's examine possible options for Γ_5.

Given that $\Gamma_0 = \gamma_0 + \phi kA_{\gamma j}\gamma_j$ and $\Gamma_j = \gamma_j + \phi kA_{\gamma j}\gamma_0$ in (2.14), we anticipate the general form for Γ_5 to be $\Gamma_5 \equiv \gamma_X + Y$ in which we define two unknowns to be determined using (3.6). First, *X* is one of the indexes 0, 1, 2, 3 or 5 of a Dirac matrix. Second, *Y* is a complete unknown which we anticipate will also contain a Dirac matrix as do the operators in (2.14). Using this in (3.6) we obtain:

$$\begin{aligned} 0 &= \tfrac{1}{2}\{\gamma_j\Gamma_5 + \Gamma_5\gamma_j\} = \tfrac{1}{2}\{\gamma_j\gamma_X + \gamma_j Y + \gamma_X\gamma_j + Y\gamma_j\} = \tfrac{1}{2}\{\gamma_j, \gamma_X\} + \tfrac{1}{2}\{\gamma_j, Y\} \\ 0 + \phi &= \tfrac{1}{2}\{\gamma_0\Gamma_5 + \Gamma_5\gamma_0\} = \tfrac{1}{2}\{\gamma_0\gamma_X + \gamma_0 Y + \gamma_X\gamma_0 + Y\gamma_0\} = \tfrac{1}{2}\{\gamma_0, \gamma_X\} + \tfrac{1}{2}\{\gamma_0, Y\} \end{aligned}\,. \tag{3.7}$$

From the top line, so long as $\gamma_X \neq -Y$ which means so long as $\Gamma_5 \neq 0$, we must have both the anticommutators $\{\gamma_j, \gamma_X\} = 0$ and $\{\gamma_j, Y\} = 0$. This excludes *X* being a space index 1, 2 or 3 leaving only $\gamma_X = \gamma_0$ or $\gamma_X = \gamma_5$. Moreover, whatever Dirac operator is part of *Y* must likewise be either γ_0 or γ_5. From the bottom line, however, we must have the anticommutators $\{\gamma_0, \gamma_X\} = 0$ and $\tfrac{1}{2}\{\gamma_0, Y\} = \phi$. The former means that the only remaining choice is $\gamma_X = \gamma_5$, and the latter means that $Y = \phi\gamma_0$. Therefore, we conclude that $\Gamma_5 = \gamma_5 + \phi\gamma_0$. So, including this in (2.14) gives:

$$\Gamma_M = \begin{pmatrix} \gamma_0 + \phi kA_{\gamma j}\gamma_j & \gamma_j + \phi kA_{\gamma j}\gamma_0 & \gamma_5 + \phi\gamma_0 \end{pmatrix}. \tag{3.8}$$

With this final operator $\Gamma_5 \equiv \gamma_5 + \phi\gamma_0$, we can use all of (3.8) above in (3.1) to precisely reproduce $G_{j5} = \phi^2 k A_{\gamma j}$ and $G_{5k} = \phi^2 k A_{\gamma k}$ in (3.3). This leaves the remaining components G_{05}, G_{50} and G_{55} to which we now turn.

If we use $\Gamma_0 = \gamma_0 + \phi k A_{\gamma j}\gamma_j$ and $\Gamma_5 = \gamma_5 + \phi\gamma_0$ in (3.1) to ensure that these remaining components are also fully covariant over all five dimensions, then we determine that:

$$G_{05} = G_{50} = \tfrac{1}{2}\left\{\Gamma_0\Gamma_5 + \Gamma_5\Gamma_0\right\} = \phi\gamma_0\gamma_0 + \tfrac{1}{2}\left\{\gamma_0, \gamma_5\right\} + \tfrac{1}{2}\phi k A_{\gamma j}\left\{\gamma_j, \gamma_5\right\} + \tfrac{1}{2}\phi^2 k A_{\gamma j}\left\{\gamma_j, \gamma_0\right\} = \phi\,, \tag{3.9}$$

$$G_{55} = \Gamma_5\Gamma_5 = \left(\gamma_5 + \phi\gamma_0\right)\left(\gamma_5 + \phi\gamma_0\right) = \gamma_5\gamma_5 + \phi^2\gamma_0\gamma_0 + \phi\left\{\gamma_5\gamma_0 + \gamma_0\gamma_5\right\} = 1 + \phi^2\,. \tag{3.10}$$

These two components are now different from those in (3.3). However, in view of Dirac operator deconstruction these are required to be so to ensure that the metric tensor is completely generally-covariant across all five dimensions, just as we were required to set $A_j = A_{\gamma j}$ at (2.12) to ensure simple spacetime covariance.

Changing (3.3) to incorporate (3.9) and (3.10), we now have:

$$G_{\mathrm{MN}} = \begin{pmatrix} G_{00} & G_{0k} & G_{05} \\ G_{j0} & G_{jk} & G_{j5} \\ G_{50} & G_{5k} & G_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \phi \\ 0 & \eta_{jk} + \phi^2 k^2 A_{\gamma j} A_{\gamma k} & \phi^2 k A_{\gamma j} \\ \phi & \phi^2 k A_{\gamma k} & 1 + \phi^2 \end{pmatrix}. \tag{3.11}$$

This metric tensor is fully covariant across all five dimensions, and because it is rooted in the Dirac operators (3.8), we expect that this can be made fully compatible with Dirac's theory of the multitude of fermions observed in the natural world. Moreover, in the context of Kaluza-Klein, Dirac's Quantum Theory of the Electron [13] has also forced us to set $A_j = A_{\gamma j}$ in the metric tensor, and thereby also served up a *quantum theory of the photon.* Because of its origins in requiring Kaluza-Klein theory to be compatible with Dirac theory, we shall refer to the above as the "Dirac-Kaluza-Klein" (DKK) metric tensor, and give the same name to the overall theory based on this.

Importantly, when electrodynamics is turned off by setting $A_{\gamma j} = 0$ and $\phi = 0$ the signature of (3.11) becomes $\mathrm{diag}\left(G_{\mathrm{MN}}\right) = \left(+1, -1, -1, -1, +1\right)$ with a determinant $\left|G_{\mathrm{MN}}\right| = -1$, versus $\left|G_{\mathrm{MN}}\right| = 0$ in (1.1) as reviewed earlier. This means that the inverse obtained via $G^{\mathrm{MA}}G_{\mathrm{AN}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ will be non-singular as opposed to that in (1.1), and that there is no reliance whatsoever on having $\phi \neq 0$ in order to avoid singularity. This in turn frees G_{55} from the energy requirements of ϕ which causes the fifth dimension in (1.1) to have a spacelike signature, and in fact, we see that *the fifth dimension in (3.11) is a second timelike, not fourth spacelike, dimension*. In turn, because (3.10) shows that $G_{55} = 1 + \phi^2 = \gamma_5\gamma_5 + \phi^2$ obtains its signature from $\gamma_5\gamma_5 = 1$, it now becomes possible to fully associate the Kaluza-Klein fifth dimension with the γ_5 of Dirac theory. This is not possible when a theory based on (1.1) causes G_{55} to be spacelike even though $\gamma_5\gamma_5 = 1$ is timelike, because of this conflict between timelike and spacelike signatures. Moreover, having only $G_{55}=\phi^2$ causes G_{55} to shrink or expand or even disappear entirely, based on the magnitude of ϕ. We shall review the physics consequences of all of these matters more closely later, but for now, we will conclude this section by condensing (3.11) down to a 2x2 format in the nature of (1.1). Then in the next section we will calculate the non-singular inverse G^{MN} of (3.11).

To consolidate (3.11) to 2x2 format, we first note that $A_{\gamma 0}=0$ for the photon. So, we can restore these zeros to the spacetime components of (3.11) and consolidate these to $G_{\mu\nu}=\eta_{\mu\nu}+\phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu}$. This is exactly what is in (1.1) when $g_{\mu\nu}=\eta_{\mu\nu}$, but for the fact that the gauge symmetry has been broken to force $A_\mu = A_{\gamma\mu}$. But we also know that G_{05} and G_{j5} have been constructed so as to form a four-vector in spacetime, likewise for G_{50} and G_{5k}. Therefore, we now *define* a new four-vector

$$\Phi_\mu \equiv \begin{pmatrix} \phi & \phi^2 k A_{\gamma j} \end{pmatrix}. \tag{3.12}$$

Moreover, $G_{55}=\gamma_5\gamma_5+\phi^2\gamma_0\gamma_0$ in (3.10) teaches that the underlying timelike signature (and the metric non-singularity) is rooted in $\gamma_5\gamma_5=1$, and via $\phi^2\gamma_0\gamma_0=\phi^2$ that the square of the scalar field is rooted in $\gamma_0\gamma_0=1$ which has two time indexes. So, may now formally assign $\eta_{55}=1$ to the fifth component of the Minkowski metric signature, and we may assign $\phi^2=\Phi_0\Phi_0$ to the field in G_{55}. With all of this, and using minimal coupling to generalize $\eta_{\mathrm{MN}}\mapsto g_{\mathrm{MN}}$ which also means accounting for non-zero $g_{\mu 5}$, $g_{5\nu}$, (3.11) now may be compacted via (3.12) to the 2x2 form:

$$G_{\mathrm{MN}}=\begin{pmatrix} G_{\mu\nu} & G_{\mu 5} \\ G_{5\nu} & G_{55} \end{pmatrix}=\begin{pmatrix} g_{\mu\nu}+\phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu} & g_{\mu 5}+\Phi_\mu \\ g_{5\nu}+\Phi_\nu & g_{55}+\Phi_0\Phi_0 \end{pmatrix}. \tag{3.13}$$

Now, let's calculate the inverse of (3.13).

4. Calculation of the Inverse Dirac-Kaluza-Klein Metric Tensor

As already mentioned, the modified Kaluza-Klein metric tensor (3.13) has a *non-singular* inverse G^{MN} specified in the usual way by $G^{\mathrm{MA}}G_{\mathrm{AN}}=\delta^{\mathrm{M}}{}_{\mathrm{N}}$. We already know this because when all electromagnetic fields are turned off and $g_{\mathrm{MN}}=\eta_{\mathrm{MN}}$, we have a determinant $|G_{\mathrm{MN}}|=-1$ which is one of the litmus tests that can be used to demonstrate non-singularity. But because this inverse is essential to being able to calculate connections, equations of motion, and the Einstein field equation and related energy tensors, next important step – which is entirely mathematical – is to explicitly calculate the inverse of (3.13), which we shall now do.

Calculating the inverse of a 5x5 matrix is a very cumbersome task if one employs a brute force approach. But we can take great advantage of the fact that the tangent space Minkowski tensor $\operatorname{diag}(\eta_{\mathrm{MN}})=(+1,-1,-1,-1,+1)$ has two timelike and three spacelike dimensions when we set $A_{\gamma j}=0$ and $\phi=0$ to turn off the electrodynamic fields, by using the analytic blockwise inversion method detailed in [14]. Specifically, we split the 5x5 matrix into 2x2 and 3x3 matrices along the "diagonal", and into 2x3 and 3x2 matrices off the "diagonal." It is best to work from (3.11) which does not show the time component $A_{\gamma 0}$ of the gauge vector because this is equal to zero for a photon. We expand this out to show the entire 5x5 matrix, and we move the rows and columns so that the ordering of the indexes is not $\mathrm{M}=0,1,2,3,5$, but rather is $\mathrm{M}=0,5,1,2,3$. With all of this, (3.11) may be rewritten as:

$$G_{\mathrm{MN}}=\begin{pmatrix} G_{00} & G_{05} & G_{01} & G_{02} & G_{03} \\ G_{50} & G_{55} & G_{51} & G_{52} & G_{53} \\ G_{10} & G_{15} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{25} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{35} & G_{31} & G_{32} & G_{33} \end{pmatrix}=\begin{pmatrix} 1 & \phi & 0 & 0 & 0 \\ \phi & 1+\phi^2 & \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 1} & -1+\phi^2 k^2 A_{\gamma 1}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 2} & \phi^2 k^2 A_{\gamma 2}A_{\gamma 1} & -1+\phi^2 k^2 A_{\gamma 2}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 2}A_{\gamma 3} \\ 0 & \phi^2 kA_{\gamma 3} & \phi^2 k^2 A_{\gamma 3}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 3}A_{\gamma 2} & -1+\phi^2 k^2 A_{\gamma 3}A_{\gamma 3} \end{pmatrix}. \tag{4.1}$$

Then, we can find the inverse using the blockwise inversion relation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \tag{4.2}$$

with the matrix block assignments:

$$\begin{gathered} \mathbf{A}=\begin{pmatrix} 1 & \phi \\ \phi & 1+\phi^2 \end{pmatrix}; \quad \mathbf{B}=\begin{pmatrix} 0 & 0 & 0 \\ \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3} \end{pmatrix}; \\ \mathbf{C}=\begin{pmatrix} 0 & \phi^2 kA_{\gamma 1} \\ 0 & \phi^2 kA_{\gamma 2} \\ 0 & \phi^2 kA_{\gamma 3} \end{pmatrix}; \quad \mathbf{D}=\begin{pmatrix} -1+\phi^2 k^2 A_{\gamma 1}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 1}A_{\gamma 3} \\ \phi^2 k^2 A_{\gamma 2}A_{\gamma 1} & -1+\phi^2 k^2 A_{\gamma 2}A_{\gamma 2} & \phi^2 k^2 A_{\gamma 2}A_{\gamma 3} \\ \phi^2 k^2 A_{\gamma 3}A_{\gamma 1} & \phi^2 k^2 A_{\gamma 3}A_{\gamma 2} & -1+\phi^2 k^2 A_{\gamma 3}A_{\gamma 3} \end{pmatrix}. \end{gathered} \tag{4.3}$$

The two inverses we need to calculate are $\mathbf{A}^{-1}$ and $\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}$. The former is a 2x2 matrix which is easily inverted, see [15]. Its determinant $|\mathbf{A}|=1+\phi^2-\phi^2=1$, so its inverse is:

$$\mathbf{A}^{-1}=\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}. \tag{4.4}$$

Next, we need to calculate $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$, then invert this. We first calculate:

$$\begin{aligned}
-\mathbf{CA}^{-1}\mathbf{B} &= -\begin{pmatrix} 0 & \phi^2 k A_{\gamma 1} \\ 0 & \phi^2 k A_{\gamma 2} \\ 0 & \phi^2 k A_{\gamma 3} \end{pmatrix} \begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \phi^2 k A_{\gamma 1} & \phi^2 k A_{\gamma 2} & \phi^2 k A_{\gamma 3} \end{pmatrix} \\
&= -\begin{pmatrix} 0 & \phi^2 k A_{\gamma 1} \\ 0 & \phi^2 k A_{\gamma 2} \\ 0 & \phi^2 k A_{\gamma 3} \end{pmatrix} \begin{pmatrix} -\phi^3 k A_{\gamma 1} & -\phi^3 k A_{\gamma 2} & -\phi^3 k A_{\gamma 3} \\ \phi^2 k A_{\gamma 1} & \phi^2 k A_{\gamma 2} & \phi^2 k A_{\gamma 3} \end{pmatrix} = -\begin{pmatrix} \phi^4 k^2 A_{\gamma 1} A_{\gamma 1} & \phi^4 k^2 A_{\gamma 1} A_{\gamma 2} & \phi^4 k^2 A_{\gamma 1} A_{\gamma 3} \\ \phi^4 k^2 A_{\gamma 2} A_{\gamma 1} & \phi^4 k^2 A_{\gamma 2} A_{\gamma 2} & \phi^4 k^2 A_{\gamma 2} A_{\gamma 3} \\ \phi^4 k^2 A_{\gamma 3} A_{\gamma 1} & \phi^4 k^2 A_{\gamma 3} A_{\gamma 2} & \phi^4 k^2 A_{\gamma 3} A_{\gamma 3} \end{pmatrix}
\end{aligned} . \quad (4.5)$$

Therefore:

$$\begin{aligned}
\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B} &= \begin{pmatrix} -1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 3} \\ \left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 1} & -1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 2} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 3} \\ \left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 1} & \left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 2} & -1+\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 3} \end{pmatrix} . \\
&= \eta_{jk} + \left(\phi^2-\phi^4\right)k^2 A_{\gamma j} A_{\gamma k}
\end{aligned} \quad (4.6)$$

We can easily invert this using the skeletal mathematical relation $(1+x)(1-x)=1-x^2$. Specifically, using the result in (4.6) we may write:

$$\begin{aligned}
&\left(\eta_{jk} + \left(\phi^2-\phi^4\right)k^2 A_{\gamma j} A_{\gamma k}\right)\left(\eta_{kl} - \left(\phi^2-\phi^4\right)k^2 A_{\gamma k} A_{\gamma l}\right) \\
&= \eta_{jk}\eta_{kl} + \left(\phi^2-\phi^4\right)k^2\left(\eta_{kl} A_{\gamma j} A_{\gamma k} - \eta_{jk} A_{\gamma k} A_{\gamma l}\right) - \left(\phi^2-\phi^4\right)^2 k^4 A_{\gamma j} A_{\gamma k} A_{\gamma k} A_{\gamma l} = \delta_{jl}
\end{aligned} . \quad (4.7)$$

The $A_{\gamma j} A_{\gamma k} A_{\gamma k} A_{\gamma l}$ term zeros out because $A_{\gamma k} A_{\gamma k} = 0$ for the photon gauge vector. Sampling the diagonal $j=l=1$ term, $\eta_{k1} A_{\gamma 1} A_{\gamma k} - \eta_{1k} A_{\gamma k} A_{\gamma 1} = -A_{\gamma 1} A_{\gamma 1} + A_{\gamma 1} A_{\gamma 1} = 0$. Sampling the off-diagonal $j=1$, $l=2$ term, $\eta_{k1} A_{\gamma 2} A_{\gamma k} - \eta_{2k} A_{\gamma k} A_{\gamma 1} = -A_{\gamma 2} A_{\gamma 1} + A_{\gamma 2} A_{\gamma 1} = 0$. By symmetry, all other terms zero as well. And of course, $\eta_{jk}\eta_{kl} = \delta_{jl}$. So (4.7) taken with (4.6) informs us that:

$$\begin{aligned}
&\left(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B}\right)^{-1} = \eta_{jk} - \left(\phi^2-\phi^4\right)k^2 A_{\gamma j} A_{\gamma k} \\
&= \begin{pmatrix} -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1} A_{\gamma 3} \\ -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2} A_{\gamma 3} \\ -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 2} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3} A_{\gamma 3} \end{pmatrix}
\end{aligned} . \quad (4.8)$$

We now have all the inverses we need; the balance of the calculation is matrix multiplication.

From the lower-left block in (4.2) we use **C** in (4.3), with (4.4) and (4.8), to calculate:

$$
\begin{aligned}
&-\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} \\
&=\begin{pmatrix} 1+\left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma1} & \left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma2} & \left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma3} \\ \left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma1} & 1+\left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma2} & \left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma3} \\ \left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma1} & \left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma2} & 1+\left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma3} \end{pmatrix}\begin{pmatrix} 0 & \phi^2kA_{\gamma1} \\ 0 & \phi^2kA_{\gamma2} \\ 0 & \phi^2kA_{\gamma3} \end{pmatrix}\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}, \qquad (4.9) \\
&=\begin{pmatrix} -\phi^3kA_{\gamma1}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma1}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma1}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma1}A_{\gamma k}A_{\gamma k} \\ -\phi^3kA_{\gamma2}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma2}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma2}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma2}A_{\gamma k}A_{\gamma k} \\ -\phi^3kA_{\gamma3}-\left(\phi^2-\phi^4\right)\phi^3k^3A_{\gamma3}A_{\gamma k}A_{\gamma k} & \phi^2kA_{\gamma3}+\left(\phi^2-\phi^4\right)\phi^2k^3A_{\gamma3}A_{\gamma k}A_{\gamma k} \end{pmatrix}=\begin{pmatrix} -\phi^3kA_{\gamma1} & \phi^2kA_{\gamma1} \\ -\phi^3kA_{\gamma2} & \phi^2kA_{\gamma2} \\ -\phi^3kA_{\gamma3} & \phi^2kA_{\gamma3} \end{pmatrix}
\end{aligned}
$$

again using $A_{\gamma k}A_{\gamma k}=0$. We can likewise calculate $-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}$ in the upper-right block in (4.2), but it is easier and entirely equivalent to simply use the transposition symmetry $G_{\mathrm{MN}}=G_{\mathrm{NM}}$ of the metric tensor and the result in (4.9) to deduce:

$$
-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}=\begin{pmatrix} -\phi^3kA_{\gamma1} & -\phi^3kA_{\gamma2} & -\phi^3kA_{\gamma3} \\ \phi^2kA_{\gamma1} & \phi^2kA_{\gamma2} & \phi^2kA_{\gamma3} \end{pmatrix}, \qquad (4.10)
$$

For the upper left block in (4.2) we use $\mathbf{B}$ in (4.3), with (4.4) and (4.9) to calculate:

$$
\begin{aligned}
&\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} \\
&=\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}+\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ \phi^2kA_{\gamma1} & \phi^2kA_{\gamma2} & \phi^2kA_{\gamma3} \end{pmatrix}\begin{pmatrix} \phi^3kA_{\gamma1} & -\phi^2kA_{\gamma1} \\ \phi^3kA_{\gamma2} & -\phi^2kA_{\gamma2} \\ \phi^3kA_{\gamma3} & -\phi^2kA_{\gamma3} \end{pmatrix}, \qquad (4.11) \\
&=\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}+\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \phi^5k^2A_{\gamma k}A_{\gamma k} & -\phi^4k^2A_{\gamma k}A_{\gamma k} \end{pmatrix}=\begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}
\end{aligned}
$$

again using $A_{\gamma k}A_{\gamma k}=0$. And (4.8) already has the complete lower-right block in (4.2).

So, we now reassemble (4.8) through (4.11) into (4.2) to obtain the complete inverse:

$$
\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1}=\begin{pmatrix} 1+\phi^2 & -\phi & -\phi^3kA_{\gamma1} & -\phi^3kA_{\gamma2} & -\phi^3kA_{\gamma3} \\ -\phi & 1 & \phi^2kA_{\gamma1} & \phi^2kA_{\gamma2} & \phi^2kA_{\gamma3} \\ -\phi^3kA_{\gamma1} & \phi^2kA_{\gamma1} & -1-\left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma1} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma2} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma1}A_{\gamma3} \\ -\phi^3kA_{\gamma2} & \phi^2kA_{\gamma2} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma1} & -1-\left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma2} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma2}A_{\gamma3} \\ -\phi^3kA_{\gamma3} & \phi^2kA_{\gamma3} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma1} & -\left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma2} & -1-\left(\phi^2-\phi^4\right)k^2A_{\gamma3}A_{\gamma3} \end{pmatrix} \qquad (4.12)
$$

Then we reorder rows and columns back to the $\mathrm{M}=0,1,2,3,5$ sequence and connect this to the contravariant (inverse) metric tensor to write:

$$G^{\mathrm{MN}}=\begin{pmatrix} 1+\phi^2 & -\phi^3 kA_{\gamma 1} & -\phi^3 kA_{\gamma 2} & -\phi^3 kA_{\gamma 3} & -\phi \\ -\phi^3 kA_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 1}A_{\gamma 3} & \phi^2 kA_{\gamma 1} \\ -\phi^3 kA_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 1} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 2} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 2}A_{\gamma 3} & \phi^2 kA_{\gamma 2} \\ -\phi^3 kA_{\gamma 3} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 1} & -\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 2} & -1-\left(\phi^2-\phi^4\right)k^2 A_{\gamma 3}A_{\gamma 3} & \phi^2 kA_{\gamma 3} \\ -\phi & \phi^2 kA_{\gamma 1} & \phi^2 kA_{\gamma 2} & \phi^2 kA_{\gamma 3} & 1 \end{pmatrix}. \tag{4.13}$$

In a vitally-important contrast to the usual Kaluza-Klein G^{MN} in (1.1), this is manifestly *not singular*. This reverts to $\mathrm{diag}\left(G^{\mathrm{MN}}\right)=\mathrm{diag}\left(\eta^{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,+1\right)$ when $A_{\gamma\mu}=0$ and $\phi=0$ which is exactly the same signature as G_{MN} in (3.11). Then we consolidate to the 3x3 form:

$$G^{\mathrm{MN}}=\begin{pmatrix} G^{00} & G^{0k} & G^{05} \\ G^{j0} & G^{jk} & G^{j5} \\ G^{50} & G^{5k} & G^{55} \end{pmatrix}=\begin{pmatrix} 1+\phi^2 & -\phi^3 kA_{\gamma k} & -\phi \\ -\phi^3 kA_{\gamma j} & \eta^{jk}-\left(\phi^2-\phi^4\right)k^2 A_{\gamma j}A_{\gamma k} & \phi^2 kA_{\gamma j} \\ -\phi & \phi^2 kA_{\gamma k} & 1 \end{pmatrix}. \tag{4.14}$$

Now, the photon gauge vectors $A_{\gamma j}$ in (4.14) still have lower indexes, and with good reason: We cannot simply raise these indexes of objects *inside the metric tensor* at will as we might for any other tensor. Rather, we must use the metric tensor (4.14) itself to raise and lower indexes, by calculating $A_\gamma{}^{\mathrm{M}}=G^{\mathrm{MN}}A_{\gamma\mathrm{N}}$. Nonetheless, it would be desirable to rewrite (4.14) with all upper indexes inside, which will simplify downstream calculations. Given that $A_{\gamma 0}=0$ for the photon and $A_{\gamma 5}=0$, and raising indexes for $A_\gamma{}^0$ and $A_\gamma{}^5$ while sampling $A_\gamma{}^1$ we may calculate:

$$\begin{aligned} A_\gamma{}^0 &= G^{0\mathrm{N}}A_{\gamma\mathrm{N}}=G^{01}A_{\gamma 1}+G^{02}A_{\gamma 2}+G^{03}A_{\gamma 3}=-\phi^3 kA_{\gamma k}A_{\gamma k}=0 \\ A_\gamma{}^1 &= G^{1\mathrm{N}}A_{\gamma\mathrm{N}}=G^{11}A_{\gamma 1}+G^{12}A_{\gamma 2}+G^{13}A_{\gamma 3}=-A_{\gamma 1}-\left(\phi^2+\phi^4\right)k^2 A_{\gamma 1}A_{\gamma k}A_{\gamma k}=-A_{\gamma 1}\ , \\ A_\gamma{}^5 &= G^{5\mathrm{N}}A_{\gamma\mathrm{N}}=G^{51}A_{\gamma 1}+G^{52}A_{\gamma 2}+G^{53}A_{\gamma 3}=-\phi^2 kA_{\gamma k}A_{\gamma k}=0 \end{aligned} \tag{4.15}$$

once again employing $A_{\gamma k}A_{\gamma k}=0$. The middle result applies to other space indexes, so that:

$$A_\gamma{}^j=\eta^{jk}A_{\gamma k}, \tag{4.16}$$

which is the usual way of raising indexes in flat spacetime. As a result, with $g^{\mu\nu}=\eta^{\mu\nu}$ we may raise the index in (3.12) to obtain:

$$\Phi^\mu=\begin{pmatrix}\phi & \phi^2 kA_\gamma{}^j\end{pmatrix}=\begin{pmatrix}\phi & -\phi^2 kA_{\gamma j}\end{pmatrix}. \tag{4.17}$$

And we may then use (4.17) to write (4.14) as:

$$G^{\mathrm{MN}}=\begin{pmatrix} G^{00} & G^{0k} & G^{05} \\ G^{j0} & G^{jk} & G^{j5} \\ G^{50} & G^{5k} & G^{55} \end{pmatrix}=\begin{pmatrix} 1+\phi^2 & -\phi^3 kA_{\gamma k} & -\Phi^0 \\ -\phi^3 kA_{\gamma j} & \eta^{jk}-\left(\phi^2-\phi^4\right)k^2 A_{\gamma j}A_{\gamma k} & -\Phi^j \\ -\Phi^0 & -\Phi^k & 1 \end{pmatrix}. \tag{4.18}$$

Now we focus on the middle term which is expanded to $\eta^{jk}-\phi^2 k^2 A_{\gamma j}A_{\gamma k}+\phi^4 k^2 A_{\gamma j}A_{\gamma k}$. Working from (4.17) we calculate:

$$\Phi^0\Phi^0=\phi^2;\quad \Phi^0\Phi^k=-\phi^3 kA_{\gamma k};\quad \Phi^j\Phi^0=-\phi^3 kA_{\gamma j};\quad \Phi^j\Phi^k=\phi^4 k^2 A_{\gamma j}A_{\gamma j}. \tag{4.19}$$

Therefore, we may use (4.19) in (4.18) to write:

$$G^{\mathrm{MN}}=\begin{pmatrix} G^{00} & G^{0k} & G^{05} \\ G^{j0} & G^{jk} & G^{j5} \\ G^{50} & G^{5k} & G^{55} \end{pmatrix}=\begin{pmatrix} 1+\Phi^0\Phi^0 & \Phi^0\Phi^k & -\Phi^0 \\ \Phi^j\Phi^0 & \eta^{jk}-\phi^2 k^2 A_{\gamma j}A_{\gamma k}+\Phi^j\Phi^k & -\Phi^j \\ -\Phi^0 & -\Phi^k & 1 \end{pmatrix}. \tag{4.20}$$

Finally, again taking advantage of the fact that $A_{\gamma 0}=0$ and again using (4.16) to raise the indexes in $A_{\gamma j}A_{\gamma k}=A_\gamma{}^j A_\gamma{}^k$, while using $1=\eta_{55}=\eta^{55}$, we may consolidate this into the 2x2 format:

$$G^{\mathrm{MN}}=\begin{pmatrix} G^{\mu\nu} & G^{\mu 5} \\ G^{5\nu} & G^{55} \end{pmatrix}=\begin{pmatrix} \eta^{\mu\nu}-\phi^2 k^2 A_\gamma{}^\mu A_\gamma{}^\nu+\Phi^\mu\Phi^\nu & -\Phi^\mu \\ -\Phi^\nu & \eta^{55} \end{pmatrix}. \tag{4.21}$$

This is the inverse of (3.13), and it is a good exercise to check and confirm that in fact, $G^{\mathrm{MA}}G_{\mathrm{AN}}=\delta^{\mathrm{M}}{}_{\mathrm{N}}$.

The final step is to apply minimal coupling to generalize $\eta^{\mathrm{MN}}\mapsto g^{\mathrm{MN}}$, with possible non-zero $g_{\mu 5}$, $g_{5\nu}$, $g^{\mu 5}$ and $g^{5\nu}$. Finally, with all this, (3.13) and (4.21) become:

$$\boxed{G_{\mathrm{MN}}=\begin{pmatrix} g_{\mu\nu}+\phi^2 k^2 A_{\gamma\mu}A_{\gamma\nu} & g_{\mu 5}+\Phi_\mu \\ g_{5\nu}+\Phi_\nu & g_{55}+\Phi_0\Phi_0 \end{pmatrix};\quad G^{\mathrm{MN}}=\begin{pmatrix} g^{\mu\nu}-\phi^2 k^2 A_\gamma{}^\mu A_\gamma{}^\nu+\Phi^\mu\Phi^\nu & g^{\mu 5}-\Phi^\mu \\ g^{5\nu}-\Phi^\nu & g^{55} \end{pmatrix}}. \tag{4.22}$$

These are the direct counterparts to the Kaluza-Klein metric tensors (1.1). This inverse, in contrast to that of (1.1), is manifestly non-singular.

Finally, we commented after (2.6) that it would have been possible to choose minus rather than plus signs in the tetrad / field assignments. We make a note that had we done so, this would have carried through to a sign flip in all the $\varepsilon_k{}^0$ and $\varepsilon_0{}^k$ tetrad components in (2.12), it would have changed (2.14) to $\Gamma_\mu = \begin{pmatrix} \gamma_0 - \phi k A_{\gamma j}\gamma_j & \gamma_j - \phi k A_{\gamma j}\gamma_0 \end{pmatrix}$, and it would have changed (3.8) to include $\Gamma_5 = \gamma_5 - \phi\gamma_0$. Finally, for the metric tensors (4.22), all would be exactly the same, except that we would have had $G_{\mu 5} = G_{5\mu} = g_{\mu 5} - \Phi_\mu$ and $G^{\mu 5} = G^{5\mu} = g^{\mu 5} + \Phi^\mu$, with the vectors in (3.12) and (4.17) instead given by $\Phi_\mu = \begin{pmatrix} \phi & -\phi^2 k A_{\gamma j} \end{pmatrix}$ and $\Phi^\mu = \begin{pmatrix} \phi & -\phi^2 k A_\gamma{}^j \end{pmatrix}$. We note this because in a related preprint by the author at [16], this latter sign choice was required at [14.5] in a similar circumstance to ensure limiting-case solutions identical to those of Dirac's equation, as reviewed following [19.13] therein. Whether a similar choice may be required here cannot be known for certain without calculating detailed correspondences with Dirac theory based on the Γ_{M} in (3.8).

Before we conclude this section, it is illustrative to rewrite the new vector (4.17) as:

$$\Phi^\mu = \begin{pmatrix} \phi + \phi^2 k A_\gamma{}^0 & \phi^2 k A_\gamma{}^j \end{pmatrix}, \tag{4.23}$$

taking advantage of $A_\gamma{}^0 = 0$ to explicitly display the spacetime covariance of $A_\gamma{}^\mu$ notwithstanding that the gauge symmetry has been broken to that of a photon. We then calculate the antisymmetric tensor defined by the bivector $\mathrm{P}^{\mu\nu} \equiv \partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu$ in two separate parts as follows:

$$\begin{aligned} \mathrm{P}^{0k} &= \partial^0 \Phi^k - \partial^k \Phi^0 = \partial^0 \left(\phi^2 k A_\gamma{}^k\right) - \partial^k \left(\phi + \phi^2 k A_\gamma{}^0\right) \\ &= \phi^2 k \left(\partial^0 A_\gamma{}^k - \partial^k A_\gamma{}^0\right) + 2\partial^0 \phi k A_\gamma{}^k - 2\partial^k \phi k A_\gamma{}^0 - \partial^k \phi, \\ &= \phi^2 k F_\gamma{}^{0k} + \left(2kA_\gamma{}^k \partial^0 - 2kA_\gamma{}^0 \partial^k - \partial^k\right)\phi \end{aligned} \tag{4.24a}$$

$$\begin{aligned} \mathrm{P}^{jk} &= \partial^j \Phi^k - \partial^k \Phi^j = \partial^j \left(\phi^2 k A_\gamma{}^k\right) - \partial^k \left(\phi^2 k A_\gamma{}^j\right) \\ &= \phi^2 k \left(\partial^j A_\gamma{}^k - \partial^k A_\gamma{}^j\right) + 2\partial^j \phi k A_\gamma{}^k - 2\partial^k \phi k A_\gamma{}^j. \\ &= \phi^2 k F_\gamma{}^{jk} + \left(2kA_\gamma{}^k \partial^j - 2kA_\gamma{}^j \partial^k\right)\phi \end{aligned} \tag{4.24b}$$

We see the emergence of the field strength tensor $\phi^2 k F_\gamma{}^{jk}$ in its usual Kaluza-Klein form, modified merely to indicate that this arises from taking $F_\gamma{}^{\mu\nu} = \partial^\mu A_\gamma{}^\nu - \partial^\nu A_\gamma{}^\mu$ for a photon. The only term which bars immediately merging both of (4.24) is the gradient $-\partial^k\phi$ in (4.24a). For this, we now define a vector $I^\mu \equiv \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix}$ and use this to form the covariant expression:

$$\begin{pmatrix} 0 \\ \partial^j \phi \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & \partial^k \phi \end{pmatrix} = \partial^\mu \phi I^\nu - \partial^\nu \phi I^\mu = -I^\mu \partial^\nu \phi + I^\nu \partial^\mu \phi = -\left(I^\mu \partial^\nu - I^\nu \partial^\mu\right)\phi. \tag{4.25}$$

We then use this to combine both of (4.24) into:

$$\begin{aligned} \mathrm{P}^{\mu\nu} &= \partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu = \phi^2 k F_\gamma^{\ \mu\nu} - \left(2kA_\gamma^{\ \mu}\partial^\nu - 2kA_\gamma^{\ \nu}\partial^\mu\right)\phi - \left(I^\mu\partial^\nu - I^\nu\partial^\mu\right)\phi \\ &= \phi^2 k F_\gamma^{\ \mu\nu} - \left(\left(I^\mu + 2kA_\gamma^{\ \mu}\right)\partial^\nu - 2k\left(I^\nu + A_\gamma^{\ \nu}\right)\partial^\mu\right)\phi \end{aligned} \tag{4.26}$$

The newly-appearing vector $I^\mu + 2kA_\gamma^{\ \mu}$ is itself of interest, because the breaking of the gauge symmetry has removed $A_\gamma^{\ 0} = \phi_\gamma = 0$ from the photon gauge vector. But $I^\mu + 2kA_\gamma^{\ \mu}$ with $I^\mu \equiv \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix}$ replaces the removed $A_\gamma^{\ 0} = 0$ by the number 1, then adds this to the rest of $A_\gamma^{\ j}$ multiplied by the constant factor $2k$ with dimensions of charge/energy.

But the main reason we have obtained (4.26), is to make clear that the field strength $F_\gamma^{\ \mu\nu}$ which is needed for the Lorentz Force motion and the Maxwell tensor, does emerge from (4.22) in the exact same form as it does from the usual Kaluza-Klein metric tensor (1.1), so that the Kaluza "miracle" is undiminished. But there is one wrinkle: $F_\gamma^{\ \mu\nu}$ is the field strength of a photon, not a general materially-sourced $F^{\mu\nu}$. Keeping in kind Heaviside's intuitions a generation before gauge theory in formulating Maxwell's original theory without gauge fields, this is where gauge symmetry comes into play: even though the gauge symmetry is broken for $A_\gamma^{\ \mu}$ and we are therefore prohibited from turning the luminous $A_\gamma^{\ \mu}$ in (4.22) back into a classical materially-sourced potential $A^\mu = \begin{pmatrix} \phi & \mathbf{A} \end{pmatrix}$ which can be transformed to a rest frame, the same prohibition *does not apply to* $F_\gamma^{\ \mu\nu} = \partial^\mu A_\gamma^{\ \nu} - \partial^\nu A_\gamma^{\ \mu}$ obtained from this $A_\gamma^{\ \mu}$, because the antisymmetry of $F_\gamma^{\ \mu\nu}$ is nevertheless *invariant under gauge transformations*. So irrespective of this $A^\mu = A_\gamma^{\ \mu}$ symmetry breaking, any luminous photon fields $F_\gamma^{\ \mu\nu}$ emerging from applying the five-dimensional Einstein equation to (4.22) can always be gauge-transformed using $F_\gamma^{\ \mu\nu} \to F^{\mu\nu}$ into those of a classical materially-sourced potential $A^\mu = \begin{pmatrix} \phi & \mathbf{A} \end{pmatrix}$. From this, the Lorentz motion and the Maxwell tensor become embedded in the five-dimensional theory just as in the usual Kaluza-Klein theory based on (1.1), without diminution. This is exactly what the author did in Section 21 of [16] to obtain the empirically-observed lepton magnetic moments at [23.5] and [23.6] of that same paper.

5. How the Dirac-Kaluza-Klein Metric Tensor Resolves the Challenges faced by Kaluza-Klein without Diminishing the Kaluza "Miracle," and Grounds the Now-Timelike Fifth Dimension in Manifestly-Observed Physical Reality

As has been previously pointed out, in the circumstance where all electrodynamic interactions are turned off by setting $A_{\gamma j} = 0$ and what is now $\Phi_\mu = 0$, then (4.22) reduces when $g_{\mu\nu} = \eta_{\mu\nu}$ to $\mathrm{diag}\left(G_{\mathrm{MN}}\right) = \left(+1, -1, -1, -1, +1\right)$. But in the same situation the usual Kaluza-Klein metric tensor (1.1) reduces to $\mathrm{diag}\left(G_{\mathrm{MN}}\right) = \left(+1, -1, -1, -1, 0\right)$ with a determinant $\left|G_{\mathrm{MN}}\right| = 0$. This of course means the Kaluza-Klein metric tensor is not-invertible and therefore becomes singular when electrodynamic interactions are turned off. Again, this may be seen directly from the fact that when we set $A_{\gamma j} = 0$ and $\phi = 0$, in (1.1) we get $G^{55} = g_{\alpha\beta} A^\alpha A^\beta + 1/\phi^2 = 0 + \infty$. This degeneracy leads to a number of interrelated ills which have hobbled Kaluza-Klein as a viable theory of the natural world for a year shy of a century.

First, the scalar field ϕ carries a much heavier burden than it should, because Kaluza-Klein relies upon this field being non-zero to ensure that the five-dimensional spacetime geometry is non-singular. This imposes constraints upon ϕ which would not exist if ϕ was not doing "double duty" as both a scalar field and as a structural element required to maintain the non-degeneracy of Minkowski spacetime extended to five dimensions. Second, this makes it next-to-impossible to account for the fifth dimension in the observed physical world. After all, the space and time of real physical experience have a flat spacetime signature $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1)$ which is structurally sound even in the absence of any fields whatsoever. But what is one to make of a signature which, when $g_{\mu\nu} = \eta_{\mu\nu}$ and $A_{\gamma k} = 0$ is given by $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,\phi^2)$ with $|\eta_{\mathrm{MN}}| = -\phi^2$? How is one to explain the physicality of a fifth dimension which contributes a $G_{55} = \phi^2$ to the Minkowski signature that is based upon a field, rather than being either a timelike +1 or a spacelike –1 Pythagorean metric component? The Minkowski signature defines the *tangent* spacetime at each event, absent curvature. How can a tangent space which by definition is not curved, be dependent upon a field ϕ which if it has even the slightest modicum of energy, will cause curvature? This is an internal logical contradiction of the Kaluza-Klein metric tensor (1.1) that had persisted for a full century, and it leads to such hard-to-justify oddities as a fifth dimension and a $|\eta_{\mathrm{MN}}| = -\phi^2$ which dilates or contracts (hence the sometime-used name "dilaton") in accordance with the behavior of ϕ^2.

Third, (4.22) is obtained by *requiring* that it be possible to deconstruct the Kaluza-Klein metric tensor into a set of Dirac matrices obeying (3.1), with full five-dimensional general covariance. What we have found is that it is *not* possible to have 5-dimensional general covariance symmetry if $G_{05} = G_{50} = 0$ and $G_{55} = \phi^2$ as in (1.1). Rather, general 5-dimensional covariance *requires* that $G_{05} = G_{50} = \phi$ and $G_{55} = 1 + \phi^2$ in (4.22). Further, even spacetime covariance in four dimensions, requires that we gauge the electromagnetic potential to that of the photon. Without these changes to the metric tensor components, it is simply not possible to make Kaluza-Klein theory compatible with Dirac theory to have general 5-covariance. This means that there is no consistent way of using the usual (1.1) to account for the fermions which are at the heart of observed matter in the material universe. Such an omission – even without any of its other known ills – most-assuredly renders the KK metric (1.1) "unphysical." Finally, there is the century-old demand which remains unmet to this date: "show me the fifth dimension!" There is no observational evidence at all to support the fifth dimension, at least in the form specified by (1.1), and in the efforts undertaken to date to remedy these problems.

But (4.22) leads to a whole other picture. First, by definition Dirac-type equations can be formed using the Γ_{M} in (3.8), so there is no problem of incompatibility with Dirac theory. Thus, all aspects of fermion physics may be fully accounted for. Second, it should be obvious to anyone familiar with the Dirac γ_μ and $\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3$ that one may easily form a five-dimensional

Minkowski tensor using the commutator $\eta_{\mathrm{MN}} \equiv \frac{1}{2}\{\gamma_{\mathrm{M}}, \gamma_{\mathrm{N}}\}$ to obtain a five-dimensional $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$ which has a Minkowski signature with two timelike and three spacelike dimensions. But it is not at all obvious how one might proceed to regard γ_5 as the generator of *a truly-physical fifth dimension* which is on an absolute par with the generators γ_μ of the four truly-physical dimensions which are time and space. This is true, notwithstanding the clear observational evidence that γ_5 has a multitude of observable physical impacts. The reality of γ_5 is most notable in the elementary fermions that contain the factor $\frac{1}{2}(1 \pm \gamma_5)$ for right- and left-chirality; in the one particle and interaction namely neutrinos acting weakly that are always left-chiral; and in the many observed pseudo-scalar mesons ($J^{PC} = 0^{-+}$) and pseudo-vector mesons ($J^{PC} = 1^{++}$ and $J^{PC} = 1^{+-}$) laid out in [17], all of which require the use of γ_5 to underpin their theoretical origins. So γ_5 is real and physical, as would therefore be any fifth dimension which can be *properly-connected* with γ_5.

But the immediate problem as pointed out in toward the end of [11], is that because $G_{55} = \phi^2$ in the Kaluza-Klein metric tensor (1.1), if we require electromagnetic energy densities to be positive, the fifth-dimension must have a *spacelike* signature. And this directly contradicts making γ_5 the generator of the fifth dimension because $\gamma_5\gamma_5 = 1$ produces a *timelike* signature. So, as physically-real and pervasive as are the observable consequences of the γ_5 matrix, the Kaluza-Klein metric tensor (1.1) does *not* furnish a theoretical basis for associating γ_5 with a fifth dimension, at the very least because of this timelike-versus-spacelike contradiction. This is yet another problem stemming from having ϕ carry the burden of maintaining the fifth-dimensional signature and the fundamental character of the Minkowski tangent space.

So, to summarize, on the one hand, Kaluza-Klein theory has a fifth physical dimension on a par with space and time, but it has been impossible to connect that dimension with actual observations in the material, physical universe, or to make credible sense of the dilation and contraction of that dimension based on the behavior of a scalar field. On the other hand, Dirac theory has an eminently-physical γ_5 with pervasive observational manifestations on an equal footing with γ_μ, but it has been impossible to connect this γ_5 with a true physical fifth dimension (or at least, with the Kaluza-Klein metric tensor (1.1) in five dimensions), at minimum because the metric tensor signatures conflict. *Kaluza-Klein has a fifth-dimension unable to connect to physical reality, while Dirac theory has a physically-real γ_5 unable to connect to a fifth dimension.* And the origin of this disconnect on both hands, is that the Kaluza-Klein metric tensor (1.1) cannot be deconstructed into Dirac-type matrices while maintaining five-dimensional general covariance according to (3.1). *To maintain general covariance and achieve a Dirac-type square root operator deconstruction of the metric tensor, <u>(1.1) must be replaced by (4.22)</u>.*

Once we use (4.22), all of these problems evaporate. Kaluza-Klein theory becomes fully capable of describing fermions, because the matrices (3.8) are merely Dirac operator square roots of the metric tensor. With $G_{55} = 1 + \phi^2$ the metric signature is decoupled from the energy requirements for ϕ. Most importantly, when $A_{\gamma j}=0$ and $\phi=0$ and $g_{\mathrm{MN}}=\eta_{\mathrm{MN}}$, because $\operatorname{diag}\left(G_{\mathrm{MN}}\right)=(+1,-1,-1,-1,+1)=\operatorname{diag}\left(\frac{1}{2}\{\gamma_{\mathrm{M}},\gamma_{\mathrm{N}}\}\right)=\operatorname{diag}\left(\eta_{\mathrm{MN}}\right)$, and because of this decoupling of ϕ from the metric signature, we now have a timelike $\eta_{55}=\gamma_5\gamma_5=+1$ which is directly generated by γ_5. As a consequence, the fifth dimension of Kaluza-Klein theory which has heretofore been disconnected from physical reality, can now be identified with a true physical dimension that has γ_5 as its generator, just as γ_0 is the generator of a truly-physical time dimension and γ_j are the generators of a truly-physical space dimensions. And again, γ_5 has a wealth of empirical evidence to support its reality.

Further, with (14.22) we now have two timelike and three spacelike dimensions, with matching tangent-space signatures between Dirac theory and the Dirac-Kaluza-Klein theory. With the fifth-dimension now being timelike not spacelike, the notion of "curling up" the fifth dimension into a tiny "cylinder" comes off the table completely, while the Feynman-Wheeler concept of "many-fingered time" returns to the table, providing a possible avenue to study future probabilities which congeal into past certainties as the arrow of time progresses forward with entropic increases. And because γ_5 is connected to a multitude of confirmed observed phenomena in the physical universe, the physical reality of the fifth dimension in the metric tensor (4.22) is now supported by every single observation ever made of the reality of γ_5 in particle physics, regardless of any other epistemological interpretations one may also arrive at for this fifth dimension.

Moreover, although the field equations obtained from (4.22) rather than (1.1) will change somewhat because now $G_{05}=G_{50}=\phi$ and $G_{55}=1+\phi^2$ and the gauge fields are fixed to the photon $A_\mu=A_{\gamma\mu}$ with only two degrees of freedom, there is no reason to suspect that the many good benefits of Kaluza-Klein theory will be sacrificed because of these changes which eliminate the foregoing problems. Rather, we simply expect some extra terms (and so expect some additional phenomenology) to emerge in the equations of motion and the field equations because of these modifications. But the Kaluza-Klein benefits having of Maxwell's equations, the Lorentz Force motion and the Maxwell-stress energy embedded in the field equations should remain fully intact when using (4.22) in lieu of (1.1), as illustrated by the derivation and discussion of (4.26).

Finally, given all of the foregoing, beyond the manifold observed impacts of γ_5 in particle physics, there is every reason to believe that using the five-dimensional Einstein equation with (4.22) will fully enable us to understand this fifth dimension, at bottom, as a *matter dimension*, along the lines long-advocated by the 5D Space-Time-Matter Consortium [18]. This may thereby bring us ever-closer to uncovering the truly-geometrodynamic theoretical foundation at the heart of all of nature.

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