On tensors and equations of the electromagnetic field

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Abstract

It is shown that the electromagnetic field is completely described by an asymmetric tensor of the second rank $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu$, which is a four-dimensional derivative of the electromagnetic potential. This tensor can be decomposed into the canonical antisymmetric and the new symmetric EMF tensor. From this tensor, in the form of its complete divergence, the EMF equations follow. One of them is an electromagnetic analog of the Lame equation for an elastic medium. It is shown that the longitudinal waves of the divergence of the vector potential propagate at a speed $\sqrt{2}$ greater than the speed of light and do not have a magnetic component.

Keywords: Electromagnetic field, asymmetric tensor, symmetric tensor, Maxwell equations, longitudinal waves.

Contents

1. Introduction

The theoretical basis of the classical theory of the electromagnetic field (EMF) is Maxwell's equations, generalizing the experimental results obtained by the end of the 18th century. The development of the classical theory of EMF led to its description in the form of an antisymmetric tensor of the second rank, from which the Maxwell equations follow. These equations played a key role in the development of theoretical physics and had a strong influence on the creation of a special theory of relativity and other theories. Already at the beginning of the twentieth century, classical electrodynamics was considered to be a complete science, and the theory of EMF was further developed in the form of quantum electrodynamics. Despite this, in the classical theory of EMF there were some vague spots and controversial issues. For example, for about a hundred years there is the Abraham-Minkowski problem, the crux of the problem lay in the absence of a common opinion about the correct energy-momentum tensor of the interaction of EMF with matter, the form of the electromagnetic pulse in matter and the existence of the Abraham electromagnetic force [1-13]. This situation leads to the search for new energy-momentum tensors of EMF [14-20]. At present, the bibliography on this problem is about 300 works [21]. Another problem is the mechanism for transferring the moment of an electromagnetic pulse by a plane electromagnetic wave [22-32]. The problem is that the canonical EMF wave equations do not describe this process. Until recently, electrodynamics did not even have wave equations for the energy and momentum of EMF. Such equations, following from the new energy-momentum tensor and describing the transfer of the angular momentum of the electromagnetic wave, were obtained by the author in work [20]. In the classical theory of EMF, Newton's third law is not always satisfied in the interaction of arbitrarily moving electric charges and non-parallel currents. This led to the hypothesis of the existence of a "scalar (potential) magnetic field" [33], the introduction of which into electrodynamics makes it possible to ensure the fulfillment of Newton's third law in all cases. The reality of the scalar magnetic field is confirmed in experiments on the longitudinal interaction of direct currents [33-35]. There is the problem of longitudinal electromagnetic waves in vacuum [36-42], which is that Maxwell's equations allow the existence of longitudinal waves of a scalar electromagnetic potential, but it has not been possible to detect such waves experimentally yet. The most obvious incompleteness of classical electrodynamics is manifested in plasma theory. Until now, there is no understanding of what electromagnetic forces hold charged particles in ball lightning, and the problem of prolonged plasma confinement in existing technical installations, despite half a century of intensive work, is far from being solved. There is no understanding of the cause of the existence of hot spots in Z-pinches, the phenomenon of the magnetic dynamo and a number of other plasma phenomena. The above mentioned problems require reasonable attention to the basics of the classical EMF theory and to the Maxwell equations themselves.

The purpose of this article is to consider the foundations of the classical theory of EMF with the aim of eliminating the existing discussion issues in it and the convergence of the classical theory with quantum electrodynamics.

One of the reasons for posing this problem was the author's opinion that the description of EMF using the canonical antisymmetric tensor is incomplete and does not ensure the mathematical correctness of the EMF sources introduction into its equations. The reason for this was the following. Maxwell's equations with sources follow from the canonical antisymmetric EMF tensor in the form of a four-dimensional divergence along one of its indices, which is equated to the source of the field in the form of a four-dimensional current density [43]. But the antisymmetric tensor of the second rank has divergences for each of the indices. These two divergences have opposite signs, so the total divergence of the antisymmetric EMF tensor as a four-dimensional rotor, is zero and cannot have a field source. Therefore, equating only one of the divergences of the antisymmetric EMF tensor to the source of the field is mathematically incorrect.

Usually, the antisymmetric EMF tensor is written in the form $F_{[\mu\nu]} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$, where A_ν is the four-dimensional electromagnetic potential. The first term of this expression is the four-dimensional derivative of the electromagnetic potential and is an asymmetric tensor of the second rank $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu$. This EMF tensor can be written in the form of its expansion into symmetric and antisymmetric tensors $F_{\mu\nu} = F_{[\mu\nu]}/2 + F_{(\mu\nu)}/2$. The first term of this expansion is the canonical antisymmetric EMF tensor $F_{[\mu\nu]} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$, and the second term represents the new symmetric EMF tensor $F_{(\mu\nu)} = \partial_\mu \mathbf{A}_\nu + \partial_\nu \mathbf{A}_\mu$. Thus, a complete description of the EMF is an asymmetric tensor of the second rank $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu$. From this tensor, in the form of its divergences, the EMF equations follow. Since the total divergence of the canonical antisymmetric tensor, as a four-dimensional rotor, is identically zero, the EMF equations in the form of a full four-dimensional divergence follow from the symmetric tensor $F_{(\mu\nu)} = \partial_\mu \mathbf{A}_\nu + \partial_\nu \mathbf{A}_\mu$. This divergence of the symmetric tensor should be attributed to EMF sources in the form of charges and currents.

In this article EMF and its sources are considered in a vacuum. The geometry of space-time is taken in the form of pseudo-Euclidean Minkowski space in the form (ct, ix, iy, iz) and differentiation between covariant and contravariant indexes is irrelevant. The four-dimensional electromagnetic potential is defined as $\mathbf{A}_\nu(\varphi / c, i\mathbf{A})$, where φ and A are the scalar and vector potentials of the EMF. The four-dimensional current density is defined as $\mathbf{J}_\nu(\rho \cdot c, i\mathbf{J})$, where ρ and J are the electric charge density and current density.

2. Asymmetric and symmetric electromagnetic field tensors

The asymmetric EMF tensor $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu$, which is a four-dimensional derivative of the electromagnetic potential, is written in the matrix form:

$$F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu = \begin{pmatrix} \frac{1}{c^2}\partial_t\varphi & \frac{1}{c}i\cdot\partial_t A_x & \frac{1}{c}i\cdot\partial_t A_y & \frac{1}{c}i\cdot\partial_t A_z \\ -\frac{1}{c}i\cdot\partial_x\varphi & \partial_x A_x & \partial_x A_y & \partial_x A_z \\ -\frac{1}{c}i\cdot\partial_y\varphi & \partial_y A_x & \partial_y A_y & \partial_y A_z \\ -\frac{1}{c}i\cdot\partial_z\varphi & \partial_z A_x & \partial_z A_y & \partial_z A_z \end{pmatrix} \tag{1}$$

This asymmetric tensor can be decomposed into antisymmetric and symmetric tensors $F_{\mu\nu} = F_{[\mu\nu]}/2 + F_{(\mu\nu)}/2$. The antisymmetric EMF tensor is well known in electrodynamics [43] and it is not given here. We write the symmetric tensor EMF $F_{(\mu\nu)}$ in the matrix form:

$$F_{(\mu\nu)} = \begin{pmatrix} 2\frac{1}{c^2}\partial_t\varphi & \frac{1}{c}i\cdot(\partial_t A_x - \partial_x\varphi) & \frac{1}{c}i\cdot(\partial_t A_y - \partial_y\varphi) & \frac{1}{c}i\cdot(\partial_t A_z - \partial_z\varphi) \\ \frac{1}{c}i\cdot(\partial_t A_x - \partial_x\varphi) & 2\partial_x A_x & (\partial_x A_y + \partial_y A_x) & (\partial_x A_z + \partial_z A_x) \\ \frac{1}{c}i\cdot(\partial_t A_y - \partial_y\varphi) & (\partial_x A_y + \partial_y A_x) & 2\partial_y A_y & (\partial_y A_z + \partial_z A_y) \\ \frac{1}{c}i\cdot(\partial_t A_z - \partial_z\varphi) & (\partial_x A_z + \partial_z A_x) & (\partial_y A_z + \partial_z A_y) & 2\partial_z A_z \end{pmatrix} \tag{2}$$

The canonical antisymmetric EMF tensor $F_{[\mu\nu]}$ describes the four-dimensional rotation of the EMF. Then, by analogy with a continuous medium, the symmetric tensor $F_{(\mu\nu)}$ describes the four-dimensional deformation of the EMF. The members of its diagonal describe the volume deformation of the EMF expansion/contraction, and the remaining terms describe four-dimensional shear deformations.

3. Equations of the electromagnetic field without field sources

Let us write down four-dimensional divergences with respect to the indices μ and ν of the asymmetric tensor $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu$ (differentiation between covariant and contravariant indexes is irrelevant):

$$\partial_\mu F_{\mu\nu} = \partial_\mu(\partial_\mu \mathbf{A}_\nu) \quad \text{and} \quad \partial_\nu F_{\mu\nu} = \partial_\nu(\partial_\mu \mathbf{A}_\nu) \tag{3}$$

We write these Eqs. in expanded form:

$$\frac{1}{c^2}\partial_{tt}\varphi - \Delta\varphi = 0 \tag{4}$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \Delta\mathbf{A} = 0 \tag{5}$$

$$\partial_t(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}) = 0 \tag{6}$$

$$-\nabla(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}) = 0 \tag{7}$$

Eqs. (4) and (5) represent Maxwell's canonical equations in the Lorentz gauge $\partial_t\varphi/c^2 + \nabla\cdot\mathbf{A} = 0$. Eqs. (6) and (7) represent, respectively, the derivatives with respect to time and space of the Lorentz gauge condition. We obtain the complete divergence of the asymmetric EMF tensor by adding Eqs. (4) and (6), and also (5) and (7):

$$2\frac{1}{c^2}\partial_{tt}\varphi+\partial_t\nabla\cdot\mathbf{A}-\Delta\varphi=0 \tag{8}$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A}-\frac{1}{c^2}\partial_t\nabla\varphi-\nabla(\nabla\cdot\mathbf{A})-\Delta\mathbf{A}=0 \tag{9}$$

Let us write down four-dimensional divergences with respect to the indices μ and ν of the symmetric tensor $F_{(\mu\nu)}=\partial_\mu\mathbf{A}_\nu+\partial_\nu\mathbf{A}_\mu$. Since the tensor $F_{(\mu\nu)}$ is symmetric, these divergences are equal:

$$\partial_\mu(\partial_\mu\mathbf{A}_\nu+\partial_\nu\mathbf{A}_\mu)=\partial_\nu(\partial_\mu\mathbf{A}_\nu+\partial_\nu\mathbf{A}_\mu)=0$$

Writing these equations in expanded form, we obtain two Eqs. (8) and (9), completely describing the motions of the EMF. These two equations replace Maxwell's canonical equations in potentials:

$$\nabla\cdot(-\nabla\varphi-\partial_t\mathbf{A})=\nabla\cdot\mathbf{E}=0$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A}+\frac{1}{c^2}\partial_t\nabla\varphi+\nabla\times\nabla\times\mathbf{A}=-\frac{1}{c^2}\partial_t\mathbf{E}+\nabla\times\mathbf{B}=0$$

For the static case, Eq. (8) describes the Gaussian law for an electric field without sources $\nabla\cdot\mathbf{E}=-\Delta\varphi=0$. Eq. (9) can be written in the form:

$$\partial_{tt}\mathbf{A}-2c^2\nabla(\nabla\cdot\mathbf{A})+c^2\nabla\times\nabla\times\mathbf{A}=-\partial_t\nabla\varphi \tag{10}$$

This equation is an electromagnetic analog of the Lame equation (or the dynamic Navier-Stokes equation), known in the linear theory of elasticity and describing the wave motion of a continuous elastic medium [44]:

$$\partial_{tt}\mathbf{U}-\upsilon_1^2\cdot\nabla(\nabla\cdot\mathbf{U})+\upsilon_2^2\cdot\nabla\times\nabla\times\mathbf{U}=\mathbf{G}$$

where U is the displacement vector of the medium, υ_1 is the velocity of longitudinal waves, υ_2 is the velocity of transverse waves, and G is the external force. Comparison of this equation with Eq. (10) shows that the velocity of the longitudinal EMF waves is $\sqrt{2}$ greater than the transverse wave velocity, i.e. speed of light. The right-hand side of Eq. (10) describes the EMF wave source in the form of an alternating potential electric field. This source of electromagnetic waves is an electric dipole in the form of an electric capacitor deployed in space.

Eq. (8) can be written in the form:

$$\frac{2}{c^2}\partial_{tt}\varphi-\Delta\varphi=\nabla\cdot(-\partial_t\mathbf{A}) \tag{11}$$

This equation can be interpreted as a wave equation for a scalar electromagnetic potential. It follows from this that the waves of the scalar electromagnetic potential propagate at a speed that is $\sqrt{2}$ times slower than the speed of light. The source of the waves of the scalar potential is the divergence of the time-varying vector potential or the divergence of the vortex electric field.

4. Equations of the electromagnetic field with field sources

Since the total four-dimensional divergence of a symmetric tensor can be nonzero, we equate it with the source of the EMF, i.e. to a four-dimensional current density of $\partial_\mu F_{(\mu\nu)} = \mathbf{J}_\nu$ or $\partial_\mu(\partial_\mu \mathbf{A}_\nu + \partial_\nu \mathbf{A}_\mu) = \mathbf{J}_\nu$. Let us write down this complete four-dimensional divergence of a symmetric tensor $F_{(\mu\nu)}$ with sources in expanded form:

$$2\frac{1}{c^2}\partial_{tt}\varphi + \partial_t \nabla\cdot\mathbf{A} - \Delta\varphi = \rho/\varepsilon_0 \tag{12}$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c^2}\partial_t\nabla\varphi - \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A} = \mu_0\cdot\mathbf{J} \tag{13}$$

Eq. (12) in statics describes the Gaussian law for an electric field with sources $\nabla\cdot\mathbf{E} = -\Delta\varphi = \rho/\varepsilon$, and Eq. (13) replaces the Ampere-Maxwell total current equation. Eq. (13) can be written in the form:

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c^2}\partial_t\nabla\varphi - 2\cdot\nabla(\nabla\cdot\mathbf{A}) + \nabla\times\nabla\times\mathbf{A} = \mu_0\cdot\mathbf{J} \tag{14}$$

In this equation, the fourth term represents the magnetic field rotor, and the third term describes the gradient of the scalar magnetic field, hypothetically introduced by Nikolaev [33]. This term ensures the fulfillment in electrodynamics of Newton's third law in the interaction of arbitrarily moving electric charges and non-parallel currents. For the stationary case, Eq. (14) can be written in the form of an equation describing the Ampere law, but in which there is a Nikolaev scalar magnetic field:

$$-2\cdot\nabla(\nabla\cdot\mathbf{A}) + \nabla\times\nabla\times\mathbf{A} = \mu_0\cdot\mathbf{J} \tag{15}$$

We take the rotor from both sides of Eq. (13) and obtain the well-known wave equation for the magnetic field

$$\frac{1}{c^2}\partial_{tt}(\nabla\times\mathbf{A}) - \Delta(\nabla\times\mathbf{A}) = \mu_0\cdot(\nabla\times\mathbf{J}) \quad \text{or} \quad \frac{1}{c^2}\partial_{tt}\mathbf{B} - \Delta\mathbf{B} = \mu_0\cdot(\nabla\times\mathbf{J})$$

We take the divergence from both sides of Eq. (14) and obtain the equation of longitudinal waves of the divergence of the vector potential:

$$\frac{1}{c^2}\partial_{tt}\nabla\cdot\mathbf{A} - \frac{1}{c^2}\partial_t\Delta\varphi - 2\cdot\Delta(\nabla\cdot\mathbf{A}) = \mu_0\nabla\cdot\mathbf{J} \quad \text{or} \quad \frac{1}{2c^2}\partial_{tt}\nabla\cdot\mathbf{A} - \Delta(\nabla\cdot\mathbf{A}) = \mu_0\nabla\cdot\mathbf{J} + \frac{1}{c^2}\partial_t\Delta\varphi$$

From this equation follows the previously made conclusion that the velocity of longitudinal EMF waves is $\sqrt{2}$ times greater than the speed of light. It also follows from this equation that in the longitudinal EMF waves there is no magnetic component and they can be called electroscalar waves. The last equation, up to a rotor of an arbitrary vector, can be written in the form

$$\partial_{tt}\mathbf{A} - 2c^2\Delta\mathbf{A} = \mu_0\mathbf{J} + \partial_t\nabla\varphi \tag{16}$$

We write Eq. (13) in the form:

$$\partial_{tt}\mathbf{A} - c^2\nabla\times\nabla\times\mathbf{A} - 2c^2\Delta\mathbf{A} = c^2\mu_0\cdot\mathbf{J} + \partial_t\nabla\varphi \tag{17}$$

Taking Eq. (16) into account, eliminating the longitudinal part from Eq. (17), we obtain the equation of transverse electromagnetic waves used in radio communication:

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \nabla\times\nabla\times\mathbf{A} = \mu_0\cdot\mathbf{J} + \frac{1}{c^2}\partial_t\nabla\varphi \tag{18}$$

Thus, when electromagnetic waves are excited in accordance with Eq. (13), because of the different propagation velocities of the longitudinal and transverse waves, they are separated in space and described by separate Eqs. (16) and (18).

Let us consider the spatial character of the transverse wave process, which is determined by the second term on the left-hand side of Eq. (18). This term is a double vector potential rotor. From the Stokes theorem it follows that the flux of the vector rotor through the surface is equal to its circulation along a closed contour on which this surface rests. Consequently, the spatial term of the wave Eq. (18) describes the double circulation of the vector **A** along a closed contour. This spatial configuration may be presented in the form of a toroid. Similar spatial configurations are known in gas-hydrodynamics and represent stable vortex formations, so-called vortex rings. In liquid helium, these vortex rings are quantum objects. This allows us to hope that Eq. (18) will help to substantiate the corpuscular properties of electromagnetic radiation.

5. Conclusion

Thus, a complete description of the EMF is an asymmetric second-rank tensor $F_{\mu\nu} = \partial_\mu\mathbf{A}_\nu$, which is a four-dimensional derivative of the electromagnetic potential. This tensor can be decomposed into the canonical antisymmetric and the new symmetric EMF tensors. From this tensor, in the form of its complete divergence, the EMF equations follow.

The canonical antisymmetric electromagnetic field tensor has four-dimensional divergences with opposite signs for each of the indices, so the introduction of the field source into its divergence by only one of the indices is incorrect. The total divergence of the antisymmetric tensor, as a four-dimensional rotor, is zero and does not have a source of EMF. Since it cannot be attributed to the field sources, they must be attributed to the complete four-dimensional divergence of the symmetric tensor.

The EMF equations that replace Maxwell's equations follow from the symmetric EMF tensor. These equations describe transverse and longitudinal EMF waves. One of these equations is an electromagnetic analog of the Lame equation for an elastic medium. Longitudinal waves do not have a magnetic component and propagate at a speed $\sqrt{2}$ times greater than the speed of light.

The field equation replacing the Ampere-Maxwell equation includes a hypothetical scalar magnetic field that ensures the fulfillment of Newton's third law in electrodynamics.

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