# Boundary Matrices and the Marcus-de Oliveira Determinantal Conjecture

Ameet Sharma

**ARTICLE HISTORY**

Compiled April 21, 2019

**ABSTRACT**

We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region $\Delta$. This paper focuses on boundary matrices of $\Delta$. We prove 2 theorems regarding these boundary matrices. This paper uses ideas from [1].

**KEYWORDS**

determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

## 1. Introduction

Marcus [2] and de Oliveira [3] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2...a_n$ and $b_1, b_2...b_n$ respectively, $det(A + B)$ lies within the region:

$$co\{ \prod (a_i + b_{\sigma(i)}) \}$$

where $\sigma \in S_n$. $co$ denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = diag(a_1, a_2...a_n)$ and $B_0 = diag(b_1, b_2...b_n)$, let:

$$\Delta = \big\{ det(A_0 + UB_0U^*) : U \in U(n) \big\} \tag{1}$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

---

**Conjecture 1.1** (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq co\{ \prod(a_i + b_{\sigma(i)})\} \tag{2}$$

Let

$$M(U) = det(A_0 + UB_0U^*). \tag{3}$$

The paper is organized as follows. In section 2 we define terms and functions that will be used in the rest of the paper. These definitions are necessary to state our results. In section 3, we state 3 lemmas and 2 theorems that form the bulk of the paper. We state them in the order they are proved.

## 2. Preparatory definitions

### 2.1. Terms

For the purposes of this paper we define an **ordinary point** of a curve as a point on the curve that has the following properties:

- It is not a cusp.
- It is not a point of self-intersection.
- It is not an isolated point.
- The curve has a unique tangent at that point.

Given a unitary matrix U and square, diagonal matrices $A_0$ and $B_0$ all of dimension $n \times n$,

- If $M(U)$ is a point on $\partial\Delta$ (the boundary of $\Delta$), we call $M(U)$ a boundary point of $\Delta$ and we call U a **boundary matrix** of $\Delta$. See eq. (1) and eq. (3).
- We define the **B-matrix** of U as $UB_0U^*$.
- We define the **C-matrix** of U as $A_0 + UB_0U^*$.
- We define the **F-matrix** of U as $C^{-1}A_0 - A_0C^{-1}$ where C is the C-matrix of U. Note that the F-matrix is only defined when C is invertible, or equivalently when $det(C) = M(U) \neq 0$. See eq. (3). Also note that since $A_0$ is diagonal, the F-matrix is a zero-diagonal matrix. The idea for using the F-matrix comes from [1], Theorem 4, p.27.

Throughout the rest of the paper, we'll assume $A_0$ and $B_0$ are defined, even if we don't explicitly mention them.

### 2.2. Functions given a unitary matrix U

Given a unitary matrix U with B-matrix B, C-matrix C and F-matrix F. For every skew-hermitian matrix Z, we define the following functions

let

$$U_Z(t) = (e^{Zt})U \tag{4}$$

where $t$ is any real number.

Since the exponential of a skew-hermitian matrix is unitary, $U_Z(t)$ is a function of unitary matrices.

let

$$B_Z(t) = U_Z(t)B_0U_Z^*(t) \tag{5}$$

let $C_Z(t) = A_0 + B_Z(t)$

We note that $B_Z(0) = B$ and $C_Z(0) = C$.

let

$$R_Z(t) = det(C_Z(t)) \tag{6}$$

We can see by eq. (1) that $R_Z(t) \subseteq \Delta$.

$R_Z(0) = A_0 + UB_0U^*$

So by eq. (3) we see that $R_Z(0) = M(U)$.

So all the $R_Z(t)$ functions go through $M(U)$ at $t = 0$.

We shall refer to these functions in the rest of the paper with the same notation (for example $R_Z(t)$ for a skew-hermitian matrix Z. $R_{Z_1}(t)$ for a skew-hermitian matrix $Z_1$). Note that $R_Z(t)$ requires $A_0, B_0, U$ and $Z$ in order to be defined. But we won't explicitly mention $A_0$ and $B_0$. All the results in this paper assume there are two diagonal matrices $A_0$ and $B_0$ defined in the background.

### 2.3. Skew-Hermitian matrices $Z^{ab}$ and $Z^{ab,i}$

Given two integers a,b where $1 \leq a, b \leq n$ and $a \neq b$.

We define the $n \times n$ skew-hermitian matrix $Z^{ab}$ as follows. $Z_{ab}^{ab} = -1$ (the element at the ath row and bth column is -1.) $Z_{ba}^{ab} = 1$ (the element at the bth row and ath column is 1.) And all other elements are 0. Note that $Z^{ab} = -Z^{ba}$.

We define the $n \times n$ skew-hermitian matrix $Z^{ab,i}$ as follows. $Z_{ab}^{ab,i} = i$ and $Z_{ba}^{ab,i} = i$.

All other elements are zero. Note that $Z^{ab,i} = Z^{ba,i}$.

It is straightforward to verify that $Z^{ab}$ and $Z^{ab,i}$ are skew-hermitian.

## 3. Main Results

**Lemma 3.1.** *Given a unitary matrix U with $M(U) \neq 0$. Let F be its F-matrix. Then $R'_Z(0) = M(U)tr(ZF)$ for any skew-hermitian matrix Z.*

**Lemma 3.2.** *Given an $n \times n$ zero-diagonal matrix W. Given $tr(Z^{ab}W) = 0$ and $tr(Z^{ab,i}W) = 0$ for all pairs (a,b) where $1 \leq a, b \leq n$ and $a \neq b$. Then W is the zero-matrix.*

**Lemma 3.3.** *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\partial\Delta$ at $M(U)$ with direction vector v. Then for every skew-hermitian matrix Z, $tr(ZF) = cv$ where c is some real number.*

**Theorem 3.4.** *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Then F can be written uniquely in the form $F = e^{i\theta}H$ where H is a zero-diagonal hermitian matrix and $0 \leq \theta < \pi$.*

**Theorem 3.5.** *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\Delta$ at $M(U)$. By the previous theorem we know that $F = e^{i\theta}H$ for some real $0 \leq \theta < \pi$. Then L makes an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis.*

## 4. Proof of lemma 3.1

The proof given here uses ideas from [1], Theorem 4, p.26-27. But the proof given here is complete on its own.

**Proof.** We're given a unitary matrix U where $M(U) \neq 0$. So its F-matrix is well-defined and we call it F. Let B be its B-matrix, and C be its C-matrix. Given an arbitrary skew-hermitian matrix Z.

We can use Jacobi's formula [4] on eq. (6) to find $R'_Z(t)$

$$R'_Z(t) = tr(det(C_Z(t))C_Z^{-1}(t)C'_Z(t)) \tag{7}$$

$R'_Z(0) = tr(det(C_Z(0))C_Z^{-1}(0)C'_Z(0))$

We can substitute C for $C_Z(0)$.

$R'_Z(0) = tr(det(C)C^{-1}C'_Z(0))$

$R'_Z(0) = det(C)tr(C^{-1}C'_Z(0))$

We know that $C'_Z(t) = B'_Z(t)$ so

$R'_Z(0) = det(C)tr(C^{-1}B'_Z(0))$

By section 2.1 and eq. (3) we know that $det(C) = M(U)$

$$R'_Z(0) = M(U)tr(C^{-1}B'_Z(0)) \tag{8}$$

Using eq. (5),

$$B'_Z(t) = \frac{dU_Z(t)}{dt}B_0U^*_Z(t) + U_Z(t)B_0\frac{dU^*_Z(t)}{dt} \tag{9}$$

Using eq. (4),

$\frac{dU_Z(t)}{dt} = Ze^{Zt}U$

$U^*_Z(t) = (U^*)e^{-Zt}$

$\frac{dU^*_Z(t)}{dt} = -(U^*)Ze^{-Zt}$

Substitute these and eq. (4) into eq. (9)

$B'_Z(t) = Ze^{Zt}UB_0(U^*)e^{-Zt} - (e^{Zt})UB_0(U^*)Ze^{-Zt}$

$B'_Z(0) = ZUB_0U^* - UB_0(U^*)Z$

Using the definition of the C-matrix in section 2.1

$B'_Z(0) = Z(C - A_0) - (C - A_0)Z$

$C^{-1}B'_Z(0) = C^{-1}ZC - C^{-1}ZA_0 - Z + C^{-1}A_0Z$

$tr(C^{-1}B'_Z(0)) = tr(C^{-1}ZC) - tr(C^{-1}ZA_0) - tr(Z) + tr(C^{-1}A_0Z)$

The first and third terms cancel since similar matrices have the same trace.

$tr(C^{-1}B'_Z(0)) = -tr(C^{-1}ZA_0) + tr(C^{-1}A_0Z)$.

Using the idea that $tr(XY) = tr(YX)$

$tr(C^{-1}B'_Z(0)) = -tr(ZA_0C^{-1}) + tr(ZC^{-1}A_0)$

$tr(C^{-1}B'_Z(0)) = tr(Z(C^{-1}A_0 - A_0C^{-1}))$

$tr(C^{-1}B'_Z(0)) = tr(ZF)$

Substitute this into eq. (8) to get

$$R'_Z(0) = M(U)tr(ZF) \tag{10}$$

5

This proves lemma 3.1. □

## 5. Proof of lemma 3.2

***Proof.*** Given an $n \times n$ zero-diagonal matrix W. Given that for every pair (a,b) where $1 \le a, b \le n$ and $a \ne b$,

$tr(Z^{ab}W) = 0.$

$tr(Z^{ab,i}W) = 0$

(See section 2.3 for definitions of $Z^{ab}$ and $Z^{ab,i}$).

by direct computation we see that

$tr(Z^{ab}W) = W_{ab} - W_{ba} = 0$

$tr(Z^{ab,i}W) = (W_{ab} + W_{ba})i = 0$

Solving these, we get that $W_{ab} = 0$ and $W_{ba} = 0$. So all the off-diagonal elements of W are zero. Hence W is the zero-matrix. □

## 6. Proof of lemma 3.3

***Proof.*** Given a boundary matrix U with $M(U) \ne 0$ and with F-matrix $F \ne 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\partial\Delta$ at $M(U)$. Let v be the direction vector of the line L. Note that v is just a non-zero complex number.

Let Z be a skew-hermitian matrix. By lemma 3.1 we know that $R'_Z(0) = M(U)tr(ZF)$.

Since $R_Z(t) \subseteq \Delta$ and $R_Z(0) = M(U)$, we know that $R'_Z(0) = kv$ for some real number k. (since L is the unique tangent to $\partial\Delta$ at $M(U)$, then it must the tangent to every curve that lies in $\Delta$, goes through $M(U)$ and has a well-defined derivative at $M(U)$).

So, $M(U)tr(ZF) = kv$

$tr(ZF) = (\frac{k}{M(U)})v$ □

## 7. Proof of theorem 3.4

***Proof.*** Given a boundary matrix U with $M(U) \ne 0$ and with F-matrix $F \ne 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$.

We pick an arbitrary pair (a,b) such that $1 \le a, b \le n$ and $a \ne b$

We have two skew-hermitian matrices $Z^{ab}$ and $Z^{ab,i}$ defined as per section 2.3.

By direct computation we see that

$$tr(Z^{ab}F) = F_{ab} - F_{ba}$$

$$tr(Z^{ab,i}F) = (F_{ab} + F_{ba})i$$

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$. (note that these are not tensors. $F_{ab,r}$ is just the real component of $F_{ab}$ and $F_{ab,i}$ is just the imaginary component.) We can substitute this in to get

$$tr(Z^{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i}) \tag{11}$$

$$tr(Z^{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r}) \tag{12}$$

We know by lemma 3.3 that these are collinear vectors in the complex plane.

So we know that

$$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$$

We can simplify this to get:

$$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$$

$$|F_{ab}| = |F_{ba}|$$

We can write:

$$F_{ab} = |F_{ab}| \angle \theta_{ab}$$

$$F_{ba} = |F_{ab}| \angle \theta_{ba}$$

Note that we are given that $F \neq 0$. Note that we already know by section 2.1 that F is zero-diagonal.

We will divide the possible values of F into multiple cases. First we split all cases into two. The first is when only one pair of elements of the F-matrix, $F_{ab}$ and $F_{ba}$ is nonzero. The second case is when multiple pairs of elements of the F-matrix are nonzero. We shall further subdivide the second case using the fact that all tr(ZF) values are collinear. We can divide these cases into 3 possibilities: 1. All nonzero $tr(ZF)$ values are imaginary. 2. All nonzero $tr(ZF)$ values are real. 3. All nonzero $tr(ZF)$ values are not real or imaginary. (note that since F is nonzero, we don't have to deal with the possibility that tr(ZF) is 0 for all skew-hermitian matrices Z. see lemma 3.2).

So we have 4 cases to deal with.

**Case 1: $|F_{ab}|$ is non-zero for only one pair $\{a, b\}$ where $a \neq b$**

In this case,

$H = e^{-(\theta_{ab} + \theta_{ba})/2}F$ is a hermitian matrix, and we're finished.

7

**Case 2: $|F_{ab}|$ is non-zero for multiple pairs $\{a, b\}$ where $a \neq b$. For any skew-hermitian Z, when tr(ZF) is non-zero, it is imaginary.**

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = -\theta_{ba}$. This holds for all distinct pairs {a,b}, so our F-matrix is already hermitian, and we're done.

**Case 3: $|F_{ab}|$ is non-zero for multiple pairs $\{a, b\}$ where $a \neq b$. For any skew-hermitian Z, when tr(ZF) is non-zero, it is real.**

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = \pi - \theta_{ba}$. This holds for all distinct pairs {a,b}

$H = e^{-(\frac{\pi}{2})}F$ is hermitian and we're done.

**Case 4: $|F_{ab}|$ is non-zero for multiple pairs $\{a, b\}$ where $a \neq b$. For any skew-hermitian matrix Z, when tr(ZF) is non-zero, it isn't real or imaginary.**

Suppose $|F_{ab}| \neq 0$ and $|F_{cd}| \neq 0$

if $tr(Z^{ab}F) \neq 0$, then

slope of $tr(Z^{ab}F) = \frac{\sin(\theta_{ab}) - \sin(\theta_{ba})}{\cos(\theta_{ab}) - \cos(\theta_{ba})} = -\cot(\frac{\theta_{ab} + \theta_{ba}}{2})$

if $tr(Z^{ab,i}F) \neq 0$:

slope of $tr(Z^{ab,i}F) = \frac{\cos(\theta_{ab}) + \cos(\theta_{ba})}{-\sin(\theta_{ab}) - \sin(\theta_{ba})} = -\cot(\frac{\theta_{ab} + \theta_{ba}}{2})$

We know that since $|F_{ab}| \neq 0$, at least one of $tr(Z^{ab}F)$ or $tr(Z^{ab,i}F)$ is non-zero.

similarly,

if $tr(Z^{cd}F) \neq 0$, then

slope of $tr(Z^{cd}F) = -\cot(\frac{\theta_{cd} + \theta_{dc}}{2})$

if $tr(Z^{cd,i}F) \neq 0$:

slope of $tr(Z^{cd,i}F) = -\cot(\frac{\theta_{cd} + \theta_{dc}}{2})$

We know that since $|F_{cd}| \neq 0$, at least one of $tr(Z^{cd}F)$ or $tr(Z^{cd,i}F)$ is non-zero.

So we have:

$\cot(\frac{\theta_{cd} + \theta_{dc}}{2}) = \cot(\frac{\theta_{ab} + \theta_{ba}}{2})$ (lemma 3.3)

therefore:

$\frac{\theta_{cd} + \theta_{dc}}{2} = \frac{\theta_{ab} + \theta_{ba}}{2} + n\pi$ for some integer n.

We can freely adjust $\theta_{cd}$ by $-2n\pi$. It makes no difference since $|F_{cd}|\angle\theta_{cd} = |F_{cd}|\angle(\theta_{cd} - 2n\pi)$

So after the adjustment we have:

$\frac{\theta_{cd} + \theta_{dc}}{2} = \frac{\theta_{ab} + \theta_{ba}}{2}$.

We make the same adjustment for any pair $\{c, d\} \neq \{a, b\}$ where $|F_{cd}| \neq 0$

We set $\beta = \frac{\theta_{ab} + \theta_{ba}}{2}$

8

let $H = e^{-i\beta}F$

For some pair (x,y) where $x \neq y$ and $|H_{xy}| \neq 0$,

$H_{xy} = |H_{xy}|\angle\alpha_{xy}$

$\alpha_{xy} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{xy}$

$\alpha_{yx} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{yx}$

But because of our adjustments,

$\frac{\theta_{ab}+\theta_{ba}}{2} = \frac{\theta_{xy}+\theta_{yx}}{2}$

Plugging this into the above two formulas we have

$\alpha_{xy} = \frac{\theta_{xy}-\theta_{yx}}{2}$

$\alpha_{yx} = -(\frac{\theta_{xy}-\theta_{yx}}{2})$

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

So in all 4 cases we can write $F = e^{i\beta}H$ for some hermitian matrix H and some real $\beta$. But we've not arrived at a unique representation for F yet.

Suppose

$F = e^{i\beta_1}H_1 = e^{i\beta_2}H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2 = H_2^* = e^{i(\beta_2-\beta_1)}H_1^* = e^{i(\beta_2-\beta_1)}H_1$

So

$(e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)})H_1 = 0$

Since $F \neq 0$, we know $H_1 \neq 0$ so

$e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)} = 0$

$e^{i(\beta_1-\beta_2)} = e^{i(\beta_2-\beta_1)}$

Then

$\beta_1 - \beta_2 = \beta_2 - \beta_1 + 2k\pi$, for any integer k

$\beta_1 = \beta_2 + k\pi$

So if we restrict all $\beta$ to $0 \leq \beta < \pi$, we have a unique representation since k is forced to 0.

This completes our proof of theorem 3.4. $\qquad\square$

## 8. Proof of theorem 3.5

Given a boundary matrix U with $M(U) \neq 0$ and F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\partial\Delta$ at $M(U)$.

**Proof.** By theorem 3.4 we know that

$$F = e^{i\theta}H \tag{13}$$

for some real $0 \leq \theta < \pi$ and some zero-diagonal hermitian matrix H.

We can substitute eq. (13) into eq. (11) and eq. (12) and simplify to get:

$$tr(Z^{ab}F) = 2H_{ab,i}e^{i(\theta+\pi/2)} \tag{14}$$

$$tr(Z^{ab,i}F) = 2H_{ab,r}e^{i(\theta+\pi/2)} \tag{15}$$

By lemma 3.2 we know that at least one of the above equations is nonzero for some pair (a,b). So then using lemma 3.1 we know that $R'_Z(0) = M(U)tr(ZF) \neq 0$ for some skew-hermitian matrix Z.

So by eq. (14) and eq. (15) we see that for some skew-hermitian matrix Z, $tr(ZF)$ forms an angle of $(\theta + \pi/2)$ or $(\theta + 3\pi/2)$ with the positive real axis (depending on whether the coefficient is negative or not). Therefore $R'_Z(0)$ forms an angle $arg(M(U)) + \theta + \pi/2$ or $arg(M(U)) + \theta + 3\pi/2$ with the positive real axis.

Therefore the line L forms an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis (since this is a line as opposed to a vector, a rotation of $\pi$ makes no difference).

This completes our proof of theorem 3.5. $\qquad\square$

### References

[1] Bebiano N, Querió J. The determinant of the sum of two normal matrices with prescribed eigenvalues. Linear Algebra and its Applications. 1985;71:23–28.

[2] Marcus M. Derivations, plücker relations and the numerical range. Indiana University Math Journal. 1973;22:1137–1149.

[3] de Oliveira GN. Research problem: Normal matrices. Linear and Multilinear Algebra. 1982; 12:153–154.

[4] Wikipedia contributors. Jacobi's formula — Wikipedia, the free encyclopedia ; 2019. [Online; accessed 13-February-2019]; Available from: https://en.wikipedia.org/w/index.php?title=Jacobi%27s_formula&oldid=880845059.