question 452: the number pi , elementary formulas

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abstract

This note presents some elementary formulas involving π .

1. Introduction

The number π is defined by

$$\pi = 4\int_0^1 \sqrt{1-x^2}\, dx = 3.14159265\ ... \tag{1}$$

this note presents some elementary formulas involving π.

2. Formulas

Let $n > 1$, $u \geq 0$, $v = (1+u^n)^{-1}$, then

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} + \frac{u}{1+u^n} = \int_0^u \frac{1}{1+x^n}\, dx + \int_0^v \sqrt[n]{\frac{1-x}{x}}\, dx \tag{2}$$

Let $n > 1$, $0 < u \leq 1$, then

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} + \frac{u}{1+u^n} = \sum_{k=0}^{\infty} \frac{(-1)^k u^{n k+1}}{n k+1} + n (1+u^n)^{1/n} \sum_{k=0}^{\infty} \frac{(-1/n)_k (1+u^n)^{-k-1}}{k! (n k+n-1)} \tag{3}$$

Let $n > 1$, $u < 1$, $v > 1$, $z = (1+u^n)^{-1}$, $w = (1+v^n)^{-1}$, then

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} = v w - u z + \sum_{k=0}^{\infty} \frac{(-1)^k u^{n k+1}}{n k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k v^{-n k-n+1}}{n k+n-1} + n \sum_{k=0}^{\infty} \frac{(-1/n)_k \left(z^{k-\frac{1}{n}+1} - w^{k-\frac{1}{n}+1}\right)}{k! (n k+n-1)} \tag{4}$$

Let $n > 1$, $u > 0$, then

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} + \frac{u}{1+u^n} = u F\left(1, \frac{1}{n}, 1+\frac{1}{n}, -u^n\right) + \frac{n}{n-1} (1+u^n)^{-1+\frac{1}{n}} F\left(-\frac{1}{n}, 1-\frac{1}{n}, 2-\frac{1}{n}, (1+u^n)^{-1}\right) \tag{5}$$

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} + \frac{u}{1+u^n} = \frac{u}{1+u^n} F\left(1, 1, 1+\frac{1}{n}, \frac{u^n}{1+u^n}\right) + \frac{n}{n-1} (1+u^n)^{-1+\frac{1}{n}} F\left(-\frac{1}{n}, 1-\frac{1}{n}, 2-\frac{1}{n}, (1+u^n)^{-1}\right) \tag{6}$$

$$\frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} + \frac{u}{1+u^n} = \frac{u}{(1+u^n)^{1/n}} F\left(\frac{1}{n}, \frac{1}{n}, 1+\frac{1}{n}, \frac{u^n}{1+u^n}\right) + \frac{n}{n-1} (1+u^n)^{-1+\frac{1}{n}} F\left(-\frac{1}{n}, 1-\frac{1}{n}, 2-\frac{1}{n}, (1+u^n)^{-1}\right) \tag{7}$$

Let $n > 1$, $u > 1$, then

$$\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}+\frac{u}{1+u^n}=\frac{u}{1+u^n}\,F\left(1,\,1,\,1+\frac{1}{n},\,\frac{u^n}{1+u^n}\right)+\frac{n}{n-1}\,\frac{u}{1+u^n}\,F\left(1,\,-\frac{1}{n},\,2-\frac{1}{n},\,-\frac{1}{u^n}\right) \tag{8}$$

$$\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}+\frac{u}{1+u^n}=\frac{u}{(1+u^n)^{1/n}}\,F\left(\frac{1}{n},\,\frac{1}{n},\,1+\frac{1}{n},\,\frac{u^n}{1+u^n}\right)+\frac{n}{n-1}\,u^{-n+1}\,F\left(1-\frac{1}{n},\,2,\,2-\frac{1}{n},\,-\frac{1}{u^n}\right) \tag{9}$$

Let $n>1$, $u<1$, $v>1$, $z=(1+u^n)^{-1}$, $w=(1+v^n)^{-1}$, then

$$\begin{aligned}\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}=&\,v\,w-u\,z+u\,F\left(1,\,\frac{1}{n},\,1+\frac{1}{n},\,-u^n\right)+\frac{v^{-n+1}}{n-1}\,F\left(1,\,1-\frac{1}{n},\,2-\frac{1}{n},\,-v^{-n}\right)+\\&\frac{n}{n-1}\,z^{1-\frac{1}{n}}\,F\left(-\frac{1}{n},\,1-\frac{1}{n},\,2-\frac{1}{n},\,z\right)-\frac{n}{n-1}\,w^{1-\frac{1}{n}}\,F\left(-\frac{1}{n},\,1-\frac{1}{n},\,2-\frac{1}{n},\,w\right)\end{aligned} \tag{10}$$

Let $n>1$, $\phi=\dfrac{1+\sqrt{5}}{2}$, then

$$\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}+\phi^{-1-\frac{1}{n}}=\phi^{-1/n}\sum_{k=0}^{\infty}\frac{(-1)^k\,\phi^{-k}}{n\,k+1}+n\,\phi^{-1+\frac{1}{n}}\sum_{k=0}^{\infty}\frac{(-1/n)_k\,\phi^{-k}}{k\,!\,(n\,k+n-1)} \tag{11}$$

Let $v=\left(2\left(9+\sqrt{849}\right)\right)^{1/3}-8\left(\dfrac{3}{9+\sqrt{849}}\right)^{1/3}$, $u=\dfrac{1}{2\sqrt[3]{6}}\left(-\sqrt{v}+\sqrt{\dfrac{12}{\sqrt{v}}-v}\right)$, then

$$\frac{2\,\pi}{3\sqrt{3}}+u^2=\sum_{k=0}^{\infty}\frac{(-1)^k\,u^{3\,k+1}}{3\,k+1}+3\,u^{2/3}\sum_{k=0}^{\infty}\frac{(-1/3)_k\,u^k}{k\,!\,(3\,k+2)} \tag{12}$$

Let $u=\dfrac{1}{3}\left(-1+\left(\dfrac{25-3\sqrt{69}}{2}\right)^{1/3}+\left(\dfrac{25+3\sqrt{69}}{2}\right)^{1/3}\right)$, then

$$\frac{\pi}{2\sqrt{2}}+u^2=\sum_{k=0}^{\infty}\frac{(-1)^k\,u^{4\,k+1}}{4\,k+1}+4\,u^{3/4}\sum_{k=0}^{\infty}\frac{(-1/4)_k\,u^k}{k\,!\,(4\,k+3)} \tag{13}$$

Let $u=\dfrac{1}{6}\left(\left(2\left(9+\sqrt{93}\right)\right)^{1/3}-2\left(\dfrac{3}{9+\sqrt{93}}\right)^{1/3}\right)^{3/2}$, then

$$\frac{3\,\pi}{16}+u^2=\sum_{k=0}^{\infty}\frac{(-3/2)_k\,u^{2\,k+1}}{k\,!\,(2\,k+1)}+3\sum_{k=0}^{\infty}\frac{(-1/2)_k\,u^{\frac{2}{3}\,k+1}}{k\,!\,(2\,k+3)} \tag{14}$$

Let $u=0.732464\,...,$ $u^7+u^6+u-1=0$, then

$$\frac{\pi}{3}+\frac{\ln\left(2+\sqrt{3}\right)}{\sqrt{3}}-1+u^2=u-2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,u^{6\,k+1}}{6\,k+1}+6\sqrt[6]{2}\sum_{k=0}^{\infty}\frac{(-1/6)_k\left((1+u)^{k+\frac{5}{6}}-1\right)}{k\,!\,(6\,k+5)\,2^k} \tag{15}$$

Let $m\in\mathbb{N}=\{1,\,2,\,3,\,...\}$, $0<u<1$, $u\left(1+u^2\right)^{m+1}=1$, then

$$\frac{\pi}{2^{2\,m+1}}\binom{2\,m}{m}+u^2=u\,F\left(m+1,\,\frac{1}{2},\,\frac{3}{2},\,-u^2\right)+\frac{2\,(m+1)}{2\,m+1}\,u^{\frac{2\,m+1}{2\,m+2}}\,F\left(-\frac{1}{2},\,m+\frac{1}{2},\,m+\frac{3}{2},\,u^{\frac{1}{m+1}}\right) \tag{16}$$

Let $u=\left(\dfrac{1}{6}\left(100+12\sqrt{69}\right)^{1/3}+\dfrac{2}{3}\left(100+12\sqrt{69}\right)^{-1/3}-\dfrac{1}{3}\right)^2$, then

$$\frac{\pi}{4}+u^2=u\,F\left(2,\,\frac{1}{2},\,\frac{3}{2},\,-u^2\right)+\frac{4}{3}\,u^{3/4}\,F\left(-\frac{1}{2},\,\frac{3}{2},\,\frac{5}{2},\,u^{1/2}\right) \tag{17}$$

Let $u = 1 - \left(\frac{1}{2} + \frac{1}{18}\sqrt{93}\right)^{1/3} - \left(\frac{1}{2} - \frac{1}{18}\sqrt{93}\right)^{1/3}$, then

$$1 - \frac{\pi}{4} + u^2 - u = \sum_{n=0}^{\infty} \frac{2^{-n-1} c_n u^{n+1}}{n+1} - \frac{2}{3} u^{3/2} F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, u\right) \tag{18}$$

$c_{n+2} = 2(c_{n+1} - c_n)$, $n = 1, 2, 3, \ldots$, $\{c_n, n \geq 0\} = \{1, -2, -2, 0, 4, 8, 8, 0, \ldots\}$.

Let $0 < p < 1$, $u > 1$, then

$$\frac{\pi}{\sin(p\pi)} = \frac{1}{1-p}\left(\frac{1}{u}\right)^{1-p} F\left(1, 1-p, 2-p, -\frac{1}{u}\right) + \frac{2u^p}{2+u} \sum_{n=0}^{\infty} \left(\frac{u}{2+u}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{k+p} \tag{19}$$

$$\frac{\pi}{\sin(p\pi)} = \frac{u^p}{(1-p)(1+u)} F\left(1, 1, 2-p, \frac{1}{1+u}\right) + \frac{2u^p}{2+u} \sum_{n=0}^{\infty} \left(\frac{u}{2+u}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{k+p} \tag{20}$$

$$\frac{\pi}{\sin(p\pi)} = \frac{1}{(1-p)(1+u)^{1-p}} F\left(1-p, 1-p, 2-p, \frac{1}{1+u}\right) + \frac{2u^p}{2+u} \sum_{n=0}^{\infty} \left(\frac{u}{2+u}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{k+p} \tag{21}$$

Let $0 < p < 1$, $0 < u < 1$, $v > 1$, $w = \frac{u + u^2 + v + v^2}{2}$, then

$$\frac{\pi}{\sin(p\pi)} = \frac{1}{p} u^p F(1, p, 1+p, -u) + \frac{1}{1-p}\left(\frac{1}{v}\right)^{1-p} F\left(1, 1-p, 2-p, -\frac{1}{v}\right) + \frac{1}{w} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \frac{(-1)^k \left(v^{k+m+p+1} - u^{k+m+p+1}\right)}{w^k (k+m+p+1)} \tag{22}$$

Let $0 < p < 1/2$, $i = \sqrt{-1}$, then

$$p\pi = \frac{1}{\sin(p\pi) + i\cos(p\pi)} F(1, 1, 2, -\cos(p\pi) - i\sin(p\pi)) + \frac{1}{\sin(p\pi) - i\cos(p\pi)} F(1, 1, 2, -\cos(p\pi) + i\sin(p\pi)) \tag{23}$$

$$p\pi = \left(\frac{\sin(p\pi)}{2 + 2\cos(p\pi)} - \frac{i}{2}\right) F\left(1, 1, 2, \frac{1}{2} + \frac{i\sin(p\pi)}{2 + 2\cos(p\pi)}\right) + \left(\frac{\sin(p\pi)}{2 + 2\cos(p\pi)} + \frac{i}{2}\right) F\left(1, 1, 2, \frac{1}{2} - \frac{i\sin(p\pi)}{2 + 2\cos(p\pi)}\right) \tag{24}$$

$$\frac{\pi}{2} = \left(\frac{1}{\sqrt{2}+1} - i\right) F\left(1, 1, 2, \frac{1}{2} + \frac{i}{2\sqrt{2}+2}\right) + \left(\frac{1}{\sqrt{2}+1} + i\right) F\left(1, 1, 2, \frac{1}{2} - \frac{i}{2\sqrt{2}+2}\right) \tag{25}$$

Remark : $F(a, b, c, x) = {}_2F_1(a, b; c; x)$ is the hypergeometric function.

References

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