question 450: Some Integrals Involving the Euler's Constant

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abstract

This note presents some integrals involving the Euler-Mascheroni constant:

$$\gamma = \lim(H_n - \ln n) = 0.577215 \ldots$$

1. Introduction. Basic examples

$$\ln \pi - 2 \ln 2 - \gamma = \int_0^1 \ln \tanh^{-1} x \, dx \tag{1}$$

$$\ln \pi - 2 \ln 2 - \gamma = \int_0^\infty (\operatorname{sech} x)^2 \ln x \, dx \tag{2}$$

$$\ln \pi - 2 \ln 2 - \gamma = \int_{-\infty}^\infty (\operatorname{sech} e^x)^2 \, x \, e^x \, dx \tag{3}$$

$$\ln \pi - \ln 2 - \gamma = \int_0^1 \ln \ln\left(\frac{1+x}{1-x}\right) dx \tag{4}$$

$$\ln \pi - \ln 2 - \gamma = \int_0^1 \ln \ln\left(\frac{2-x}{x}\right) dx \tag{5}$$

$$\frac{1}{2}(\ln \pi - \ln 2 - \gamma) = \int_1^\infty \frac{\ln \ln x}{(1+x)^2} \, dx \tag{6}$$

$$\frac{1}{2}(\ln \pi - \ln 2 - \gamma) = \int_0^\infty \frac{e^{-x} \ln x}{(1+e^{-x})^2} \, dx \tag{7}$$

$$\frac{1}{2}(\ln \pi - \ln 2 - \gamma) = \int_{-\infty}^\infty \frac{x \, e^{x-e^x}}{(1+e^{-e^x})^2} \, dx \tag{8}$$

$$\frac{1}{2}(\ln \pi - \ln 2 - \gamma) = -\int_{-\infty}^\infty \frac{x \, e^{-x-e^{-x}}}{(1+e^{-e^{-x}})^2} \, dx \tag{9}$$

2. Integrals

$$\ln \pi - \ln 2 - \gamma = \int_0^\infty \frac{2}{1+e^{e^x}} \, dx - \int_0^\infty \frac{1-e^{-e^{-x}}}{1+e^{-e^{-x}}} \, dx \tag{10}$$

$$\ln\pi-\ln 2-\gamma=\int_1^{\infty}\frac{2}{x\,(1+e^x)}\,dx-\int_1^{\infty}\frac{1-e^{-1/x}}{x\left(1+e^{-1/x}\right)}\,dx \tag{11}$$

$$\ln\pi-\ln 2-\gamma=\int_0^{1}\frac{2}{x\left(1+e^{1/x}\right)}\,dx-\int_0^{1}\frac{1-e^{-x}}{x\,(1+e^{-x})}\,dx \tag{12}$$

$$\ln\pi-\ln 2-\gamma=\int_{-\infty}^{\infty}\frac{2\,e^x}{1+e^{e^{e^x}}}\,dx-\int_{-\infty}^{\infty}\frac{\left(1-e^{-e^{-e^x}}\right)e^x}{1+e^{-e^{-e^x}}}\,dx \tag{13}$$

$$\ln\pi-\ln 2-\gamma=\int_{-\infty}^{\infty}\frac{2\,e^{-x}}{1+e^{e^{e^{-x}}}}\,dx-\int_{-\infty}^{\infty}\frac{\left(1-e^{-e^{-e^{-x}}}\right)e^{-x}}{1+e^{-e^{-e^{-x}}}}\,dx \tag{14}$$

$$\ln\pi-\ln 2-\gamma=\int_0^{\infty}e^{-e^x/2}\,\mathrm{sech}\left(\frac{e^x}{2}\right)dx-\int_0^{\infty}\tanh\left(\frac{e^{-x}}{2}\right)dx \tag{15}$$

$$\ln\pi-\ln 2-\gamma=v^2-u^2+\int_0^{u}\left(\frac{2}{1+e^{e^x}}+\ln\ln\left(\frac{2-x}{x}\right)\right)dx-\int_0^{v}\left(\frac{1-e^{-e^{-x}}}{1+e^{-e^{-x}}}-\ln\ln\left(\frac{1+x}{1-x}\right)\right)dx \tag{16}$$

where $u=0.377398$ .., is root of the equation : $u=2\left(1+e^{e^u}\right)^{-1}$, and $v=0.341205$ ..., is root of the equation : $v=\dfrac{1-e^{-e^{-v}}}{1+e^{-e^{-v}}}$.

$$\ln\pi-2\ln 2-\gamma=\int_0^{\infty}(1-\tanh e^x)\,dx-\int_0^{\infty}\tanh e^{-x}\,dx \tag{17}$$

$$\ln\pi-2\ln 2-\gamma=\int_1^{\infty}\frac{1-\tanh x}{x}\,dx-\int_1^{\infty}\frac{1}{x}\tanh\left(\frac{1}{x}\right)dx \tag{18}$$

$$\ln\pi-2\ln 2-\gamma=\int_0^{1}\frac{1}{x}\left(1-\tanh\left(\frac{1}{x}\right)\right)dx-\int_0^{1}\frac{\tanh x}{x}\,dx \tag{19}$$

$$\ln\pi-2\ln 2-\gamma=\int_0^{\infty}(1-\tanh(2\cosh x))\,dx-\int_0^{\infty}\tanh(2\cosh x)\tanh(e^x)\tanh(e^{-x})\,dx \tag{20}$$

$$\ln\pi-2\ln 2-\gamma=u^2-v^2-\int_0^{u}\left(\tanh e^{-x}-\ln\tanh^{-1}x\right)dx+\int_0^{v}\left(1-\tanh e^x+\ln\tanh^{-1}(1-x)\right)dx \tag{21}$$

where $u=0.529221$ ..., is root of the equation : $u=\tanh e^u$, and $v=0.170644$ ..., is root of the equation : $v=1-\tanh e^v$.

References

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