Mixed generalized multifractal densities for vector valued quasi-Ahlfors measures

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Abstract

In the present work we are concerned with some density estimations of vector valued measures in the framework of the so-called mixed multifractal analysis. We precisely consider some Borel probability measures satisfying a weak quasi-Alfors regularity. Mixed multifractal generalizations of densities are then introduced and studied in a framework of relative mixed multifractal analysis.

Key words: Hausdorff and packing measures, Hausdorff and packing dimensions, Multifractal formalism, Mixed cases, Hölderian Measures.
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1 Introduction

In the present work we are concerned with some density estimations of vector valued measures in the framework of the so-called mixed multifractal analysis. The latter is a natural extension of multifractal analysis of single objects such as measures, fuctions, statistical data, distributions... It is developed quite recently (since 2014) in the pure mathematical point of view. In physics and statistics, it was appearing on different forms but not really and strongly linked

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to the mathematical theory. See ...... In many applications such as clustering topics, each attribute in a data sample may be described by more than one type of measure. This leads researchers to apply measures well adopted for mixed-type data. See for example [15].

We aim to consider some cases of simultaneous behaviors of measures instead of a single measure as in the classic or original multifractal analysis of measures and to introduce a simultaneous density characteristic of such measures relatively to one .... This is important as it permits for example to characterize fractal or irregular sets such as Moran ones. The present work will provide a natural extension of [3], [4], [7], [6], [12], [20], [1]. For backgrounds and details on multifractal dimensions, Moran sets, the readers are asked to review [22], [23], [25], [26], [27], [36], [39], [40], [41], [43], [43], [42], [45], [46], [47]).

2 Comments and Corridgum of Some Existing Works

In this section we review some existing works that already investigated similar problems as the one(s) investigated here. So, we firstly recall that the present work may form with [1], [2], [4], [5], [7], [17], [22], [24] a quite full study of the concepts of multifractal densities of measures.

However, we noticed that there has been some lack in hypothesis in the last recent works [1] and [7]. Although the developments in [1] are in some parts based on [7] which also refers to [5], the authors did not pay attention to the fact that general probability measures (eventhough being doubling) may not lead to multifractal dimensions. Indeed, it is already mentioned in [5] that

- for a given Borel probability measure, the infimum for the μ-Hausdorff measure (and thus the supremum for μ-packing measures) extends over μ-ρ-coverings (packings). A μ-ρ-covering being a covering by cylinders C with $\mu(C) < \rho$.
- The measure μ is nonatomic, since otherwise there may be no μ-ρ-covering at all.

It is therefore questionable for both [1] and [7] the existence of multifractal dimensions in a general framework not taking into account some control of the measure of balls by means of their diameters.

To overcome these lacks, we proposed in the present work to assume some weak hypothesis on the measures applied. It consists of a weak form of the so-called Alhfors measures. For more details on such measures, we may refer to citeEdgar, [19], [31].

Definition 1 *A borel probability measure ν on $\mathbb{R}^d$ is said to be quasi-Ahlfors with index (regularity) $\alpha > 0$ if there*

$$\limsup_{|U|\longrightarrow 0} \frac{\mu(U)}{|U|^\alpha} < +\infty.$$

Using this assumption, the multifractal generalizations of Hausdorff and packing measures introduced in [1] and [7] induce in a usual way multifractal generalizations of Hausdorff and packing dimensions. Otherwise, the task remains questionable and thus the set of coverings applied there may be empty!!!

In the following section we will review in brief this problem and show how the assumption of beig quasi Alhlfors induce usual dimensions.

3 Preliminaries

In this section, we aim to introduce the general tools that will be applied next. We will review in brief the notion of mixed multifractal generalisations of Hausdorff and packing measures already introduced in [14] and next introduce the mixed multifractal generalisations of ....

Denote $\mathcal{P}(\mathbb{R}^n)$ the set of probability measures on $\mathbb{R}^n$, $n \geq 1$. Consider a vector of probability measures $\mu = (\mu_1, \mu_2, ..., \mu_k)$ on $\mathbb{R}^n$. Denote

$$\mu(B(x,r)) = (\mu_1(B(x,r)), ..., \mu_k(B(x,r))).$$

and for $q = (q_1, q_2, ..., q_k) \in \mathbb{R}^n$,

$$[\mu(B(x,r)]^q = [\mu_1(B(x,r)]^{q_1} \times ... \times [\mu_k(B(x,r)]^{q_k}$$

Let finally $\nu \in \mathcal{P}(\mathbb{R}^n)$. The mixed multifractal generalisations of Hausdorff measure relatively to μ and ν is introduced in [14] as follows. For $E \subset \mathbb{R}^n$,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) = \inf\{ \sum_i (\mu(B(x_i,r_i)))^q (\nu(B(x_i,r_i)))^t \},$$

where the inf is taken over the set of all centred δ-coverings $(B(x_i,r_i)))_i$ of E. Next,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E) = \lim_{\delta\downarrow 0} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) = \sup_{\delta>0} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E)$$

and finally,

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F).$$

Similarly, to the mixed multifractal generalisation of Hausdorrf measure, the mixed generalized multifractal packing measure relatively to μ and ν has been introduced in [14] as follows. For $E \subset \mathbb{R}^n$,

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) = \sup\{ \sum_i (\mu(B(x_i, r_i)))^q (\nu(B(x_i, r_i))^t\},$$

where the sup is taken over the set of all centred δ-packings $(B(x_i, r_i)))_i$ of E. Let next

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \lim_{\delta \downarrow 0} \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) = \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E)$$

and

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) = \inf_{E \subseteq \underset{i}{\cup} E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E_i).$$

The following Theorem proved in [14] resumes the characteristics of the mixed multifractal generalisations $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$.

Theorem 3.1 ([14]) • *The functions $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$ are metric outer measure and thus measures on the Borel family of subsets of $\mathbb{R}^n$.*

- *For all $E \subset \mathbb{R}^n$, there exists unique extended reel numbers denoted $dim^q_{\mu,\nu}(E)$, $Dim^q_{\mu,\nu}(E)$, $\Delta^q_{\mu,\nu}(E) \in [-\infty, +\infty]$ satisfying respectively*

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty & \text{if } t < dim^q_{\mu,\nu}(E), \\ 0 & \text{if } t > dim^q_{\mu,\nu}(E), \end{cases}$$

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty & \text{if } t < Dim^q_{\mu,\nu}(E), \\ 0 & \text{if } t > Dim^q_{\mu,\nu}(E), \end{cases}$$

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty & \text{if } t < \Delta^q_{\mu,\nu}(E), \\ 0 & \text{if } t > \Delta^q_{\mu,\nu}(E). \end{cases}$$

For the convinience and to prove the necessity of the assumption of quasi-Ahlfors regularity mentioned above we recall in brief the proof of the first point in Theorem 4.2. We claim firstly that

Lemma 2 *$\forall\, E \subseteq \mathbb{R}^d$ and $\forall q \in \mathbb{R}^k$, the set*

$$\Gamma_q = \left\{ t \ ; \ \mathcal{H}^{q,t}_{\mu,\nu}(E) < +\infty \right\} \neq \emptyset$$

Indeed, let $M \in \mathbb{R}^*_+$ such that

$$\overline{\lim_{|U| \to 0}} \frac{\nu(U)}{|U|^\alpha} < M.$$

There exists $\delta > 0$ such that $\forall\ r$, $0 < r < \delta$,

$$\nu(U) \leq M\, |U|^\alpha \, ; \forall\ U, \ \ |U| < r.$$

Let next $(B(x_i,r_i))_i$ a $\varepsilon-$ covering of E and consider the $\xi-$ families defined by the Besicovitch covering theorem. We get

$$\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t \leq \sum_{i=1}^{\xi}\sum_j \mu(B(x_{ij},r_{ij}))^q \nu(B(x_{ij},r_{ij}))^t.$$

Whenever $q \geq 0$, the right hand term is bounded by

$$\sum_{i=1}^{\xi}\sum_j \nu(B(x_{ij},r_{ij}))^t.$$

For $t=1$, this becomes

$$\sum_{i=1}^{\xi}\sum_j \nu(B(x_{ij},r_{ij})).$$

As the $(B(x_{ij},r_{ij}))_j$ are disjoint, the last quantity will be bounded by

$$\sum_{i=1}^{\xi} \nu\left(\underset{j}{\cup} B(x_{ij},r_{ij})\right) \leq \xi\nu(\mathbb{R}^d) = \xi.$$

Consequently

$$\mathcal{H}^{q,1}_{\mu,\nu}(E) < +\infty.$$

Assume now that there exist i, $1 \leq i \leq k$ such that $q_i \leq 0$.

$$\nu(B(x_i,r_i))^t \leq M^t r_i^{\alpha t}, \forall\ i.$$

Consequently

$$\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t \leq 2^{-\alpha} M^t \sum_i \mu(B(x_i,r_i))^q (2r_i)^{\alpha t}.$$

Let next $t > \frac{1}{\alpha}\left[\max\left(1, dim^q_\mu(E)\right)\right]$. We obtain

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) \leq 2^{-\alpha} M^t \mathcal{H}^{q,\alpha t}_{\mu}(E) < +\infty.$$

Lemma 3 *i)* $\mathcal{H}^{q,t}_{\mu,\nu}(E) < +\infty \Rightarrow \mathcal{H}^{q,s}_{\mu,\nu}(E) = 0$, $\forall\ s > t$.
ii) $\mathcal{H}^{q,t}_{\mu,\nu}(E) > 0 \Rightarrow \mathcal{H}^{q,s}_{\mu,\nu}(E) = +\infty$, $\forall\ s < t$.

TO prove assertion i) let $(B(x_i,r_i))_i$ a $\delta-$ covering of E

$$\begin{aligned}
&\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^s \\
=&\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t \nu(B(x_i,r_i))^{s-t} \\
\leq& M^{s-t}\delta^{s-t}\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t.
\end{aligned}$$

Consequently
$$\overline{\mathcal{H}}^{q,s}_{\mu,\nu,\delta}(E) \leq M^{s-t}\delta^{s-t}\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E).$$
Hence
$$\mathcal{H}^{q,s}_{\mu,\nu}(E) = 0.$$
ii) Using the same arguments, we get
$$\overline{\mathcal{H}}^{q,s}_{\mu,\nu,\delta}(E) \geq M^{s-t}\delta^{s-t}\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) \quad (\text{as } s-t<0).$$
Consequently,
$$\mathcal{H}^{q,s}_{\mu,\nu}(E) = +\infty.$$
The two Lemmas above permits to introduce the generalised mixed multifractal Hausdorff dimension as
$$dim^{q}_{\mu,\nu}(E) = \inf\left\{t,\ \mathcal{H}^{q,t}_{\mu,\nu}(E)=0\right\} = \sup\left\{t,\ \mathcal{H}^{q,t}_{\mu,\nu}(E)=+\infty\right\}.$$

We now introduce the mixed density of measures. The original definition are introduced in [7] and [6]). It is next re-studied by several authors and extension to generalized multifractal Hausdorff and Packing measures already introduced in [22] are developed in [1]. In the present work, we conduct an extension of these works to the case of mixed multifractal generalizations of Hausdorff and packing measures introduced in [20] and their most recent generalizations in [14].

Definition 3.1 *Let $\theta \in \mathcal{P}(\mathbb{R}^d)$, $x \in \mathcal{S}_\theta$, $q=(q_1,q_2,...,q_k)\in\mathbb{R}^k$ and $t\in\mathbb{R}$. The upper and lower (q,t)-density of θ with respect to μ and ν are defined respectively by*
$$\overline{d}^{q,t}_{\mu,\nu}(x,\theta) = \limsup_{r\to 0}\frac{[\theta(B(x,r)]}{(\mu(B(x,r)))^q(\nu(B(x,r))^t}$$
and
$$\underline{d}^{q,t}_{\mu,\nu}(x,\theta) = \liminf_{r\to 0}\frac{[\theta(B(x,r)]}{(\mu(B(x,r)))^q(\nu(B(x,r))^t}.$$
Whenever $\overline{d}^{q,t}_{\mu,\nu}(x,\theta) = \underline{d}^{q,t}_{\mu,\nu}(x,\theta)$, we denote the common value by $d^{q,t}_{\mu,\nu}(x,\theta)$ and we call it the (q,t)-density of θ with respect to μ and ν.

Next, for a single measure $\mu\in\mathcal{P}(\mathbb{R}^n)$ and $a>1$, write
$$P_a(\mu) = \limsup_{r\downarrow 0}\left(\sup_{x\in\operatorname{sup}p\mu}\frac{\mu(B(x,ar))}{\mu(B(x,r))}\right)$$
and for a vector valued measure $\mu=(\mu_1,\mu_2,...,\mu_k)\in\Big(\mathcal{P}(\mathbb{R}^n)\Big)^k$,
$$P_a(\mu) = \bigcap_{i=1}^{k} P_a(\mu_i).$$

Finally, define the set of the so-called doubling vector valued measures on $\mathbb{R}^n$, by

$$P_D(\mathbb{R}^n) = \bigcup_{a>1} \{\mu \in P(\mathbb{R}^n);\ P_a(\mu) < \infty\}.$$

4 Main results

The first result of of the present work deals with the establishement of lower and upper bounds for the mixed multifractal density introduced above by means of the mixed multifractal generalizations of Hausdorff and packing measures. We will see that such bounds permits to obtain the multifractal formalism already introduced in [22] and re-considered next in [1], [6], [14], [20], ...

Theorem 4.1 *There exists constants $C_1, C_2 > 0$ such that for all $E \subseteq \mathcal{S}_\mu$, a Borel set, we have*

$$C_1 \mathcal{H}^{q,t}_{\mu,\nu}(E) \inf_{x\in E} \overline{d}^{q,t}_{\mu,\nu}(x,\theta) \leq \theta(E) \leq C_2 \mathcal{H}^{q,t}_{\mu,\nu}(E) \sup_{x\in E} \overline{d}^{q,t}_{\mu,\nu}(x,\theta), \tag{1}$$

whenever $\mathcal{H}^{q,t}_{\mu,\nu}(E) < \infty$ and

$$C_1 \mathcal{P}^{q,t}_{\mu,\nu}(E) \inf_{x\in E} \underline{d}^{q,t}_{\mu,\nu}(x,\theta) \leq \theta(E) \leq C_2 \mathcal{P}^{q,t}_{\mu,\nu}(E) \sup_{x\in E} \underline{d}^{q,t}_{\mu,\nu}(x,\theta), \tag{2}$$

whenever $\mathcal{P}^{q,t}_{\mu,\nu}(E) < \infty$.

Remark 4.1 *Whenever $\mu, \nu \in P_D(\mathbb{R}^n)$, we may choose $C_1 = C_2 = 1$ in Theorem 4.1.*

As a result of the estimations above of the new mixed multifractal densities, we aim in the next step to show that such estimations permit in some special cases to compute the mixed multifractal generalizations of both Hausdorff and packing dimensions of sets characterized by the existence of some suitable measure(s) supported on.

For a borel set $E \subset \mathbb{R}^d$ define

$$\overline{D}^{q,t}_{\mu,\nu}(x,E) = \overline{d}^{q,t}_{\mu,\nu}(x, \mathcal{H}^{q,s}_{\mu,\nu}(E))$$

and

$$\underline{D}^{q,t}_{\mu,\nu}(x,E) = \underline{d}^{q,t}_{\mu,\nu}(x, \mathcal{H}^{q,s}_{\mu,\nu}(E)).$$

Define similarly

$$\overline{\Delta}^{q,t}_{\mu,\nu}(x,E) = \overline{d}^{q,t}_{\mu,\nu}(x, \mathcal{P}^{q,t}_{\mu,\nu}(E))$$

and

$$\underline{\Delta}^{q,t}_{\mu,\nu}(x,E) = \underline{d}^{q,t}_{\mu,\nu}(x, \mathcal{P}^{q,t}_{\mu,\nu}(E)).$$

As usually, whenever

$$\overline{D}_{\mu,\nu}^{q,t}(x,E)=\underline{D}_{\mu,\nu}^{q,t}(x,E)$$

we write $D_{\mu,\nu}^{q,t}(x,E)$ the commun value and similarly, whenever

$$\overline{\Delta}_{\mu,\nu}^{q,t}(x,E)=\underline{\Delta}_{\mu,\nu}^{q,t}(x,E)$$

we write $\Delta_{\mu,\nu}^{q,t}(x,E)$ the commun value.

Denote next

$$\underline{K}=\{\,x\in\,E,\ \underline{D}_{\mu,\nu}^{q,t}(x,E)=1\},\ \ \overline{K}=\{\,x\in\,E,\ \overline{D}_{\mu,\nu}^{q,t}(x,E)=1\},$$

$$\underline{T}=\{\,x\in\,E,\ \underline{\Delta}_{\mu,\nu}^{q,t}(x,E)=1\},\ \ \overline{T}=\{\,x\in\,E,\ \overline{\Delta}_{\mu,\nu}^{q,t}(x,E)=1\},$$

$$K=\underline{K}\cap\overline{K}\ \text{ and }\ T=\overline{T}\cap\underline{T}.$$

The following results provides a description of these sets by means of their mixed multifractal generalizations of Hausdorff and packing dimensions.

Theorem 4.2 *Let E be a borel set such that $E\subset\mathcal{S}_\mu\cap\mathcal{S}_\nu$.*

(1) Whenever $\mathcal{H}_{\mu,\nu}^{q,t}(E)<\infty$ and $\mu,\nu\in P_D(\mathbb{R}^d)$, there holds that

$$dim_{\mu,\nu}^q(\overline{K})=t.$$

(2) Whenever $\mathcal{P}_{\mu,\nu}^{q,t}(E)<\infty$, there holds that

$$Dim_{\mu,\nu}^q(\underline{T})=t.$$

(3) Whenever $\mathcal{P}_{\mu,\nu}^{q,t}(E)<\infty$ and $\mu,\nu\in P_D(\mathbb{R}^d)$, the following assertions are equivalent.

a. $\mathcal{H}_{\mu,\nu}^{q,t}=\mathcal{P}_{\mu,\nu}^{q,t}$.

b. $\overline{D}_{\mu,\nu}^{q,t}(x,E)=\underline{D}_{\mu,\nu}^{q,t}(x,E)=1$ *for* $\mathcal{P}_{\mu,\nu}^{q,t}-a.a.x\in E$.

c. $\overline{\Delta}_{\mu,\nu}^{q,t}(x,E)=\underline{\Delta}_{\mu,\nu}^{q,t}(x,E)=1$ *for* $\mathcal{P}_{\mu,\nu}^{q,t}-a.a.x\in E$.

(4) Whenever $\mathcal{H}_{\mu,\nu}^{q,t}=\mathcal{P}_{\mu,\nu}^{q,t}<\infty$ and $\mu,\nu\in P_D(\mathbb{R}^d)$, there holds that

$$dim_{\mu,\nu}^q(K)=dim_{\mu,\nu}^q(T)=t.$$

This result is imortant as it consists of a first information leading to the computation of the multifractal spectrum due to the densities introduced. Indeed, related to the origins of the multifractal spectrum, such as in [3], [4], [7], [6], [14], [20], [22], ... a starting point in the classical case is to establish an estimation of the form

$$\theta(B(x,r))\sim(\mu(B(x,r))^q(2r)^{t\pm\varepsilon},\ \ r\to0.$$

Which by considering a somehow holderian probability measure

$$\nu(B(x,r))\sim(2r)^{t\pm\varepsilon},\ \ r\to0$$

means that the densities considered above are all equals 1. These last assumptions permits to compute the multifractal spectrum (evaluated as the Hausdorff dimension of the level sets of the densities) by means of a Legendre transform of a convex function issued from the multifractal generalized dimensions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Delta_{\mu,\nu}$.

In the following part we aim to provide in a preparatory step a caraterization of the sets of points in the support(s) of the relative measure(s) with the same density.

Theorem 4.3 *Let $\mu,\nu \in P_D(\mathbb{R}^n)$ and E a Borel subset of $supp\mu\cap\ supp\nu$. Consider the sets*

$$E_1 = \{\ x\in E, \overline{D}^{q,t}_{\mu,\nu}(x,E) = \underline{D}^{q,t}_{\mu,\nu}(x,E)\}$$

and

$$E_2 = \{\ x\in E, \overline{\Delta}^{q,t}_{\mu,\nu}(x,E) = \underline{\Delta}^{q,t}_{\mu,\nu}(x,E)\}.$$

1) If $\mathcal{H}^{q,t}_{\mu,\nu}(E)<\infty$ then

a) $\overline{D}^{q,t}_{\mu,\nu}(x,E_1) = \underline{D}^{q,t}_{\mu,\nu}(x,E_1)$, for $\mathcal{H}^{q,t}_{\mu,\nu}-a.\ e$ on E_1.

b) $\overline{D}^{q,t}_{\mu,\nu}(x,E\diagdown E_1) = \underline{D}^{q,t}_{\mu,\nu}(x,E\diagdown E_1)$, for $\mathcal{H}^{q,t}_{\mu,\nu}-a.\ e$ on $E\diagdown E_1$.

2) If $\mathcal{P}^{q,t}_{\mu,\nu}(E)<\infty$ then

a) $\overline{\Delta}^{q,t}_{\mu,\nu}(x,E_2) = \underline{\Delta}^{q,t}_{\mu,\nu}(x,E_2)$, for $\mathcal{P}^{q,t}_{\mu,\nu}-a.\ e$ on E_2.

b) $\overline{\Delta}^{q,t}_{\mu,\nu}(x,E\diagdown E_2) = \underline{\Delta}^{q,t}_{\mu,\nu}(x,E\diagdown E_2)$, for $\mathcal{P}^{q,t}_{\mu,\nu}-a.\ e$ on $E\diagdown E_2$.

5 Proof of Main Results

5.1 Proof of Theorem 4.1.

We firstly show the left-hand side inequality of (1). So, denote $\overline{d} = \inf\limits_{x\in E}\overline{d}^{q,t}_{\mu,\nu}(x,\theta)$. Whenever $\overline{d}=0$, the inequality is obvious. So, assume that $\overline{d}>0$ and let η be such that $0<\eta<\overline{d}$ and denote $\overline{d}_\eta=\overline{d}-\eta$. Let finally $F\subset E$ be closed, $H\subset F$. Finally, for $\delta>0$, let

$$B_\delta(F) = \{\ x\in\mathbb{R}^d,\ dist(F,x)\leq\delta\}.$$

It is straightforward that $B_\delta(F) \downarrow F$ whenever $\delta \downarrow 0$. Therefore, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\theta(B_\delta(F)) \leq \theta(F) + \varepsilon, \ \forall\, \delta, \ 0 < \delta < \delta_0.$$

On the other hand, as $\mathcal{H}^{q,t}_{\mu,\nu}(H) < \infty$, we may also write

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(H) - \varepsilon \leq \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(H), \ \forall\, \delta, \ 0 < \delta < \delta_0.$$

Denote next

$$\Gamma^{q,t}_{\mu,\nu}(B(x,r)) = [\mu(B(x,r))]^q [\nu(B(x,r))]^t$$

and consider the set

$$\mathcal{B}_\delta = \Big\{ B(x,r);\, x \in H,\, 0 < r < \delta, \ \ \theta(B(x,r)) \geq \overline{d}_\eta \Gamma^{q,t}_{\mu,\nu}(B(x,r)) \Big\}.$$

From the definition of $\overline{d}$ and $\overline{d}_\eta$, there exists $\delta_0 > 0$ such that

$$\frac{[\theta(B(x,r)]}{(\mu(B(x,r)))^q (\nu(B(x,r))^t} \geq \overline{d}_\eta; \ \ \forall r, \, 0 < r < \delta_0.$$

Or equivalently,

$$\theta(B(x,r)) \geq \overline{d}_\eta \Gamma^{q,t}_{\mu,\nu}(B(x,r)); \ \ \forall r, \, 0 < r < \delta_0.$$

Henceforth, $\mathcal{B}_\delta \neq \emptyset$. So, let next N be the number of countable subfamilies $(\mathcal{B}_i)_{1 \leq i \leq N} = (B(x_{ij}, r_{ij}))_j$ of $\mathcal{B}_\delta$ defined in Besicovitch covering Theorem, Recall that for all i, $\mathcal{B}_i$ is composed of pairwise disjoint balls $B(x_{ij}, r_{ij})$. Moreover,

$$H \subset \bigcup_{1 \leq i \leq N} \bigcup_j B(x_{ij}, r_{ij}).$$

It follows that

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(H) &\leq \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}\Big(\bigcup_{i=1}^{N} \bigcup_j B(x_{ij}, r_{ij})\Big) \\
&\leq \sum_{i=1}^{N} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}\Big(\bigcup_j B(x_{ij}, r_{ij})\Big) \\
&\leq \sum_{i=1}^{N} \sum_j [\mu(B(x_{ij}, r_{ij}))]^q [\nu(B(x_{ij}, r_{ij}))]^q \\
&\leq \frac{1}{\overline{d}_\eta} \sum_{i=1}^{N} \sum_j \theta(B(x_{ij}, r_{ij})) \\
&\leq \frac{1}{\overline{d}_\eta} \sum_{i=1}^{N} \theta(B_\delta(F)) \\
&\leq \frac{N}{\overline{d}_\eta} \theta(B_\delta(F)).
\end{aligned}$$

Consequenlty, we obtain

$$\mathcal{H}^{q,t}_{\mu,\nu}(H) \leq \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(H) + \varepsilon \leq \frac{N}{\overline{d}_\eta}\theta(B_\delta(F)) + \varepsilon.$$

This is valid for all $\varepsilon > 0$. So by letting $\varepsilon \to 0$, and observing that $\overline{d}_\eta = \overline{d} - \eta$, we get

$$(\overline{d} - \eta)\mathcal{H}^{q,t}_{\mu,\nu}(H) \leq N\theta(B_\delta(F)); \quad \forall \eta > 0, \ 0 < \eta < \overline{d}.$$

By letting similarly $\eta \downarrow 0$, we get

$$\overline{d}\mathcal{H}^{q,t}_{\mu,\nu}(H) \leq N\theta(B_\delta(F)).$$

Now, whenever $\delta \downarrow 0$ and taking the sup on $H \subset F$, we obtain

$$\overline{d}\mathcal{H}^{q,t}_{\mu,\nu}(F) \leq N\theta(F).$$

This is valid for all closed $F \subset E$. As a result, taking the sup on F and replacing $\overline{d}$ by its exact form, we obtain

$$\mathcal{H}^{q,t}_{\mu,\nu}(E)\inf_{x\in E}\overline{d}^{q,t}_{\mu,\nu}(x,\theta) \leq N\theta(E).$$

We now proceed to show the right-hand side part of inequality (1). As previously, let $\overline{D} = \sup\limits_{x\in E}\overline{d}^{q,t}_{\mu,\nu}(x,\theta)$, $\eta > 0$ and denote $\overline{D} + \eta$. For $\delta > 0$ consider the set

$$E_\delta = \{x \in E;\ \overline{D}_\eta\Gamma^{q,t}_{\mu,\nu}(B(x,r)) \geq \theta(B(x,r),\ 0 < r < \delta\}.$$

It is straightforward that for all $\delta > 0$,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E_\delta) \leq \mathcal{H}^{q,t}_{\mu,\nu}(E_\delta) < \infty.$$

On the other hand, for all $\varepsilon > 0$, there exists a δ-covering $(B(x_i,r_i))_i$ of E_δ we such that

$$\sum_i [\mu(B(x_i,r_i))]^q[\nu(B(x_i,r))]^t \leq \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E_\delta) + \varepsilon.$$

Consequently,

$$\begin{aligned}
\theta(E_\delta) &\leq \theta(\textstyle\bigcup_i B(x_i,r_i)) \\
&\leq \sum_i \theta(B(x_i,r_i)) \\
&\leq \overline{D}_\eta \sum_i [\mu(B(x_i,r_i))]^q[\nu(B(x_i,r))]^t \\
&\leq \overline{D}_\eta \left[\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E_\delta) + \varepsilon\right].
\end{aligned}$$

As $E_\delta \subset E$, we get

$$\theta(E_\delta) \leq \overline{D}_\eta \left[\mathcal{H}^{q,t}_{\mu,\nu}(E) + \varepsilon\right].$$

Whenever $\delta \downarrow 0$, we get

$$\theta(E) \leq \overline{D}_\eta \left[\mathcal{H}^{q,t}_{\mu,\nu}(E) + \varepsilon\right].$$

This is true for all $\varepsilon, \eta > 0$. Consequently,

$$\theta(E) \leq \overline{D}\mathcal{H}^{q,t}_{\mu,\nu}(E)$$

or equivalently

$$\theta(E) \leq \sup_{x\in E} \overline{d}^{q,t}_{\mu,\nu}(x,\theta)\mathcal{H}^{q,t}_{\mu,\nu}(E).$$

Now, we will prove the left-hand side of (2). Denote similarly to the previous case $\underline{d} = \inf_{x\in E} \underline{d}^{q,t}_{\mu,\nu}(x,\theta)$ and assume herealso that $\underline{d} > 0$. Let next η be such that $0 < \eta < \underline{d}$ and denote $\underline{d}_\eta = \underline{d} - \eta$. Let also $F \subset E$ be closed and denote for $\delta > 0$,

$$B_\delta(F) = \{x \in \mathbb{R}^n;\ dist(F,x), \leq \delta\}$$

It is straightforward that, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ satisfying

$$\theta(B_\delta(F)) \leq \theta(F) + \varepsilon;\ \forall\delta,\ 0 < \delta < \delta_0.$$

Denote next

$$F_\delta = \left\{x \in F;\ \underline{d}\Gamma^{q,t}_{\mu,\nu}(B(x,r)) \geq \theta(B(x,r)),\ 0 < r < \delta\right\}.$$

Let $(B(x_i,r_i))_i$ be a centered δ-packing of F_δ. It holds that

$$\begin{aligned}
\underline{d}_\eta \sum_i [\mu(B(x_i,r_i))]^q[\nu(B(x_i,r_i))]^t &\leq \sum_i \theta(B(x_i,r_i))\\
&\leq \sum_i \theta(B(x_i,r_i))\\
&\leq \theta(\bigcup_i B(x_i,r_i))\\
&\leq \theta(B_\delta(F))\\
&\leq \theta(F) + \varepsilon\\
&\leq \theta(E) + \varepsilon.
\end{aligned}$$

Which yields that

$$\underline{d}_\eta \mathcal{P}^{q,t}_{\mu,\nu}(F_\delta) \leq \underline{d}_\eta \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F_\delta) \leq \underline{d}_\eta \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F_\delta) \leq \theta(E) + \varepsilon.$$

Finally, letting δ, ε and $\eta \to 0$, we get

$$\underline{d}\mathcal{P}^{q,t}_{\mu,\nu}(F) \leq \theta(E).$$

It remains to check the right-hand side inequality of (2). Let $\overline{D} = \sup_{x\in E} \underline{d}^{q,t}_{\mu,\nu}(x,\theta)$, $F \subset E$ and $\varepsilon, \eta, \delta > 0$

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) + \varepsilon.$$

Denote next $\overline{D}_\eta = \overline{D} + \eta$ and consider the set

$$\mathcal{B}_\delta = \left\{ B(x,r),\ x \in F,\ 0 < r < \delta,\ \ \overline{D}_\eta \Gamma^{q,t}_{\mu,\nu}(B(x,r)) \leq \theta(B(x,r)) \right\}.$$

It follows from Vitali's Theorem **??** that there exists a δ-packing $(B(x_i,r_i))_i \subset \mathcal{B}_\delta$ of F satisfying

$$\theta\left(F \setminus \left(\bigcup_i B(x_i,r_i)\right)\right) = 0.$$

Furthermore, we have

$$\begin{aligned}
\theta(F) &= \theta(\bigcup_i (F \cap B(x_i,r_i)) \\
&= \theta((F \cap \bigcup_i B(x_i,r_i)) \\
&\leq \sum_i \theta(F \cap B(x_i,r_i)) \\
&\leq \sum_i \theta(B(x_i,r_i)) \\
&\leq \overline{D}_\eta \sum_i [\mu(B(x_i,r_i))]^q [\nu(B(x_i,r_i))]^t \\
&\leq \overline{D}_\eta \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F) \\
&\leq \overline{D}_\eta \left[\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) + \varepsilon\right].
\end{aligned}$$

Whenever $\varepsilon \to 0$ and $\eta \to 0$, we obtain

$$\theta(F) \leq \overline{D}\,\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F), \quad \forall F \subset E.$$

And thus, the desired inequality follows.

5.2 Proof of Theorem 4.2.

1. We shall prove that

$$0 < \mathcal{H}^{q,t}_{\mu,\nu}(\overline{K}) < \infty. \tag{3}$$

Consider in a first step the set

$$F = \{\ x \in E; \overline{D}^{q,t}_{\mu,\nu}(x,E) > 1\}$$

and its decomposition into a sequence of nested sets

$$F_m = \{\ x \in E; \overline{D}^{q,t}_{\mu,\nu}(x,E) > 1 + \frac{1}{m}\}, \quad m \in \mathbb{N},$$

as $F = \underset{m}{\cup} F_m$. It follows from Theorem 4.1 and Remark 4.1 that

$$\mathcal{H}^{q,t}_{\mu,\nu}(F_m)(1+\frac{1}{m}) \leq \mathcal{H}^{q,t}_{\mu,\nu}(F_m),\ \forall\, m.$$

Which yields that

$$\mathcal{H}^{q,t}_{\mu,\nu}(F_m) = 0, \ \forall \ m.$$

Consequently,

$$\mathcal{H}^{q,t}_{\mu,\nu}(F) = 0$$

and thus

$$\overline{D}^{q,t}_{\mu,\nu}(x, E) \leq 1, \ \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.\ a.\ x \in E \ . \tag{4}$$

Next, consider similarly the set

$$G \ = \{ \ x \in E; \overline{D}^{q,t}_{\mu,\nu}(x, E) < 1 \ \}$$

and analogously the sequence of nested sets

$$G_m = \{ \ x \in E; \overline{D}^{q,t}_{\mu,\nu}(x, E) \leq 1 - \frac{1}{m}\}.$$

It follows from Theorem 4.1 and Remark 4.1 that

$$\mathcal{H}^{q,t}_{\mu,\nu}(G_m)(1 - \frac{1}{m}) \geq \mathcal{H}^{q,t}_{\mu,\nu}(G_m), , \ \forall \ m.$$

Consequently,

$$\mathcal{H}^{q,t}_{\mu,\nu}(G_m) = 0, \ \forall \ m.$$

Since $G = \bigcup_m G_m$, we obtain

$$1 \leq \overline{D}^{q,t}_{\mu,\nu}(x, E), \ \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.\ a.\ x \in E \ . \tag{5}$$

Equations (4) and (5) yield that

$$\overline{D}^{q,t}_{\mu,\nu}(x, E) = 1, \ \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.\ a.\ x \in E \ .$$

As a result

$$\mathcal{H}^{q,t}_{\mu,\nu}(\{ \ x \in E, \ \overline{D}^{q,t}_{\mu,\nu}(x, E) = 1\}) > 0.$$

Observing now that

$$\mathcal{H}^{q,t}_{\mu,\nu}(\{ \ x \in E, \ \overline{D}^{q,t}_{\mu,\nu}(x, E) = 1\}) \leq \mathcal{H}^{q,t}_{\mu,\nu}(E) < \infty,$$

it followe that

$$0 < \mathcal{H}^{q,t}_{\mu,\nu}(\{ \ x \in E, \ \overline{D}^{q,t}_{\mu,\nu}(x, E) = 1\}) < \infty.$$

Hence (3) holds and

$$dim^q_{\mu,\nu}(\overline{K}) = t.$$

Assertion **2.** may be cheked by similar techniques.

3. We shall prove that $a) \Rightarrow b) \Rightarrow c) \Rightarrow a)$.
$a) \Rightarrow b)$. From assertion **1.** above, it follows that

$$\overline{D}^{q,t}_{\mu,\nu}(x, E) = 1, \ \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.\ a.\ x \in E.$$

So, as $\mathcal{H}^{q,t}_{\mu,\nu} = \mathcal{P}^{q,t}_{\mu,\nu}$, we get

$$\overline{D}^{q,t}_{\mu,\nu}(x, E) = 1, \text{ for } \mathcal{P}^{q,t}_{\mu,\nu} - a.\ a.\ x \in E. \tag{6}$$

Now, proceeding as in the proof of equations (4) and (5) in assertion **1.** above, we get

$$\underline{D}^{q,t}_{\mu,\nu}(x, E) = 1, \text{ for } \mathcal{P}^{q,t}_{\mu,\nu} - a.a.x \in E,$$

which by the hypothesis $\mathcal{H}^{q,t}_{\mu,\nu} = \mathcal{P}^{q,t}_{\mu,\nu}$ yields that

$$\underline{D}^{q,t}_{\mu,\nu}(x, E) = 1, \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.a.x \in E.$$

So, assertion **b.** is proved.
$b) \Rightarrow c)$. Using assertion **b.**, it follows from Theorem 4.1 and Remark 4.1 that, for any ball $B(x, r)$,

$$\mathcal{H}^{q,t}_{\mu,\nu}(B(x, r)) = \mathcal{P}^{q,t}_{\mu,\nu}(B(x, r)).$$

Again, using assertion **b.** above, we get

$$\mathcal{H}^{q,t}_{\mu,\nu}(B(x, r)) \sim \Gamma^{q,t}_{\mu,\nu}(B(x, r)), \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.a.x \in E.$$

Consequently,

$$\mathcal{P}^{q,t}_{\mu,\nu}(B(x, r)) \sim \Gamma^{q,t}_{\mu,\nu}(B(x, r)), \text{ for } \mathcal{P}^{q,t}_{\mu,\nu} - a.a.x \in E.$$

Hence,

$$\underline{\Delta}^{q,t}_{\mu,\nu}(x, E) = \overline{\Delta}^{q,t}_{\mu,\nu}(x, E) = 1, \text{ for } \mathcal{P}^{q,t}_{\mu,\nu} - a.a.x \in E.$$

So as assertion **c.**
$c) \Rightarrow a)$. Applying Theorem 4.1 and Remark 4.1, we get for all borel set $F \subset E$,

$$\mathcal{H}^{q,t}_{\mu,\nu}(F) \leq \mathcal{H}^{q,t}_{\mu,\nu}(F) \leq \mathcal{P}^{q,t}_{\mu,\nu}(F).$$

Consequently,

$$\mathcal{P}^{q,t}_{\mu,\nu} = \mathcal{H}^{q,t}_{\mu,\nu}.$$

4) is an immediate consequence from assertions **1.**, **2.** and **3.**

5.3 Proof of Theorem 4.3

We will prove the assertion **1.a.** The remaining assertions may be proved by following similar techniques.
1.a. We claim that for all $\mathcal{H}^{q,t}_{\mu,\nu}$-measurable set $F \subset E$, we have

$$\overline{D}^{q,t}_{\mu,\nu}(x, E) = \overline{D}^{q,t}_{\mu,\nu}(x, F) \text{ and } \underline{D}^{q,t}_{\mu,\nu}(x, E) = \underline{D}^{q,t}_{\mu,\nu}(x, F);\ \mathcal{H}^{q,t}_{\mu,\nu} - a.e \text{ on } F. \tag{7}$$

Indeed, denote as in [1], [6], [37] for $\theta \in P(\mathbb{R}^n)$

$$\theta_E(A) = \theta(E \cap A) \text{ and } \lambda_E(A) = \theta(A \cap E^c) \;\text{ for all Borel set } A.$$

It is straightforward that

$$\overline{d}^{q,t}_{\mu,\nu}(x, \theta_E) \leq \overline{d}^{q,t}_{\mu,\nu}(x, \theta)$$

and

$$\underline{d}^{q,t}_{\mu,\nu}(x, \theta_E) \leq \underline{d}^{q,t}_{\mu,\nu}(x, \theta).$$

On the other hand, again a straightforward computation yields that

$$\underline{d}^{q,t}_{\mu,\nu}(x, \theta) \leq \underline{d}^{q,t}_{\mu,\nu}(x, \theta_E) + \overline{d}^{q,t}_{\mu,\nu}(x, \lambda_E)$$

and

$$\overline{d}^{q,t}_{\mu,\nu}(x, \theta) \leq \overline{d}^{q,t}_{\mu,\nu}(x, \theta_E) + \overline{d}^{q,t}_{\mu,\nu}(x, \lambda_E).$$

We now claim that

$$\overline{d}^{q,t}_{\mu,\nu}(x, \lambda_E) = 0. \tag{8}$$

Indeed, the set $G = \{x;\ \overline{d}^{q,t}_{\mu,\nu}(x, \lambda_E) \neq 0\}$ is a countable union of

$$G_k = \{\ x \in E;\ \overline{d}^{q,t}_{\mu,\nu}(x, \lambda) \geq \frac{1}{k}\};\ k \geq 1.$$

From Theorem 4.1 (and Remark 4.1), we get

$$\lambda_E(G_k) \geq \frac{1}{k}\mathcal{H}^{q,t}_{\mu,\nu}(G_k).$$

Consequently,

$$\mathcal{H}^{q,t}_{\mu,\nu}(G_k) = 0;\ \forall k$$

and thus

$$\mathcal{H}^{q,t}_{\mu,\nu}(G) = 0.$$

Therefore,

$$\overline{d}^{q,t}_{\mu,\nu}(x, \lambda_E) = 0 \text{ for } \mathcal{H}^{q,t}_{\mu,\nu} - a.\ a \text{ on } E,$$

which leads to (8). As a result, we get

$$\underline{d}^{q,t}_{\mu,\nu}(x, \theta) \leq \underline{d}^{q,t}_{\mu,\nu}(x, \theta_E)$$

and

$$\overline{d}^{q,t}_{\mu,\nu}(x, \theta) \leq \overline{d}^{q,t}_{\mu,\nu}(x, \theta_E).$$

These estimations together yield claim (7).

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