Malmsten's Integral

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abstract

This note presents some formulas related with Malmsten's integral.

1. Introduction.

This note presents some formulas related with Malmsten's integral (1842):

$$\int_1^\infty \frac{\ln \ln x}{1+x^2}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{1}$$

two related integrals :

$$\int_0^1 \frac{1}{1+x^2}\ln\ln\frac{1}{x}\,dx = \int_{\pi/4}^{\pi/2}\ln\ln\tan x\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{2}$$

Integral (1) was first evaluated by Carl Johan Malmsten and colleagues in 1842.

2. Related Integrals

$$\int_0^\infty \frac{e^x \ln x}{1+e^{2x}}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{3}$$

$$\int_0^\infty \frac{\ln x}{2\cosh x}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{4}$$

$$\int_{-\infty}^\infty \frac{x\,e^x}{2\cosh e^x}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{5}$$

$$\int_1^\infty \frac{\ln \cosh^{-1} x}{2x\sqrt{x^2-1}}\,dx = \int_1^\infty \frac{\ln\left(\ln\left(x+\sqrt{x^2-1}\right)\right)}{2x\sqrt{x^2-1}}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{6}$$

$$\int_0^\infty \frac{\ln \sinh^{-1} x}{2(1+x^2)}\,dx = \int_0^\infty \frac{\ln\left(\ln\left(x+\sqrt{x^2+1}\right)\right)}{2(1+x^2)}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{7}$$

$$\int_{-\infty}^\infty \frac{x\,e^{x-e^x}}{1+e^{-2e^x}}\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{8}$$

$$\int_{\ln(1+\sqrt{2})}^{\infty} \frac{\ln \ln \sinh x}{\cosh x} \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{9}$$

$$\int_0^{\infty} \frac{2 x \ln x}{\cosh x^2} \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{10}$$

$$\int_0^{\infty} \frac{x}{2} \left(\frac{e^x}{\cosh e^x} - \frac{e^{-x}}{\cosh e^{-x}} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{11}$$

$$\frac{1}{2} \int_0^1 \left(\operatorname{sech} x \ln x + \frac{1}{x^2} \operatorname{sech} \frac{1}{x} \ln \frac{1}{x} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{12}$$

$$\frac{1}{2} \int_1^{\infty} \left(\operatorname{sech} x \ln x + \frac{1}{x^2} \operatorname{sech} \frac{1}{x} \ln \frac{1}{x} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{13}$$

$$\int_0^{\pi/2} \frac{1}{2} \ln(\ln(\sec x + \tan x)) \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{14}$$

$$\int_0^{\infty} \left(\tan^{-1}\left(e^{-e^x}\right) - \tan^{-1}\left(\frac{1 - e^{-e^{-x}}}{1 + e^{-e^{-x}}} \right) \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{15}$$

$$\int_0^{\infty} \left(\tan^{-1}\left(e^{-e^{-x}}\right) - \tan^{-1}\left(\frac{1 - e^{-e^{x}}}{1 + e^{-e^{x}}} \right) \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{16}$$

$$\int_0^{\infty} \left(\tan^{-1}\left(e^{-e^x}\right) + \tan^{-1}\left(e^{-e^{-x}}\right) - \frac{\pi}{4} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{17}$$

$$\int_0^{\infty} \tan^{-1}\left(e^{-e^x}\right) dx - \int_0^{\infty} \left(\frac{\pi}{4} - \tan^{-1}\left(e^{-e^{-x}}\right) \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{18}$$

$$\int_0^{\infty} \tan^{-1}\left(\frac{e^{-e^x} + e^{-e^{-x}} + e^{-e^{-x} - e^x} - 1}{e^{-e^x} + e^{-e^{-x}} - e^{-e^{-x} - e^x} + 1} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{19}$$

$$\int_1^{\infty} \frac{1}{x} \tan^{-1}\left(\frac{e^{-x} + e^{-1/x} + e^{-x-1/x} - 1}{e^{-x} + e^{-1/x} - e^{-x-1/x} + 1} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{20}$$

$$\int_0^{1} \frac{1}{x} \tan^{-1}\left(\frac{e^{-x} + e^{-1/x} + e^{-x-1/x} - 1}{e^{-x} + e^{-1/x} - e^{-x-1/x} + 1} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{21}$$

$$\int_0^{e^{-1}} \frac{-1}{x \ln x} \tan^{-1}\left(\frac{x + e^{1/\ln x} + x\, e^{1/\ln x} - 1}{x + e^{1/\ln x} - x\, e^{1/\ln x} + 1} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{22}$$

$$\int_{e^{-1}}^{1} \frac{-1}{x \ln x} \tan^{-1}\left(\frac{x + e^{1/\ln x} + x\, e^{1/\ln x} - 1}{x + e^{1/\ln x} - x\, e^{1/\ln x} + 1} \right) dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{23}$$

$$\frac{e^2}{2} \int_0^{\infty} \frac{x \ln x \sinh(e\, x)}{(\cosh(e\, x))^2} \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{24}$$

$$-\frac{\pi}{4} + \frac{1}{2} \int_0^{\infty} \frac{x \ln x \sinh x}{(\cosh x)^2} \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{25}$$

$$\frac{1}{2} \int_0^{1} \frac{\ln x}{\cosh x} \, dx + \int_1^{\infty} \frac{\tan^{-1} e^{-x}}{x} \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{26}$$

$$\frac{1}{2}\int_0^{e^{-1}} \frac{\ln x}{\cosh x}\, dx + \frac{1}{2}\int_e^{\infty} \frac{\ln x}{\cosh x}\, dx + \int_{e^{-1}}^{e} \frac{\tan^{-1} e^{-x}}{x}\, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) + \tan^{-1} e^{-e} + \tan^{-1} e^{-e^{-1}} \tag{27}$$

$$\int_0^{e^{-e}} \frac{\ln(-\ln x)}{1+x^2}\, dx + \int_{e^{-1}}^{1} \frac{\ln(-\ln x)}{1+x^2}\, dx - \int_{e^{-e}}^{e^{-1}} \frac{\tan^{-1} x}{x \ln x}\, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) + \tan^{-1} e^{-e} \tag{28}$$

$$\int_1^{e} \frac{\ln\ln x}{1+x^2}\, dx + \int_{e^e}^{\infty} \frac{\ln\ln x}{1+x^2}\, dx - \int_e^{e^e} \frac{\tan^{-1} x}{x \ln x}\, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) - \tan^{-1} e^{e} \tag{29}$$

3. Series

$$\frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) = \frac{\tan^{-1}(\sinh a) \ln a}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n E_n a^{2n+1}}{(2n+1)(2n+1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \ln(2n+1)\, e^{-(2n+1)a}}{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_{(2n+1)a}^{\infty} e^{-x} \ln x\, dx \quad , 0 < a < \frac{\pi}{2} \tag{30}$$

In (30) E_n are the Euler's numbers:

$$E_n = \{1,\ 1,\ 5,\ 61,\ 1385,\ \ldots\} \tag{31}$$

$$\frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n E_n}{(2n+1)(2n+1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \ln(2n+1)}{(2n+1)\, e^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_{2n+1}^{\infty} e^{-x} \ln x\, dx \tag{32}$$

$$\frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) = \frac{1}{2}\int_0^1 \frac{\ln x}{\cosh x}\, dx + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{(e-1)\, e^{-n-1}}{1+e^{-2n-1}}\right) \ln n + \frac{1}{2}\sum_{n=1}^{\infty} \int_0^1 \operatorname{sech}(x+n) \ln\left(1+\frac{x}{n}\right) dx \tag{33}$$

4. Identities

$$\frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) + v^2 - u^2 - \frac{\pi}{2} v - \frac{\pi}{4} u = \int_0^u \left(\ln\ln\tan\left(\frac{\pi}{4}+x\right) - \tan^{-1} e^{e^{-x}}\right) dx + \int_0^v \left(\ln\ln\tan\left(\frac{\pi}{2}-x\right) - \tan^{-1} e^{e^{x}}\right) dx \tag{34}$$

where $u = 0.331544\ldots$, is root of the equation :

$$u = \tan^{-1}\left(\frac{1-e^{-e^{-u}}}{1+e^{-e^{-u}}}\right) \tag{35}$$

and $v = 0.265151\ldots$, is root of the equation :

$$v = \tan^{-1} e^{-e^{v}} \tag{36}$$

other formula is:

$$\frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi}\right) + v^2 - u^2 = \int_0^u \left(\ln\ln\tan\left(\frac{\pi}{4}+x\right) - \tan^{-1}\left(\frac{1-e^{-e^{-x}}}{1+e^{-e^{-x}}}\right)\right) dx + \int_0^v \left(\ln\ln\tan\left(\frac{\pi}{2}-x\right) + \tan^{-1} e^{-e^{x}}\right) dx \tag{37}$$

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