Particle in a quantum δ-function potential

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Abstract

A quantum potential $V(x,t)$ of δ-function type is introduced, to describe the inertial motion of a particle. Quantum-mechanically, it is in a bound state, though classically one seems to be free. The motion of the object (micro- or macroscopic) takes place according to the Huygens-Fresnel principle. The new position of the object (wave front) plays the role of the secondary sources that maintain the propagation. The mean value of the potential energy is $-mc^2$. We found that the de Broglie - Bohm quantum potential is the difference between the bound energy $E = -mc^2/2$ from the stationary case and our potential $V(x,t)$.

1 Introduction

It is well-known that Quantum Field Theory (QFT) combines classical fields, Special Relativity (SR) and Quantum Mechanics (QM). It treats particles (called field quanta) as excited states of an underlying field. The way it works mathematically is by placing a quantum harmonic oscillator at each point in space [1]. As Griffiths has noticed [2], observers not only disturb what is to be measured, but also produce it. We enforce the particle to have a definite position. The measurement process "creates" a particular value of the quantity to be measured (see also [3]). Araya and Bars [4] use analyticity properties of quantum fields to uncover a new multiverse geometry when the classical field has horizon or singularities. They applied their ideas to extended Rindler space which is classically equivalent to flat Minkowski space. Quantum-mechanically, the sheet structure is naturally interpreted as an infinite stack of identically Minkowski geometries, or "universes". They also showed that information does not flow from one Rindler sheet to another.

We study in this paper the motion of a particle (or a macroscopic object) in a δ-Dirac function well, following Griffiths' reasoning [2]. We consider, however, the velocity v from the expression of the potential well to be the particle velocity

[1] and not the velocity of the well. We compute the r.m.s. of the position and momentum operators and find that they obey the Uncertainty Principle. In our model, the "strength" of the potential well depends on $\hbar$ but the mean value of the potential V is classical, with $< V(x,t) >= -mc^2$, m being the particle mass and c - the velocity of light. It is worth to mention that we shall make use of the fundamental constants c and $\hbar$, from SR and QM, and not the Newton constant G. Nevertheless, the mean value of $V(x,t)$ will be interpreted as a constant gravitational potential. Thanks to the Huygens-Fresnel principle, we show that an inertial particle moves like a wave: each new position of the particle on the wave front represents, in our view, another identical particle that acts as a new source which propagates unidirectionally, with velocity v.

2 δ-function potential

We begin with the time-dependent Schrodinger equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t), \tag{2.1}$$

describing a particle moving with velocity v along the x-axis. We assume that V is given by

$$V(x,t) = -a\delta(x - vt), \tag{2.2}$$

where a is a positive constant and δ is the Dirac function (distribution). Griffiths found that the solution of (2.1) with V given by (2.2) has the exact (normalizable) expression

$$\Psi(x,t) = \frac{\sqrt{ma}}{\hbar}e^{-\frac{ma|x-vt|}{\hbar^2}}e^{-\frac{i}{\hbar}[(E+\frac{mv^2}{2})t-mvx]}, \tag{2.3}$$

where $E = -ma^2/2\hbar^2$ (which is unique) is the bound-state energy of the stationary δ-function, $V(x) = -a\delta(x)$. Griffiths gives no estimation of the value of the "strength" a. From (2.2) it is clear that a must have units of energy $\times$ distance. As the problem we treat here is general, a should depend on fundamental constants. Having only two fundamental constant at our disposal, $\hbar$ and c, the single option is $a = \hbar c$. Therefore, (2.2) becomes

$$V(x,t) = -\hbar c\ \delta(x - vt). \tag{2.4}$$

Here is the first equation where we make use of SR, through the constant c. In addition, classically, $V = 0$. In other words, even though the particle is (macroscopically) inertial, it is not free but embedded in the potential (2.4). Quantum-mechanically, it is in a bound state. The mean value of the potential

[1]Velocity is defined as $v =< p > /m$, where $< p >$ is the expectation value of the momentum of the particle in the state described by the wave function $\Psi(x,t)$.

gives us

$$
\begin{aligned}
< V(x,t) >= -\hbar c \int_{-\infty}^{\infty} \Psi^* \delta(x - vt)\Psi dx \\
= -mc^2 \int_{-\infty}^{\infty} e^{-b|x-vt|}\delta(x - vt)dx = -mc^2,
\end{aligned}
\tag{2.5}
$$

where the fundamental property of the δ-function has been used in the last step and $b = 2mc/\hbar$. One sees that the expectation value of V is proportional to the mass m of the object and plays the role of a potential energy. The bound state of the stationary case becomes now $E = -mc^2/2$. It is worth noting that the rest (potential) energy mc^2 is the only element we have used from the SR, the rest being non-relativistic QM. We also observe that the rest energy is not a correction to some formula from classical mechanics at high velocities where v/c is not negligible; it is there even at small velocities. Even in Newtonian Mechanics we have the freedom to add a constant potential mc^2 to $V(r) = -Gm/r$ [5]. In a previous paper [6] $(-mc^2)$ signified a gravitational potential energy, due to Mach's principle. Therefore, we assume that inertial motion is, in fact, a motion in a constant gravitational potential, given by (2.5). That interpretation justifies the appearance of the inertial forces in accelerated frames: the variation of the gravitational potential generates those forces. Therefore, quantum-mechanically, an inertial object is not free but bound, though macroscopically it appears to be free.

The probability density of localization of the particle $P = |\Psi(x,t)|^2$ is not differentiable at $x = vt$ because of the modulus at the exponent. The width of the curve at half of P_{max} (at constant t) is $(mc/\hbar)\ ln2$. A plot of $P(x)$ versus x has a similar form with $\Psi(x)$ from the static situation (see Ref.2, p.54).

3 Mean Hamiltonian

Our next task is to compute the expectation values of the main physical quantities and compare them with the corresponding classical expressions. Let us calculate, firstly, the expectation value of the Hamiltonian operator in the state Ψ from (2.3). We have

$$
< \hat{H} >= \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx = i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial t} dx. \tag{3.1}
$$

To find $\partial\Psi/\partial t$ we remind that $(\partial/\partial t)|x - vt| = -v\ sgn(x - vt)$, where $sgn(x)$ is -1 for $x < 0$ and equals 1 for $x > 0$. But $sgn(x)$ is an odd function, so that

$$
\int_{-\infty}^{\infty} e^{-b|x-vt|} sgn(x - vt)dx = 0. \tag{3.2}
$$

The other integral gives us

$$
\int_{-\infty}^{\infty} e^{-b|x-vt|}dx = 2\int_{0}^{\infty} e^{-b|x-vt|}dx = \frac{2}{b}, \tag{3.3}
$$

where the substitution $y = x - vt$ has been used. One obtains finally

$$< \hat{H} >= -\frac{mc^2}{2}(1 - \frac{v^2}{c^2}) = E(1 - \frac{v^2}{c^2}). \tag{3.4}$$

One verifies that, when $v = 0, < \hat{H} >= E$, the stationary case. The same result emerges when $v << c$. Moreover, $v \to c$ leads to $< \hat{H} > \to 0$, as if the particle were not bound. This is equivalent to the massless case (for example, a photon) [2]. Anyway, an increasing velocity gives an increasing energy and the particle becomes less bound (the "effective" bound energy $E(1 - v^2/c^2)$ tends to zero).

Another aim concerns the r.m.s. of the Hamiltonian $\hat{H}$. Firstly we consider

$$< \hat{H}^2 >= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial t^2} dx. \tag{3.5}$$

To find $\partial^2 \Psi / \partial t^2$, we remind that the 1st time derivative gave $sgn(y)$. The 2nd derivative leads therefore to $\partial(sgn(y))/\partial y = 2\delta(y)$, with $y = x - vt$. Hence

$$\frac{\partial^2 \Psi}{\partial t^2} = \left[-bv^2 \delta(y) + \frac{b^2 v^2}{4} - \frac{(E + \frac{mv^2}{2})^2}{\hbar^2} - \frac{ibv}{\hbar}(E + \frac{mv^2}{2}) sgn(y) \right] \Psi. \tag{3.6}$$

Using similar calculations as before, we have finally

$$< \hat{H}^2 >= \left(\frac{mc^2}{2} \right)^2 \left(1 + \frac{v^2}{c^2} \right), \tag{3.7}$$

whence

$$\Delta H = \sqrt{< \hat{H}^2 > - < \hat{H} >^2} = mcv. \tag{3.8}$$

In other words, $\Delta H = v \Delta p$.

4 Uncertainty relation between position and momentum

Our next goal is to find the mean values of the particle position and momentum. We have

$$< \hat{x} >= \int_{-\infty}^{\infty} \Psi^* x \Psi dx = \frac{mc}{\hbar} \int_{-\infty}^{\infty} x e^{-b|x - vt|} dx. \tag{4.1}$$

We change the variable of integration from x to $y = x - vt$ and obtain

$$< \hat{x} >= \frac{mc}{\hbar} \left(\int_{-\infty}^{\infty} y e^{-b|y|} dy + vt \int_{-\infty}^{\infty} e^{-b|y|} dy \right). \tag{4.2}$$

The 1st integral is null. Hence, (4.2) yields

$$< \hat{x} >= vt. \tag{4.3}$$

[2]The massless situation is not, of course, valid to our problem, because Schrodinger's equation applies to massive particles only.

For the momentum $<\hat{p}>$ we get

$$
\begin{aligned}
<\hat{p}>= \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial \Psi}{\partial x}\right) dx \\
= \frac{-ima}{\hbar} \int_{-\infty}^{\infty} \left[\frac{-ma}{\hbar^2} sgn(x-vt) + \frac{i}{\hbar} mv\right] e^{-b|x-vt|} dx.
\end{aligned}
\tag{4.4}
$$

The 1st integral gives zero and from the 2nd one obtains

$$<\hat{p}>= \frac{bmv}{2} \int_{-\infty}^{\infty} e^{-b|x-vt|} dx = mv. \tag{4.5}$$

From (4.3) and (4.5) it is clear that v represents the classical velocity of the particle. Moreover, $d<\hat{p}>/dt = 0$, though the potential V is time dependent.

Having the above results we can compute the r.m.s. deviations Δx and Δp. We have

$$<\hat{x^2}>= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx = \frac{b}{2} \int_{-\infty}^{\infty} x^2 e^{-b|x-vt|} dx. \tag{4.6}$$

Passing to the variable y and using integration by parts, the result appears as

$$<\hat{x^2}>= \frac{2}{b^2} + v^2 t^2 = \frac{1}{2}\lambda_C^2 + v^2 t^2, \tag{4.7}$$

where $\lambda_C = \hbar/mc$ is the Compton wavelength of the particle. One now obtains

$$\Delta x = \sqrt{<\hat{x^2}> - <\hat{x}>^2} = \frac{\lambda_C}{\sqrt{2}} \tag{4.8}$$

To calculate $<\hat{p^2}>$, we make use of the expression (3.4)

$$<\hat{p^2}>= 2m(<\hat{H}> - <V>) = m^2 v^2 + m^2 c^2, \tag{4.9}$$

whence

$$\Delta p = \sqrt{<\hat{p^2}> - <\hat{p}>^2} = mc. \tag{4.10}$$

We notice that Δx and Δp are not time dependent (no spreading), as for a Gaussian wave packet [3]. This has its origin in the fact that $\Psi(x,t)$ represents a monochromatic wave (the energy E is unique - no energetic spectrum, continuous or discret). It is therefore similar with the de Broglie (dB) plane wave. However, $\Psi(x,t)$ is normalized to one, in contrast with the dB wave, which is not. We see that our results applies better to macroscopic objects. From (4.8) and (4.10) we find that

$$\Delta x \Delta p = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2}, \tag{4.11}$$

[3] Do not forget that a Gaussian-type packet is a choice: we suppose that a particle is described by it - a conjecture that is in accordance with experimental facts.

and the Heisenberg relation is satisfied. It is worth to observe that Δp is very large, as a consequence of a very tiny Δx (of the order of the Compton wavelength). A large spread in momentum produces a large uncertainty in wavelength (from the dB relation) and so the particle is very localized. That corresponds to the Fig.1.7 from Griffiths book [2], where a sudden jerk has been given to the rope and a very narrow lump (our particle) travels along the x - direction. That is in the spirit of the δ-function source. From the mean values of the momentum and potential energy, we get for the wave number k and the frequency ω

$$k = \frac{<\hat{p}>}{\hbar} = \frac{mv}{\hbar}, \quad \omega = \frac{1}{\hbar} < \frac{\hat{p^2}}{2m} >= \frac{m}{2\hbar}(v^2 + c^2). \tag{4.12}$$

Note that ω varies from $mc^2/2\hbar$ at $v = 0$ to $mc^2/\hbar$ when v tends to c. The dispertion relation $\omega(k)$ appears as

$$\omega(k) = \frac{\hbar k^2}{2m} + \frac{mc^2}{2\hbar}, \tag{4.13}$$

whence the group velocity is $v_g = d\omega/dk = \hbar k/m = v$, namely the classical velocity. In addition, $0 < v_g < c$ when v varies from 0 to the velocity of light. Nevertheless, the phase velocity

$$v_f = \frac{\omega}{k} = \frac{v}{2} + \frac{c^2}{2v} \tag{4.14}$$

is always greater than c, with $c < v_f < \infty$ when v decreases from c to zero. Noting that our particle is not described by a wave packet but, nevertheless, it could be approximated with a very narrow one.

According to the viewpoint expressed until now, it turns out that our quantum object (micro- or macroscopic) does appear as being a wave, the velocity of which is determined by the initial conditions (the value of vector v). Once the wave propagates (along the x-direction) the Huygens-Fresnel principle applies: every new position of the wave-front represents a new source of propagation, as if the object were regenerating continuously. That is in the spirit of Everett's QM [7], where any object evolves into a macroscopic superposition of different versions of themselves [8]. By performing a measurement we make only visible one branch of the superposition of identical objects. This resembles the motion of small, identical objects (seated in the same direction) and floating on the surface of a pond. When a water wave propagates (by tossing into a small stone, for example), the small objects oscillate vertically. If one could see only just above the objects, we should notice one object in different locations, as if it were moving with the speed of the wave (something like "Zenon's arrow paradox"). What is actually happening is that each small object moves only vertically, appearing and disappearing from our view.

If the above recipe is valid, then we may explain the phenomenon of inertia: once put in motion with constant velocity, a particle actually propagates as a wave in accordance with Huygens-Fresnel principle and, as any wave, preserves its constant velocity of propagation, if not perturbed by external agents. Of course, we have to apply a force to stop that wave which, fortunately, propagates along one direction. This is a different property compared to standard waves, which propagate in all directions. In addition, their velocity is not given by the properties of the medium (quantum vacuum in our situation) but by initial conditions (initial velocity).

5 deB-Bohm quantum potential

Let us see now an eventual connection between our results and the deB-Bohm mechanics. For that purpose, we write $\Psi(x,t)$ as

$$\Psi(x,t) = \rho(x,t) e^{\frac{i}{\hbar} S(x,t)}, \tag{5.1}$$

with

$$\rho(x,t) = \frac{\sqrt{ma}}{\hbar} e^{-\frac{ma|x-vt|}{\hbar^2}}, \quad S(x,t) = -(E + \frac{mv^2}{2})t + mvx. \tag{5.2}$$

The Schrodinger equation (2.1) gives us [9]

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x,t) + Q(x,t), \tag{5.3}$$

and

$$\frac{\partial \rho^2}{\partial t} + \frac{1}{m}\frac{\partial}{\partial x}\left(\rho \frac{\partial S}{\partial x}\right) = 0, \tag{5.4}$$

where Q is the quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x^2}. \tag{5.5}$$

We have also

$$\frac{\partial S}{\partial t} = -\left(E + \frac{mv^2}{2}\right)t, \quad \frac{\partial S}{\partial x} = mv. \tag{5.6}$$

The expression for Q will be given by

$$Q = \frac{a}{2}\left[2\delta(x-vt) - \frac{ma}{\hbar^2}\right]. \tag{5.7}$$

With $a = \hbar c$, one obtains

$$Q = \hbar c \delta(x-vt) - \frac{mc^2}{2} = E - V. \tag{5.8}$$

Hence, the dB-Bohm quantum potential is the difference between the bound energy E [2] and our δ-function potential $V(x,t) = -\hbar c \delta(x-vt)$. In addition, we have $< Q >= E- < V >= mc^2/2$. One may easily check that, with the action S from (5.2) and Q from (5.7), Eqs. (5.3) and (5.4) are fulfilled.

6 Conclusions

QFT shows us that particles represent excited states of vacuum quantum fields. That seems to suggest that quantum vacuum is filled with particles in ground (virtual) states. We tried to extend that viewpoint to macroscopic objects, whose motion is viewed as following the Huygens-Fresnel principle. It propagates as an unidirectional wave, to which the velocity is decided by the initial conditions (initial velocity in our situation). The object is continuously regenerating as an electromagnetic wave to which propagation is determined by the secondary waves (sources) from the new wave front. We proposed an universal "strength" for the Griffiths potential, inspired by the necessity of having the correct physical units. Our model is developed on the grounds of QM and the rest energy from SR, because of the presence of the velocity of light. We found that our (normalizable) wave function is not spreading (the wave is monochromatic, like the deB plane wave), even though $< x >$ is time-dependent: Δx and Δp are constant and obey the position-momentum Heisenberg relation.

References

[1] B. Shoshany, arXiv: 1803.07098.

[2] D. J. Griffiths, Introduction to Quantum Mechanics, Prentice Hall, 1995, p.3

[3] N. D. Mermin, Physics Today, April 1985, p.38.

[4] I. J. Araya and I. Bars, arXiv: 1712.01326.

[5] H. Culetu, Int. J. Theor. Phys. 54, 2855 (2015), arXiv: 1408.3334.

[6] H. Culetu, Machs Principle and Special Relativity, Proceedings of the National Colloquium on the Theory of Relativity, Iassy University, Romania (1979).

[7] H. Everett, Rev. Mod. Phys. 29, 454 (1957).

[8] A. Mandolesi, arXiv: 1704.01173.

[9] S. Goldstein and W. Struyve, J. Phys. A48, 025303 (2015), arXiv: 1312.1990.

[10] P. J. Riggs, Erkenn. 68, 21 (2008).