An Algebraic Invariant of Gravity

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Abstract. Newton's mechanics is simple. His equivalence principle is simple, as is the inverse square law of gravitational force. A simple theory should have simple solutions to simple models. A system of n particles, given their initial speed and positions along with their masses, is such a simple model. Yet, solving for $n > 2$ is not simple.
This paper discusses, why that is a difficult problem and what could be done to get around that problem.

1. Problem Statement

Classical mechanics is essentially a linear, "first order" theory in which the dynamical quantities describe properties of the particles themselves, such as the law of inertia, $F = ma$, as well as energy and momentum conservation etc.
The graviational force, $F = (const)\nabla\frac{m_1 m_2}{|x_1 - x_2|}$, is the exception to that theory: it is a product of quantities, namely the mutual interaction the masses, disguised as a linear first order quantity F. That makes it complicated to even deal with a gravitational interaction of two particles, necessitating elliptic integrals, Legendre polynoms, Bessel functions, and all that, in order to derive its solutions. But it can be done, and it involves some beautiful mathematics and calculations, which explains, why it's done in physics first place up to this day. The result is that the particles move around the center of mass in all curves given by the intersection of a plane with a cone.
That is mathematically interesting, as it allows to describe the set of solutions through a hyperbolic, quadratic equation, namely that of the cone itself. And it straight leads to the question, if not a quadratic approach to the dynamics might be simpler to describe gravitational interaction.

2. The Cone

The picture of that cone is always that of a two-dimensional surface in three dimensions, because it is easy to visualize, but, even given the fact that one angular, cyclic coordinate can be eliminated, this is still inconvenient:
The horizontal angle in the two-dimensional cone should also be factored out. And with this we are left with the diagonals in a two-dimensional coordinate system with x and y as horizontal and verticle axis, say. The diagonals then is the set of points (x, y), for which $x^2 - y^2 = 0$ holds, and the cone sections reduce to the points of intersection of lines with the diagonals. Mathematically, the relation $x^2 - y^2 = 0$ is an equivalence relation, and it does not depend on the sign of x, nor on the sign of y. The statement of all solutions of masses in a gravitational field being cone sections therefore reduces to the existence of two dynamical quantities x, y, for which $x^2 - y^2$ is conserved throughout, i.e. it is a dynamic invariant.
Now, as the field is conservative, it is generally the best to take the particle's energy E for one coodinate, x, say. We know that E is an invariant, so the second quantity must be an invariant either, in order to preserve $E^2 - y^2 = const$, and this quantity must have the dimension of energy. That would be $V_1 := V(r \equiv 1)$, where $V(r) = (const)\frac{mM}{r}$ is the potential of the particle of mass m, M is the total mass of the system, an r is the radial distance between the center of mass M (that we place in the origin of the coordinate system), and the particle itself.
In all, the 2-particle problem of gravitation reduces to $E^2 - V_1^2 = const$. Moreover, given that $x^2 - y^2 = const$ is maintained, we can deliberately replace x with E and y with V_1, if $|E| \leq |V_1|$, and the other way round, otherwise.

3. The Implications

That's a remarkable thing: We have found another dynamic invariant of the two-body problem. Albeit not a cyclic coordinate for the Lagrangian or Hamiltonian mechanics, but in terms of squares of energy: it's $E^2 - V_1^2$. And because it is a cyclic coordinate for two particles, it is for any n-particle system with gravitational interaction:

Given n gravitationally interacting particles, we may always assume that the center of mass of that system exists and is at rest. (Energy and momentum conservation, isometry of space, etc. all lead to other well-known cyclic coodinates, which are needed to ensure that this can be done.)
Now, taking the first moving particle, we can extract $E_1^2 - V_{1,1}^2$ as a constant. Then we proceed with the next ones, and end up with the sum of m constants $E_k^2 - V_{1,k}^2$ for all the moving particles and $n - m$ particles that - for what reasons ever - don't move w.r.t. the origin.
And, if the only interaction between the particles is gravity, then we can proceed with all the particles in that system, be these at rest or not. But what is left over? Is it a constant of integration or a/the vacuum?

No: $\sum_{1\le k\le n}(V_{1,k}^2 - E_k^2)$ is nothing but the square of a kinetic energy T^2 of the system plus an arbitrary constant, so it is heat plus a constant, and we can get rid of that constant by demanding that sum to go to 0 as the particle velocities converge to zero. Gravity is well known to sustain the lowest temperatures, so the extracted heat can only be the fraction of what must remain.
Still, the extractability of $T^2 = \sum_{1\le k\le n}(V_{1,k}^2 - E_k^2)$ fits nicely to explain, why temperature adds as squares and why heat can be transferred from one system to another, which was not clear before.

Pragmatically, what we know are the $V_{1,k}$, but neither do we know the E_k^2, nor do we know E^2. So, a statement like $T^2 = \sum_k (m_k/2)^2 v_k^4$ can only be a guess, and a better one would be $T^2 = \sum_k (p_k v_k)^2$, (p_k being the momenta, v_k the velocities). What we really want, are the values of the E_k^2, from which we can deduce E^2.
We also know that E^2 is an invariant, since E is. Then, as E^2 and $E^2 - V_1^2$ are invariants, V_1^2 is an invariant, and, demanding $V_1 \le 0$, V_1 must be an invariant. When we extracted T^2 from the system, we therefore extracted $\sum_k V_{1,k}$ plus an invariant from the energy E, so we have to subtract this from the energy of the n resting particle mass points $m_k = m_k(x_k)$, that remain residing in the locations $x_k \in \mathbb{R}^3$, each. There now may be internal potential energies U_k between the resting particles, even of either sign, which should not be ignored. However, if we do, we would expect $E^2 = \sum_k (m_k c^2 - V_{1,k})^2 + T^2$, where the speed of light c has been inserted to convert the mass into energy. (We can set $c \equiv 1$ for simplicity.)

Enters Gauß law: We have $\Delta V = (const)\rho$, where ρ is the mass density and Δ the Laplace operator. We know V, therefore we get ρ, and from this we get ρ^2, and integrating over the volume we get the square of the total mass at rest. Again that ignores possible additional potential fields between the rest masses.
But it gives us another way to rewrite E^2 as volume integral over an energy square density $\mathcal{E}^2(x) = (\rho(x) - V_1(x))^2 + (\vec{j}(x)\cdot\vec{v}(x))^2$, where $\vec{j} = (j_1, j_3, j_3)$ is the flux of $\rho(x)$ and $\vec{v}$ the velocity.

Else, we could weigh the system, which means to measure the potential energy between the system and the fixed earth. Now, since the system is at rest on the whole, the fluctuation of energy and momentum w.r.t. the earth cancels out. As the particles's speed raises their rest mass, the weight of the system is expected to decrease slightly as $T \to 0$. More importantly, the square of the sum of weights allows the cancelling of positive and negative pieces, and is unequal to the sum of weights to the square. And again, the weight of a substance fails to measure its absolute energetic content $\sqrt{E^2}$. (A quadratic equation simply is not a linear one.)

Again, there is something to learn from: As shown above, the $(1/r)$-dependency of V is just the necessary ingredience needed to be able to extract the kinetic energy (or heat) as cyclic coordinates from the system. That heat is not moving freely, but it is bound to the compound system. Although we are not able to include the internal binding energies U_k into the equation, which might be the key to gravity, we may relate gravity to the amount of heat, a purely gravitational system (i.e. one that is subjected only to its own gravity) sustains without decay.

Take a step back: The cone equation $a^2 - b^2 = 1$ turns into a circle by replacing $b \to ib$. That makes the cone equation a 2-layered coverage of the unit circle. So far, we restricted the discussion on the upper half plane, only (- the first sheaf) where $V(r)$ is supposed to be negative and increasing along the verical axis. The good reason for that is that the flip of upper and lower half plane means inversion of the gravitational potential V, whilst a positive potential would lead to negative masses. And masses are to be positive, always.
Though, upper and lower half plane can also be inverted through parity inversion, i.e. the inversion of the horizontal axis (or the location coordinates), achieved by a rotation of the angle π. Contraction of objects viewed from the inside (of an imaginary) sphere will be seen as expansion from the (inverted) outside. An always attracting gravitational force will allow to tell an observer the inside apart from the outside - a well-known fact addressed by cosmology through the statement that there must not be an outside to the cosmos - which solves the parity problem by declarative exclusion.

However, as was found out foremost by chemicists, matter can be transformed in interesting ways: a chunk of solid (neutral) matter that qualifies as an appreciable source of gravity can always be fragmented into a gas of small (neutral) particles, which just shows the contrary behaviour of a contraction: it always exerts pressure to the outside: like gravity, gaseous pressure breaks parity. The exclusion of an outside will therefore not rescue parity-symmetry: But when put together, gravity and gaseous pressure, then parity-symmetry could be reinstated. That way, for a (neutral) gaseous system, one would express its "anti"-gravity as the negative (kinetic) energy that was needed to stop its debris or expansion.

References

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