# On The Proving Method of Fermat's Last Theorem

Haofeng Zhang
Beijing, China

**Abstract**: In this paper the author gives an elementary mathematics method to solve ***Fermat's Last Theorem*** (FLT), in which let this equation become an one unknown number equation, in order to solve this equation the author invented a method called "Order reducing method for equations", where the second order root compares to one order root, and with some necessary techniques the author successfully proved when *x*^(*n*-1)+*y*^(*n*-1)- *z*^(*n*-1) <= *x*^(*n*-2)+*y*^(*n*-2)- *z*^(*n*-2) there are no positive solutions for this equation, and also proves with the increasing of *x* there are still no positive integer solutions for this equation when *x*^(*n*-1)+*y*^(*n*-1)- *z*^(*n*-1)<= *x*^(*n*-2)+*y*^(*n*-2)- *z*^(*n*-2) is not satisfied.

## 1. Some Relevant Theorems

There are some theorems for proving or need to be known. ***All symbols in this paper represent positive integers unless they are stated to be not.***

**Theorem 1.1.** In the equation of

$$\begin{cases} x^n + y^n = z^n \\ \gcd(x, y, z) = 1 \\ n > 2 \end{cases} \tag{1-1}$$

$x, y, z$ meet

$x \neq y$ ,

$x + y > z$ ,

and if

$x > y$

then

$z > x > y$ .

**Proof**: Let

$x = y$ ,

we have

$$2x^n = z^n$$

and

$$\sqrt[n]{2}x = z$$

where $\sqrt[n]{2}$ is not an integer and $x$, $z$ are all positive integers, so $x \neq y$ .

Since

$$(x+y)^n = x^n + C_n^1 x^{n-1} y + ... + C_n^{n-1} x y^{n-1} + y^n > z^n ,$$

so we get

$$x + y > z.$$

Since

$$x^n + y^n = z^n ,$$

so we have

$$z^n > x^n, z^n > y^n$$

and get

$$z > x > y$$

when

$$x > y .$$

**Theorem 1.2.** In the equation of (1-1), $x, y, z$ meet

$$\gcd(x, y) = \gcd(y, z) = \gcd(x, z) = 1.$$

**Proof**: Since $x^n + y^n = z^n$ , if $\gcd(x, y) > 1$ then we have $\left(x_1^{\ n} + y^n{}_1\right) \times [\gcd(x, y)]^n = z^n$ which causes $\gcd(x, y, z) > 1$ since the left side contains the factor of $[\gcd(x, y)]^n$ then the right side must also contains this factor but contradicts against (1-1) in which $\gcd(x, y, z) = 1$, so we have $\gcd(x, y) = 1.$ Using the same way we have $\gcd(x, z) = \gcd(y, z) = 1$ .

**Theorem 1.3.** If there is no positive integer solution for

$$x^p + y^p = z^p$$

when $p > 2$ is a prime number then there is also no positive integer solution for

$$\left(x^p\right)^k + \left(y^p\right)^k = \left(z^p\right)^k .$$

**Proof:** Since $x^p + y^p = z^p$ has no positive integer solution, so there still no positive integer solution for

$$\left(x^k\right)^p + \left(y^k\right)^p = \left(z^k\right)^p$$

which means there is also no positive integer solution for

$$\left(x^p\right)^k + \left(y^p\right)^k = \left(z^p\right)^k .$$

So we only need to prove there is no positive integer solution for equation (1-1) when $n$ is a prime number.

**Theorem 1.4.** There are no positive integer solutions for equation (1-1) when $x$ or $y$ is a prime number .

**Proof:** When $x$ is a prime number, since

$$x^n = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right),$$

so we have

$$\gcd(z-y, x) = x\ ,$$

which means

$$z - y \geq x\ ,$$

we have

$$x + y \leq z\ ,$$

that contradicts against **Theorem 1.1** in which $x + y > z$ , so it is with $y$ , which means there are no positive integer solutions for equation (1-1) when $x$ or $y$ is a prime number .

**Theorem 1.5.** There are no positive integer solutions for equation (1-1) when $z$ is a prime number .

**Proof:** When $z$ is a prime number, from Theorem 1.12 we only consider the case of $n > 2$ is a prime number, since

$$x^n + y^n = z^n = (x+y)\left(x^{n-1} + \ldots + y^{n-1}\right),$$

so we have

$$\gcd(x+y, z) = z\ ,$$

from **Theorem 1.1** we know $x + y > z$ , so we get

$$x + y \geq 2z\ ,$$

that contradicts against **Theorem 1.1** in which $z > x > y \Rightarrow x + y < 2z$ , which means there are no positive integer solutions for equation (1-1) when $z$ is a prime number .

**Theorem 1.6.** There are no positive integer solutions for

$$1^n + y^n = z^n\ .$$

**Proof:** Since

$$1 = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right)$$

where

$$\begin{cases} z - y = 1 \\ \left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right) = 1 \end{cases}$$

that causes $z, y$ to be non positive integers, so there are no positive integer solutions for

$$1^n + y^n = z^n\ .$$

**Theorem 1.7.** There are no positive integer solutions for

$$2^n + y^n = z^n .$$

**Proof:** Since

$$2^n = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1}\right),$$

if

$$\begin{cases} z - y = 1 \\ z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1} = 2^n \end{cases}$$

then taking the least value for $y = 2, z = 3$, we have

$$3^{n-1} + 2\times 3^{n-2} + ... + 2^{n-1} > 2^n$$

when $n > 2$ that is impossible. If

$$\begin{cases} z - y = 2^i \\ z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1} = 2^j \\ i + j = n \\ i \geq 1 \end{cases}$$

then $z > 2$ and taking the least value of $y = 2, z = 3$, we get

$$3^{n-1} + 2\times 3^{n-2} + ... + 2^{n-1} > 2^j$$

with $n > 2$ that is also impossible, so there are no positive integer solutions for

$$2^n + y^n = z^n .$$

**Theorem 1.8.** There are no positive integer solutions for equation (1-1) when $n \to \infty$ and $x, y, z$ in equation (1-1) meet

$$\sqrt[n]{2}y < z < \sqrt[n]{2}x, x > 2, y > 1, z > 3.$$

**Proof:** Since $x^n + y^n = z^n$, let $x > y$, we get

$$\left(\frac{z}{x}\right)^n - \left(\frac{y}{x}\right)^n = 1,$$

since

$$z > x > y,$$

so we have

$$z < \sqrt[n]{2}x,$$

and

$$\lim_{n\to\infty}\left(\frac{z}{x}\right)^n-\left(\frac{y}{x}\right)^n=\infty>1$$

which means there are no positive integer solutions for equation (1-1) when $n\to\infty$.

According to **Theorem 1.6, 1.7** we have $x>2, y>1, z>3$.

If $y^n \ge \frac{z^n}{2}$, since $x>y$, so we have $x^n \ge \frac{z^n}{2}$,and

$$x^n+y^n>z^n,$$

that is impossible. So we have

$$\sqrt[n]{2}y<z.$$

**Theorem 1.9.** There are no positive integer solutions for equation (1-1) when $x, y, z \le 100$.

**Proof:** From **Theorem1.8**, we know $\sqrt[n]{2}y<z<\sqrt[n]{2}x$, so we have

$$y<\frac{100}{\sqrt[n]{2}}<x,$$

when $n=3$, we have the smallest values for $x$, so we get

$$\left(y<\frac{100}{\sqrt[3]{2}}<x\right)\Rightarrow(y<79<x),$$

from **Theorem 1.4, 1.5** we know $x, y, z$ are all not prime numbers. There are below combinations of $x, y, z$ when $x, y, z \le 100$:

$$\begin{cases}(x=80\sim 99)^n+(y=4\sim(x-1))^n=(z=81\sim 100)^n\\ x+y>z\\ x^2+y^2>z^2\\ x^j+y^j>z^j\\ j<n\end{cases}.$$

Here we take $7^n+9^n=10^n$ for example to explain how to prove. We plot the graph for this equation as showed in **Figure 1-1**.

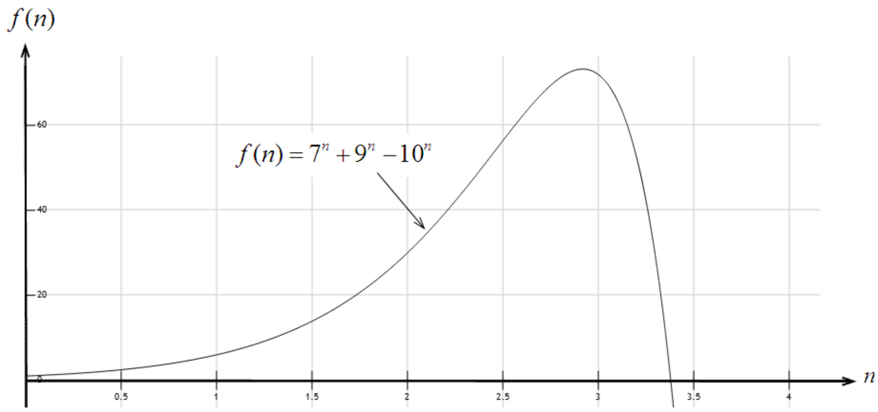


**Figure 1-1** *Graph of* $f(n)=7^n+9^n-10^n$

Obviously for equation $f(n)=7^n+9^n-10^n$ in **Figure 1-1**, we have $3<n<4$ is not an integer so there are no positive integer solutions, using this method we have the conclusion of there are no positive integer solutions for equation (1-1) when $z\le 100$.

Using the method of which we prove **Theorem 1.6, 1.7** we can prove when $x, y\le 100$, there are no positive integer solutions for equation (1-1).

**Theorem 1.10.** In the equation of (1-1), $x, y, z$ meet

$$x^{n-i}+y^{n-i}>z^{n-i},$$

$$x^{n+i}+y^{n+i}<z^{n+i},$$

where

$$n>i\ge 1.$$

This theorem holds true when $x, y, z$ are positive real numbers but $n$ must be a positive integer.

**Proof**: From equation (1-1), since

$$x^n+y^n=z^n,$$

from **Theorem 1.1**, since $z>x>y$, we have

$$x^{n-i}+y^{n-i}>\left[\left(\frac{x}{z}\right)^i x^{n-i}+\left(\frac{y}{z}\right)^i y^{n-i}=z^{n-i}\right],$$

$$x^{n+i}+y^{n+i}<\left(z^i x^{n-i}+z^i y^{n-i}=z^{n+i}\right),$$

so we have

$$x^{n-i}+y^{n-i}>z^{n-i}.$$

$$x^{n+i} + y^{n+i} < z^{n+i} .$$

This theorem means given $x, y, z$ if equation (1-1) has one positive integer solution then this solution is the only one.

**Theorem 1.11.** There are no positive integer solutions for equation (1-1) when

$$\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \leq 1.$$

And in order to have positive integer solutions for equation (1-1),

$$\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 40$$

must be satisfied.

**Proof**: In equation (1-1), let

$$\begin{cases} a = x^{n-2} \\ b = y^{n-2} \\ c = z^{n-2} \end{cases},$$

we have

$$\begin{cases} ax^2 + by^2 = cz^2 \\ a^{\frac{n-1}{n-2}} x + b^{\frac{n-1}{n-2}} y = c^{\frac{n-1}{n-2}} z \end{cases}.$$

Since we reduce the order of equation so the method is called "Order reducing method for equations". Let $x > y$ and

$$\begin{cases} y = x - f \\ z = x + e \end{cases},$$

we have

$$\begin{cases} ax^2 + b(x - f)^2 = c(x + e)^2 \\ a^{\frac{n-1}{n-2}} x + b^{\frac{n-1}{n-2}} (x - f) = c^{\frac{n-1}{n-2}} (x + e) \end{cases}$$

and

$$\begin{cases} (a + b - c)x^2 - 2(bf + ce)x + (bf^2 - ce^2) = 0 \\ a^{\frac{n-1}{n-2}} x + b^{\frac{n-1}{n-2}} (x - f) - c^{\frac{n-1}{n-2}} (x + e) = 0 \end{cases},$$

the roots are

$$x = \frac{(bf + ce) \pm \sqrt{(bf + ce)^2 - (a + b - c)(bf^2 - ce^2)}}{x^{n-2} + y^{n-2} - z^{n-2}}, \tag{1-2}$$

and

$$x=\frac{c^{\frac{n-1}{n-2}}e+b^{\frac{n-1}{n-2}}f}{a^{\frac{n-1}{n-2}}+b^{\frac{n-1}{n-2}}-c^{\frac{n-1}{n-2}}}=\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}. \tag{1-3}$$

There are two cases for $bf^2, ce^2$ when $bf^2 \geq ce^2$ and $bf^2 < ce^2$.

**Case A**: If $bf^2 \geq ce^2$, from (1-2) when

$$x=\frac{(bf+ce)+\sqrt{(bf+ce)^2-(a+b-c)(bf^2-ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}},$$

from **Theorem 1.10** we know $a+b-c=x^{n-2}+y^{n-2}-z^{n-2}>0$, so we have

$$x\leq\frac{2(bf+ce)}{x^{n-2}+y^{n-2}-z^{n-2}},$$

also from **Theorem 1.10** we have $x^{n-1}+y^{n-1}-z^{n-1}>0$, compare to (1-3) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}\leq\frac{2(bf+ce)}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\leq 1$, we have

$$bfy+cez\leq 2(bf+ce)$$

that is impossible since from **Theorem 1.8** we know $y\geq 2$ and $z>3$.

When

$$x=\frac{(bf+ce)-\sqrt{(bf+ce)^2-(a+b-c)(bf^2-ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

we have

$$x\leq\frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}},$$

compare to (1-3) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}\leq\frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\leq 1$, we have

$$bfy + cez \le bf + ce$$

that is impossible since from **Theorem 1.8** we have already known $y \ge 2$ and $z > 3$.

**Case B**: If $bf^2 < ce^2$, from (1-2) when

$$x = \frac{(bf + ce) + \sqrt{(bf + ce)^2 + (a + b - c)(ce^2 - bf^2)}}{x^{n-2} + y^{n-2} - z^{n-2}},$$

we can prove $(bf + ce)^2 > (a + b - c)(ce^2 - bf^2)$ since if not we have

$$(bf + ce)^2 \le (a + b - c)(ce^2 - bf^2)$$

and

$$[(2b + a) - c]bf^2 + 2bfce + [2c - (a + b)]ce^2 \le 0$$

that is impossible since $a + b - c > 0$ and $c > a, c > b, 2c - (a + b) > 0.$ So we have

$$x < \frac{(bf + ce)(1 + \sqrt{2})}{x^{n-2} + y^{n-2} - z^{n-2}}$$

compare to (2-4) we get

$$\frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}} < \frac{(bf + ce)(1 + \sqrt{2})}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

When $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \le 1$, we have

$$bfy + cez < (bf + ce)(1 + \sqrt{2}) < 2.5(bf + ce)$$

and

$$bf(x - f) + ce(x + e) < 2.5(bf + ce)$$

that leads to

$$x < \left[ \frac{2.5(bf + ce) + bf^2 - ce^2}{bf + ce} = 2.5 - \frac{ce^2 - bf^2}{bf + ce} \right] < 2.5$$

where possible values for $x$ are 1, 2 but according to **Theorem 1.6**, **1.7** we know there are no positive integer solutions.

When

$$x = \frac{(bf + ce) - \sqrt{(bf + ce)^2 + (a + b - c)(ce^2 - bf^2)}}{x^{n-2} + y^{n-2} - z^{n-2}}$$

is not possible since $x \le 0$.

So we have the conclusion of there are no positive integer solutions for equation (1-1) when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} \le 1$.

Obviously we have

$$bfy + cez < 2.5\left(\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\right)(bf + ce),$$

from **Theorem 1.9** we know $x, y, z \le 100$ there are no positive integer solutions for equation (1-1), so we have

$$\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 40,$$

which must be satisfied to have positive integer solutions for equation (1-1).

# 2. Proving Method

From **Theorem 1.11** we know in order to have positive integer solutions for this equation, $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$ must be satisfied. We give the graph of this equation as showed in

**Figure 2-1** when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$, where $AB // CD'$.

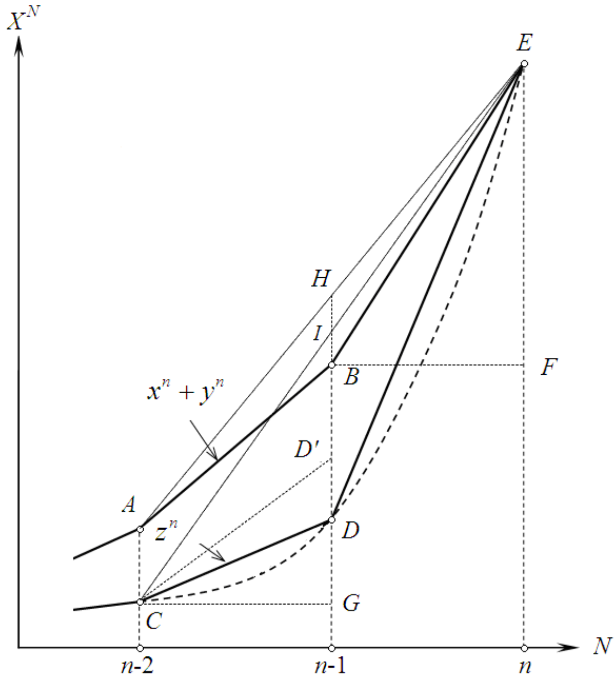

**Figure 2-1** *Graph of* $x^n + y^n = z^n$ *when* $\dfrac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$

**1.** In **Figure 2-1** we have

$$\angle CDE = 360^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1}-z^{n-2}}\right) - 90^0,$$

obviously we have

$$BD = x^{n-1} + y^{n-1} - z^{n-1},$$
$$AC = x^{n-2} + y^{n-2} - z^{n-2},$$

when $\dfrac{BD}{AC} > 1$ we have

$$\angle ABE - \angle CDE = \angle D'CD + \angle BED > 0,$$

which means

$$\angle ABE > \angle CDE.$$

It is also very clear that if $\angle ABE \le \angle CDE$ then $\dfrac{BD}{AC} < 1$.

From **Theorem 1.9** we know if $z \le 100$ then there are no positive integer solutions for equation (1-1), when $n = 3$ (*which is the worst case*) we have

$$\angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1}-z^{n-2}}\right),$$
$$= 270^0 - \arctan\left(100^3 - 100^2\right) - \arctan\left(\frac{1}{100^2 - 100}\right) > 179.99^0$$

and

$$\angle ABE > \angle CDE > 179.99^0,$$

which means $\angle ABE, \angle CDE \to 180^0$, so $ABE, CDE$ are almost lines with $z > 100, n \ge 3$, that leads to $\dfrac{BD}{AC} \to \dfrac{1}{2} < 1$, this contradicts against $BD > AC$. So when $z$ is large enough then $\dfrac{BD}{AC} = \dfrac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

**2.** For function

$$f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1}-z^{n-2}}\right),$$
$$= \frac{3}{2}\pi - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1}-z^{n-2}}\right)$$

we give the function plot for it in **Figure 2-2**.

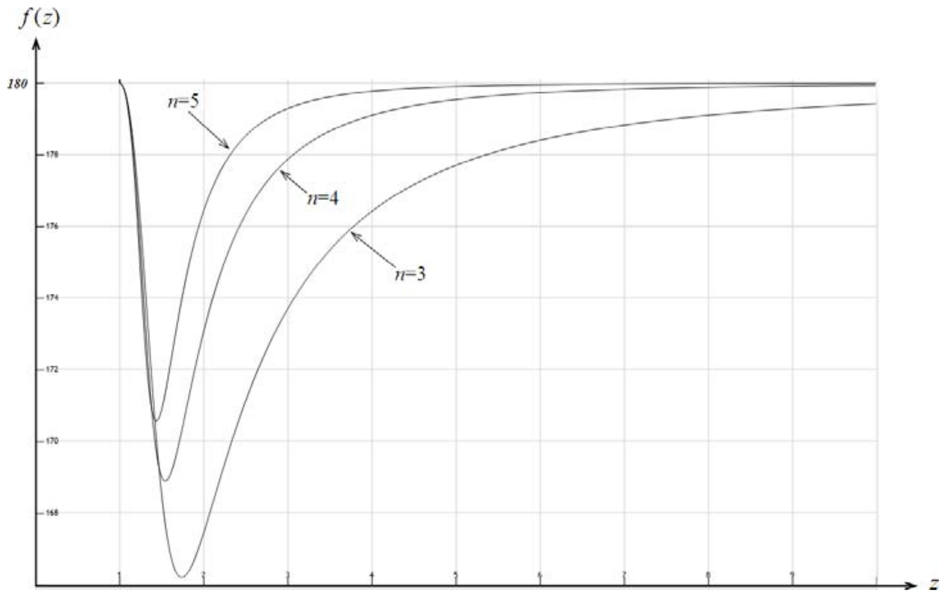


**Figure 2-2** *Graph of* $f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$

*where we take* $\pi = 3.1415926535897932$

Obviously $f(z) = \angle CDE$ is a "Monotonically increasing function" when $z \geq 3$, and with the increasing of $z$ the value of $f(z) = \angle CDE$ is close to $180^0$. It is very clear that $\angle ABE - \angle CDE$ is decreasing with the increasing of $z$, since

$$\left(\angle ABE - \angle CDE = \angle D'CD + \angle BED\right) < 180^0 - \angle CDE,$$

where $\angle CDE$ is increasing. When $n = 3$ since $\angle CDE > 179.99^0$, so we have

$$\left(\angle D'CD + \angle BED\right) < 180^0 - \angle CDE < 180^0 - 179.99^0 < 0.01^0,$$

which means

$$\angle BED, \angle D'CD < 0.01^0,$$

and when $z$ or $n$ is large enough, we have

$$\angle ABE - \angle CDE = \left(\angle BED + \angle D'CD\right) \to 0,$$

which means $BD < AC$ that contradicts against $BD > AC$. So when $z$ or $n$ is large enough then $\frac{BD}{AC} = \frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

**3.** In **Figure 2-1** we have

$$\angle ABE = \frac{3}{2}\pi - \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right),$$

$$\angle CDE = \frac{3}{2}\pi - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right),$$

so

$$\angle ABE - \angle CDE = \arctan\left(\frac{z^n - z^{n-1}}{1}\right) + \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$$
$$- \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right).$$

From (1-1) we have

$$z = \left(x^n + y^n\right)^{\frac{1}{n}},$$

we get

$$\angle ABE - \angle CDE$$
$$= \arctan\left(\frac{x^n + y^n - \left(x^n + y^n\right)^{\frac{n-1}{n}}}{1}\right) + \arctan\left(\frac{1}{\left(x^n + y^n\right)^{\frac{n-1}{n}} - \left(x^n + y^n\right)^{\frac{n-2}{n}}}\right)$$
$$- \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right).$$

We give the plot of $f(x, y) = \angle ABE - \angle CDE$ using Excel VBA program that showed below:

```
n = 3
For x = 1 To 10 ^ 5 Step 1
    For y = 1 To x - 1 Step 1
        z = (x ^ n + y ^ n) ^ (1 / n)
        t1 = z ^ n - z ^ (n - 1)
        t2 = 1 / (z ^ (n - 1) - z ^ (n - 2))
        t3 = (x ^ n + y ^ n) - (x ^ (n - 1) + y ^ (n - 1))
        t4 = 1 / ((x ^ (n - 1) + y ^ (n - 1)) - x ^ (n - 2) - y ^ (n - 2))
        CDE = Application.Atan2(t1, 1) - Application.Atan2(t2, 1)
        ABE = Application.Atan2(t3, 2) - Application.Atan2(t4, 2)
        R = CDE - ABE
         Cells(i, 1) = "z=" & z
         Cells(i, 2) = "x=" & x
         Cells(i, 3) = "y=" & y
         Cells(i, 4) = R
        i = i + 1
        If i > 65536 Then End
    Next y
```

```
Next x
```

**Figure 2-3** shows the results, obviously $f(x,y) = \angle ABE - \angle CDE, n = 3$ is decreasing.

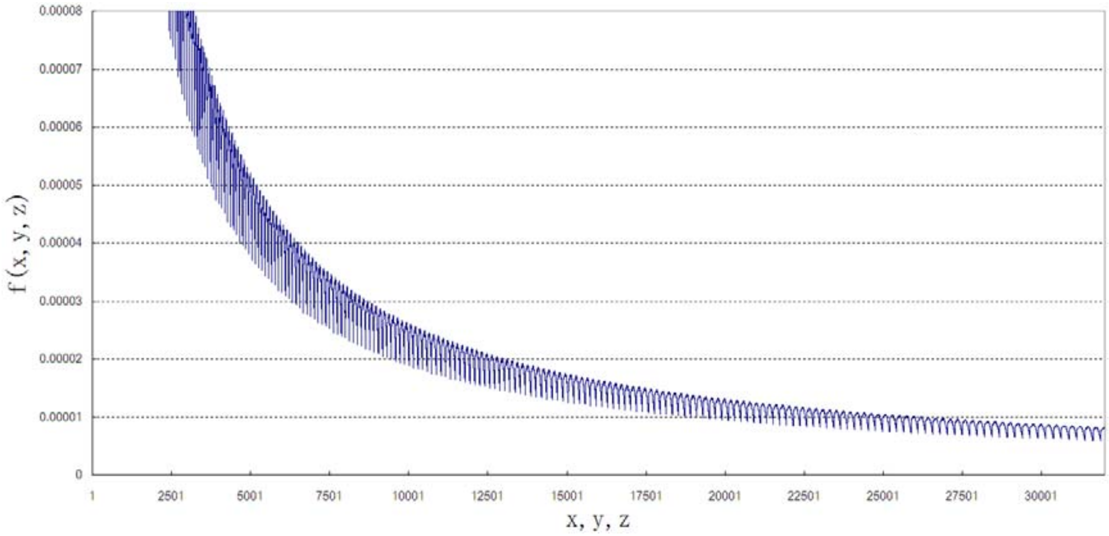


**Figure 2-3** *Graph of* $f(x,y) = \angle ABE - \angle CDE, n = 3$

**4.** In **Figure 2-1** we have

$$\begin{aligned}
&BD^2 = BE^2 + DE^2 - 2BE \times DE \times \cos(\angle BED) \\
&= \left(z^n - z^{n-1}\right)^2 + 1 + \left(x^n + y^n - x^{n-1} - y^{n-1}\right)^2 + 1 - \\
&2\sqrt{\left(z^n - z^{n-1}\right)^2 + 1} \times \sqrt{\left(x^n + y^n - x^{n-1} - y^{n-1}\right)^2 + 1} \times \\
&\cos\left[\arctan\left(\frac{1}{x^n + y^n - x^{n-1} - y^{n-1}}\right) - \arctan\left(\frac{1}{z^n - z^{n-1}}\right)\right]
\end{aligned},$$

and

$$\begin{aligned}
&AC^2 = AE^2 + CE^2 - 2AE \times CE \times \cos(\angle AEC) \\
&= \left(z^n - z^{n-2}\right)^2 + 4 + \left(x^n + y^n - x^{n-2} - y^{n-2}\right)^2 + 4 - \\
&2\sqrt{\left(z^n - z^{n-2}\right)^2 + 4} \times \sqrt{\left(x^n + y^n - x^{n-2} - y^{n-2}\right)^2 + 4} \times \\
&\cos\left[\arctan\left(\frac{2}{x^n + y^n - x^{n-2} - y^{n-2}}\right) - \arctan\left(\frac{2}{z^n - z^{n-2}}\right)\right]
\end{aligned},$$

from (1-1) we have

$$y = \left(z^n - x^n\right)^{\frac{1}{n}}.$$

We give the plot of $f(z,x) = \dfrac{BD}{AC}$ using Excel VBA program that showed below:

```
n = 3
For z = 10 ^ 7 To 10 ^ 9 Step 1
    For x = z / (2 ^ (1 / n)) To z - 1 Step 1
        y = (z ^ n - x ^ n) ^ (1 / n)
```

```
        t1 = z ^ n - z ^ (n - 1)
        t2 = x ^ n + y ^ n - x ^ (n - 1) - y ^ (n - 1)
        t3 = z ^ n - z ^ (n - 2)
        t4 = x ^ n + y ^ n - x ^ (n - 2) - y ^ (n - 2)
        BD = (t1 ^ 2 + t2 ^ 2 + 2 - 2 * Sqr((t1 ^ 2 + 1) * (t2 ^ 2 + 1)) * Cos(Application.Atan2(t2,
1) - Application.Atan2(t1, 1)))
        AC = (t3 ^ 2 + t4 ^ 2 + 8 - 2 * Sqr((t3 ^ 2 + 4) * (t4 ^ 2 + 4)) * Cos(Application.Atan2(t4,
2) - Application.Atan2(t3, 2)))
        R = (BD / AC) ^ 0.5
        Cells(j, 1) = "z=" & z
        Cells(j, 2) = "x=" & x
        Cells(j, 3) = "y=" & y
        Cells(j, 4) = R
        j = j + 1
        If j > 65536 Then End
    Next x
Next z
```

We give the plot of $f(z,x)=\dfrac{BD}{AC}, n=3$ when $z=10^7, x=\dfrac{z}{\sqrt[n]{2}}\sim z$, $n=3$, it is showed in **Figure 2-4**.

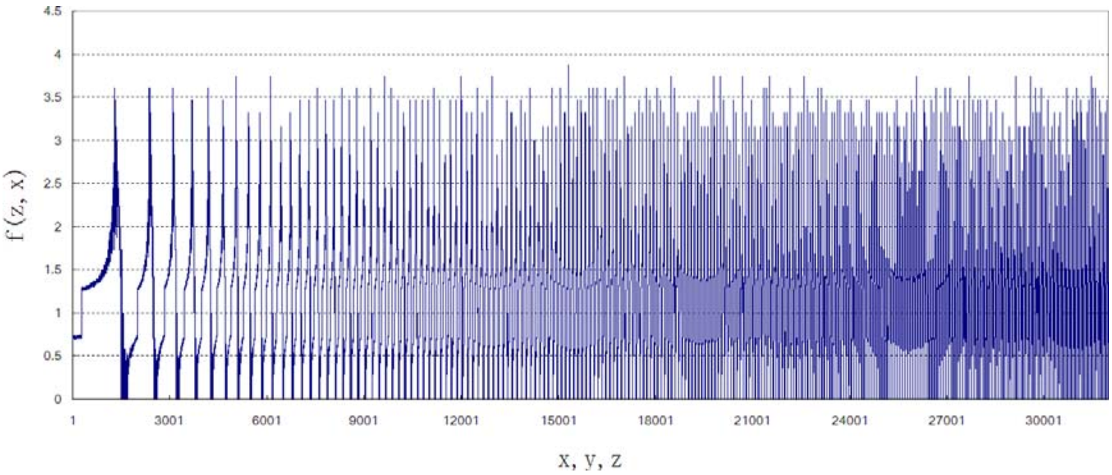


**Figure 2-4** *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3$

With the increasing of $z, n$ the value of $f(z,x)=\dfrac{BD}{AC}$ will be smaller, and we are sure of when $z, n \to \infty$ or get larger, the conclusion holds. In fact even $z=10^6$, we can still have a result of $f(z,x)=\dfrac{BD}{AC}<40$.

# 3. Conclusion

In this paper we first prove there are no positive integer solutions for equation (1-1) when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} \le 1$, and then prove with the increasing of *x* the conclusion still holds when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$ under the assumption of $z < 10^6$ there are no positive integer solutions for equation (1-1).