FORMULATION OF DIRAC EQUATION FOR AN ARBITRARY FIELD FROM A SYSTEM OF LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In our previous works we formulated Dirac equation for a free quantum particle and Maxwell field equations for the electromagnetic field from a system of linear first order partial differential equations. In this work we show that it is possible to formulate Dirac equation for the case when the quantum particle is under the influence of an external field, such as the electromagnetic field, also from a system of linear first order partial differential equations.

In our previous works on Dirac equation for a free quantum particle and Maxwell field equations for the electromagnetic field we showed that both forms of the Dirac quantum field and the Maxwell classical field could be formulated from a general system of linear first order partial differential equations by imposing a wave condition that must be satisfied by all components of a wavefunction [1,2]. The result is significant because it not only indicates the profound wave-particle property but also suggests that matter wave, which is described by Dirac equation [3], may have similar physical structure as that of the electromagnetic field, which is described by Maxwell field equations, in the sense that matter wave may also be the result of a coupling of two different physical fields. Now, since a charged particle interacts with an electromagnetic field, therefore in order to be able to describe the interacting system of a charged particle in an external field we need to extend our formulation of Dirac equation for an arbitrary field from a system of linear first order partial differential equations. When we formulated Dirac equation for a free particle we considered a general system of linear first order partial differential equations of the following forms

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial\psi_i}{\partial x_j}=\sum_{i=1}^{n} b_i^r\,\psi_i+c^r,\qquad r=1,2,\dots,n \tag{1}$$

$$\left(\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}\right)\psi=-im\sigma\psi+J \tag{2}$$

where $\psi=(\psi_1,\psi_2,\dots,\psi_n)^T$, $\partial\psi/\partial x_i=(\partial\psi_1/\partial x_i\,,\partial\psi_2/\partial x_i\,,\dots,\partial\psi_n/\partial x_i)^T$ with A_i, σ and J are matrices representing the undetermined quantities a_{ij}^r, b_{ij}^r and c^r. In order for the above systems of partial differential equations to be used to describe physical phenomena, the matrices A_i must be determined. If we apply the operator $\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}$ on the left on both sides of Equation (2) then we obtain

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)(-im\sigma\psi + J) \tag{3}$$

If we assume further that the coefficients a_{ij}^k, b_i^r and c^r are constants then Equation (3) can be shown to take the form

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = -m^2\sigma^2\psi - im\sigma J + \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{4}$$

If the matrices A_i satisfy the following conditions

$$A_i^2 = \pm 1 \tag{5}$$

$$A_iA_j + A_jA_i = 0 \quad \text{for} \quad i \neq j \tag{6}$$

then Equation (4) reduces to the following equation

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = -m^2\sigma^2\psi - im\sigma J + \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{7}$$

In particular, for a free particle in which $\sigma = 1$ and $J = 0$, Equation (2) reduces to Dirac equation

$$\gamma^\mu \partial_\mu \psi = -im\psi \tag{8}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and the γ^μ operators are given as

$$\gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{9}$$

In the following we will extend our formulation of Dirac equation from a system of linear first order partial differential equations for the case in which a particle is under the influence of an external field, such as the electromagnetic field. It is observed from the general equation given in Equation (2) that since the physical properties of a physical system are determined by the quantities m and J, which represent the mass and current density of the particle, respectively, it is reasonable to suggest that if the particle is under the influence of an external field, which can be expressed in the form of an energy, then a possible change that is needed for the description of the combined system is to change the mathematical forms that are related to the quantities m and J by introducing new physical objects such as charge and potential. Furthermore, since we showed in our previous works that the potential of a physical system can be identified with the geometric Ricci scalar, the quantities m and J could be regarded as physical manifestations of some mathematical objects [4]. With this observation we propose the following extended form for a system of linear first order partial differential equations that is used to derive Dirac equation with an external field

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial \psi_i}{\partial x_j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}^{r} V_j + c_i^{r}\right)\psi_i + d^{r}, \qquad r = 1, 2, \dots, n \tag{10}$$

The system of equations given in Equation (10) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\psi = -i\left(\sum_{i=1}^{n} qB_iV_i + m\sigma\right)\psi + J \tag{11}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i, \partial\psi_2/\partial x_i, \dots, \partial\psi_n/\partial x_i)^T$ with A_i, B_i, σ and J are matrices representing the quantities a_{ij}^r, b_{ij}^r, c_j^r and d^r, which are assumed to be constant in this work. While the quantities q, m and J represent physical entities related directly to the physical properties of the particle, the quantities V_i represent an external field, such as the potentials of an electromagnetic field. As before, if we apply the operator $\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}$ on the left on both sides of Equation (11) then we obtain

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(-i\left(\sum_{j=1}^{n} qB_jV_j + m\sigma\right)\psi + J\right) \tag{12}$$

Since the quantities A_i, B_i, σ, q, m and J are assumed to be constant, Equation (12) becomes

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi$$

$$= \left(-i\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} qB_jV_j + m\sigma\right)\right)\psi$$

$$- i\left(\sum_{i=1}^{n} qB_iV_i + m\sigma\right)\left(\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi\right) + \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i}$$

$$= -i\left(\sum_{i=1}^{n}\sum_{j=1}^{n} qA_iB_j \frac{\partial V_j}{\partial x_i}\right)\psi$$

$$- \left(\sum_{i=1}^{n}\sum_{j>i}^{n} q^2\left(B_iB_j + B_jB_i\right)V_iV_j - 2i\sum_{i=1}^{n} qmB_iV_i\,\sigma - m^2\sigma^2\right)\psi$$

$$- i\left(\sum_{i=1}^{n} qB_iV_i + m\sigma\right)J + \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \qquad (13)$$

Dirac equation for an arbitrary field can be formulated from the system of linear first order partial differential equations given in Equation (11) by setting $B_i = A_i = \gamma_i$, $\sigma = 1$, $J = 0$ and $A_iA_j + A_jA_i = 0$. In this case, in terms of the operators γ^μ, Equation (11) becomes

$$\left(\sum_{i=1}^{4} \gamma_i \frac{\partial}{\partial x_i}\right)\psi = -i\left(\sum_{i=1}^{4} q\gamma_iV_i + m\right)\psi \qquad (14)$$

Equation (14) can be written in a covariant form as Dirac equation for an arbitrary field as

$$\left(\gamma^\mu\left(i\partial_\mu - qV_\mu\right) - m\right)\psi = 0 \qquad (15)$$

Equation (13) also reduces to the following equation

$$\left(\sum_{i=1}^{4} {\gamma_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = \left(-i\sum_{i=1}^{4}\sum_{j>i}^{4} q\gamma_i\gamma_j\left(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\right) + 2i\sum_{i=1}^{4} qm\gamma_iV_i - m^2\right)\psi \qquad (16)$$

If the quantities V_i are the four-potential of an electromagnetic field given by the identification $(V_1, V_2, V_3, V_4) = \left(V, A_x, A_y, A_z\right)$ then Equation (16) can be used to determine the dynamics of the components of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_3)^T$, where the term $\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}$ are the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ which are defined in terms of the potentials in the following vector forms

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = -\left(\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right)i - \left(\frac{\partial V}{\partial y} - \frac{\partial A_y}{\partial t}\right)j - \left(\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t}\right)k \qquad (17)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) k \tag{18}$$

References

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