Analysis of Riemann's hypothesis

Let $\lambda(t) = 16e^{-\pi e^t} - 128e^{-2\pi e^t} + 704e^{-3\pi e^t}$... be the modular lambda function, and $q = e^{-\pi e^t}$, so

$$\lambda/q = 16 - 128q + 704q^2....$$

It is real valued if t is real

For each $0 < c \leq 1/2$ and r real, let $p(c, r)$ be the positive function

$$p(c, r, v) = e^{(c-1)(r+2v)} log(\frac{\lambda(r+v)}{q(r+v)}) log(\frac{\lambda(v)}{q(v)}),$$

let $b(c, r, v)$ be the symmetrization with respect to r, so we can restrict r to range over the positive reals

$$b(c, r, v) = p(c, r, v) + p(c, -r, v).$$

and let

$$f(c, r) = \int_{-\infty}^{\infty} b(c, r, v) dv,$$

and

$$g(c, r) = \frac{\partial}{\partial c} f(c, r)$$

$$= \int_{-\infty}^{\infty} (r + 2v) p(c, r, v) + (r - 2v) p(c, -r, v) dv.$$

1. Theorem. Let c be a real number such that $0 < c < 1/2$. If

$$\begin{array}{cc} f(c, r) > 0, & \frac{\partial}{\partial c} f(c, r) < 0 \\ \frac{\partial}{\partial r} f(c, r) < 0, & r \frac{\partial^2}{\partial c \partial r} f(c, r) < 0 \end{array}$$

for all $r \geq 0$ then $\zeta(c + i\omega) \neq 0$ for all ω.

Next, for ω real let

$$q(c,\omega) = ((c-1)^2+\omega^2)\int_0^\infty cos(\omega r)g(c,r)dr + (2c-2)\int_0^\infty cos(\omega r)f(c,r)dr.$$

We are mainly interested in the case when $c = 1/2$ and then

2. Conjecture

i) $q(1/2,\omega) \leq 0$ for all ω. Moreover

ii) $q(1/2,\omega) = 0 \Leftrightarrow \zeta(1/2 + i\omega) = 0$.

3. Theorem. The truth of Conjecture 1. would imply that the particular open set $U = \{c + i\omega : 0 < c < 1/2 \text{ and } q(x,\omega) < 0 \text{ for all } x \text{ such that } c \leq x \leq 1/2\}$ contains all the points of real part $< 1/2$ of some neighbourhood of every point of real part $1/2$ which is not a zero of zeta.

4. Conjecture. $q(c,\omega) \leq 0$ for $0 < c \leq 1/2$ and all real ω.

5. Theorem. The truth of Conjecture 3. would imply Riemann's hypothesis.

6. Remark. Conjecture 4 is close to just a reformulation of Riemann's question. We will relate $q(c,\omega)$ to the rate of change of the squared mangitude of an analytic function with respect to the real part of its argument. On general principles this is just the real part of the logarithmic derivative, times the non-negative function which is twice the squared magnitude of the function value itself. When the function is instead taken to be the xi function, such an equivalence as Conjecture 3 can be deduced from Hadamard's factorization theorem. Regarding Theorem 2, the consequence that there exists *some* such open set containing no zeroes of the zeta function while containing a the points of real part $< 1/2$ in a neighbourhood of each point on the line $Re(s) = 1/2$ which is not a zero, is again just a tautology. Such an open set can be obtained just by deleting the zeroes which are not on the line $Re(s) = 1/2$. Also, Conjecture 1 i) concerning the special value $c = 1/2$ has the analogue for the xi function, that the function analagous to $q(1/2,\omega)$ is non-positive for the reason that it is identically zero. Conjecture 1 ii) has no obvious analogue for the xi function.

7. Lemma. Let $L(s)$ be that L function which counts the number of ways of expressing a natural number as a sum of four squares of integers

$$L(s) = 8\zeta(s)\zeta(s-1)(1-4^{1-s})$$

Let χ be the sign character, so

$$L(s,\chi) = \frac{4-2^s}{2+2^s}L(s) = -8\zeta(s)\zeta(s-1)4^{-s}(4-2^s)(2-2^s).$$

Then for $s = c + i\omega$ with $0 \le c \le 1/2$ we have

$$q(c,\omega) = \frac{d}{dc}|L(s,\chi)\frac{\Gamma(s)}{\pi^{s-1}}\mid^2.$$

Proof of Lemma 7. From elementary properties of λ we have

$$-L(s,\chi)\Gamma(s)\pi^{1-s} = \int_0^{log(16)} e^{(s-1)t} d\ log\ \lambda/q$$

The limits of integration refer to values of $log(\lambda/q)$. If we refer to values of t the integral is taken from $-\infty$ to ∞.

Then

$$\frac{d}{dc}|L(s,\chi)\frac{\Gamma(s)}{\pi^{s-1}}\ |\ ^2=\frac{d}{dc}\ |\ \int_0^{log(16)} e^{i\omega t}e^{(c-1)t}d\ log\lambda/q.\ |^2$$

$$=\frac{d}{dc}\int_0^{log(16)} e^{i\omega u}e^{(c-1)u}d\ log\ \frac{\lambda(u)}{q(u)}\int_0^{log(16)} e^{-i\omega v}e^{(c-1)v}d\ log\ \frac{\lambda(v)}{q(v)}$$

$$=\int\int(u+v)cos((u-v)\omega)e^{(c-1)(u+v)}d\ log\ \frac{\lambda(u)}{q(u)}d\ log\ \frac{\lambda(v)}{q(v)}.$$

This step used that real part commutes with integration. Using the elementary transformation

$$r=u-v$$

$$v=v$$

this is, with the same integration limits,

$$\int\int(r+2v)cos(\omega r)e^{(c-1)(r+2v)}d\ log\ \frac{\lambda(r+v)}{q(r+v)}\ d\ log\ \frac{\lambda(v)}{q(v)}.$$

$$=\int\int cos(\omega r)\alpha$$

where α is the two-form

$$(r+2v)e^{(c-1)(r+2v)}d\ log\ (\frac{\lambda(r+v)}{q(r+v)})d\ log\frac{\lambda(v)}{q(v)}$$

If we go back and use integration by parts in both factors

$$\int e^{i\omega u}e^{(c-1)u}d\ log\ \frac{\lambda(u)}{q(u)}=\int e^{i\omega u}e^{(c-1)u}(1-c-i\omega)log(\frac{\lambda}{q})du$$

$$\int e^{-i\omega v}e^{(c-1)u}d\ log\ \frac{\lambda(v)}{q(v)}=\int e^{-i\omega v}e^{(c-1)v}(1-c+i\omega)log(\frac{\lambda}{q})dv$$

then we deduce that our desired quantity is

$$q(c,\omega)=((c-1)^2+\omega^2)\int_0^\infty cos(\omega r)g(c,r)dr+(2c-2)\int_0^\infty cos(\omega r)f(c,r)$$

where $f(c,r)$ and $g(c,r)$ are is as we have defined them.

Proof of Theorem 5.: The leftmost zeroes of $L(s,\chi)$ with positive real part are the nontrivial zeroes of Riemann's function ζ, note there is also a pole at 0. The conjecture 4 asserts that $\mid L(c+i\omega)\mid^2$ as a function of c, for each fixed $\omega > 10$, is a real analytic, non-increasing function from $(0,1/2]$ to $[0,\infty)$ which is not identically zero. Such a function can be zero only for $c = 1/2$.

Proof of Theorem 3. By definition of U and the lemma, the mangitude of $L(s,\chi)\Gamma(s)\pi^{1-s}$ is non-increasing in the real direction on U. Since the square of the magnitude is real analytic, it can only be strictly decreasing along each arc within U of constant imaginary value. The infimum value of each arc in U is attained at the rightmost limit point of the arc. For a value of ω such that $\zeta(1/2+i\omega) = 0$, Conjecture 2 asserts that also $q(1/2,\omega) < 0$. The point $1/2+i\omega$ is then the limit point of such an arc in U, by continuity of q.

8. Remark. If we think consider an infinite electrical circuit made of resistors, inductors, capacitors and linear amplifiers, whose impulse response for $t \geq 0$ is $f(1/2,t) + f(1/2,-t)$ then feeding an input signal $cos(\omega t)$ starting at time $t = 0$ and waiting until the output stabilizes to a sinosoidal type of wave, Conjecture is that the limiting phase of the output function belongs to the half of possibile phases which would tend to cancel the input if input and output were mixed together.

9. Remark. Conjecture 1 is single question about the particular real-valued function $f(1/2,\omega)$. It does seem perhaps to be a transcendental type of question, rather than one needing to return to the domain of number-theory. It may be related to the invariance of $\lambda(t)$ under $\tau \mapsto \tau + 2$ for $\tau = ie^t$ and the fact that $\lambda(-1/\tau) = 1 - \lambda(\tau)$.

Proof of Theorem 1. For such a value of c the function $f(c,r)$ as a function of $r > 0$ has positive values and its the first derivative is negative. Therefore all the values its Fourier transform have positive real part. Likewise $g(c,r)$ has negative values and its first derivative is positive when $r > 0$ so the values of its Fourier transform have negative real part. The coefficient $((c-1)^2 + w^2)$ in the definition of q is always positive while the coefficient $(2c-2)$ is always negative, thus under these conditions $q(c,w)$ is strictly negative for all w. Then we may argue as in the proof of Theorem 5, only regarding whatever particular value of c we have chosen.

Finally, let'a make Theorem 1 more explicit. Our positive function before symmetration is

$$p(c,r,v) = e^{(c-1)(2c+r)} log(\frac{\lambda}{q}(v)) log(\frac{\lambda}{q}(r+v)).$$

If we apply $1, \frac{\partial}{\partial c}, \frac{\partial}{\partial r}, \frac{\partial^2}{\partial c \partial r}$, the respective multipliers (that is, logarithmic partial derivatives) are

$$1$$

$$(r+2v)$$

$$c - 1 + \frac{\pi e^{v+r} - \frac{4}{\pi} e^{v+r} K(\lambda(v+r))^2 (1 - \lambda(v+r))}{log(\frac{\lambda}{q}(v+r)}$$

$$1 + (r+2v)(c-1) + (r+2v) \frac{\pi e^{v+r} - \frac{4}{\pi} e^{v+r} K(\lambda(v+r))^2 (1 - \lambda(v+r))}{log(\frac{\lambda}{q}(v+r))}.$$

where $K(m)$ is the complete elliptic integral of the first type of modulus m. Thus, we may restate Theorem 1 in the equivalent form

10. Theorem. Let c be real with $0 < c < 1/2$. Suppose for all $r \geq 0$

$$\int_{-\infty}^{\infty} log(\frac{\lambda}{q}(v))(e^{(c-1)(2v+r)}log(\frac{\lambda}{q}(v+r))+e^{(c-1)(2v-r)}log(\frac{\lambda}{q}(v-r)))dv > 0,$$

$$\int_{-\infty}^{\infty} log(\frac{\lambda}{q}(v))(e^{(c-1)(2v+r)}log(\frac{\lambda}{q}(v+r))(2v+r)+e^{(c-1)(2v-r)}log(\frac{\lambda}{q}(v-r))(2v-r))dv < 0,$$

$$\int_{-\infty}^{\infty} log(\frac{\lambda}{q}(v))(e^{(c-1)(2v+r)}((c-1)log(\frac{\lambda}{q}(v+r))+\pi e^{v+r}-\frac{4}{\pi}e^{v+r}K(\lambda(v+r))^2(1-\lambda(v+r)))$$
$$+e^{(c-1)(2v-r)}((1-c)log(\frac{\lambda}{q}(v-r))-\pi e^{v-r}+\frac{4}{\pi}e^{v-r}K(\lambda(v-r))^2(1-\lambda(v-r))))dv < 0,$$

and

$$\int_{-\infty}^{\infty} log(\frac{\lambda}{q}(v))(e^{(c-1)(2v+r)}$$
$$((1+(2v+r)(c-1))log(\frac{\lambda}{q}(v+r))+(2v+r)(\pi e^{v+r}-\frac{4}{\pi}e^{v+r}K(\lambda(v+r))^2(1-\lambda(v+r))$$
$$+e^{(c-1)(2v-r)}$$
$$((-1+(2v-r)(1-c))log(\frac{\lambda}{q}(v-r))+(2v-r)(-\pi e^{v-r}+\frac{4}{\pi}e^{v-r}K(\lambda(v-r))^2(1-\lambda(v-r))$$
$$)dv > 0.$$

Then $\zeta(c + i\omega) \neq 0$ for all ω.

References

1. Marcus Du Sautoy, The music of the primes: why an unsolved problem in Mathematics matters, 2004

2. Keith Ball, Rational approximations to the zeta function, June, 2017

3. Robert Mackay, email, 19 October, 2017

4. Robert Mackay, email, 20 October, 2017

5. Robert Mackay, email, 22 October, 2017

6. Robert Mackay, email, 14 December, 2017

7. Nayo Reid, conversation, Euston Place, Leamington, 23 January, 2018

8. Robert Mackay, email, 28 February, 2018

9. Robert Mackay, email, 20 March, 2018

26 March, 2018