GOLDBACH'S CONJECTURE PROOF.

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Abstract :In this paper we are going to give the proof of Goldbach conjecture by introducing a new lemma which implies Goldbach conjecture .By using Chebotarev-Artin theorem , Mertens formula and Poincare sieve we establish the lemma .

1 Introduction

The Goldbach conjecture was introduced in 1742 and has never been proven though it has been verified by computers for all numbers up to 19 digits.
It states that all, even numbers above two are the sum of two prime numbers. All studies on Goldbach conjecture have failed.So we are going to give a complete proof of Goldbach conjecture.

1.1 Principle of the Demonstration

Let n an even integer such as above 20 and denote by $\mathbb{C}_n$ the set of the composite integers of $[1, n-1]$ to what we add 1 and let f_n be the bijective mapping such that : $\begin{array}{ccc} f_n : \mathbb{C}_n & \mapsto & n - \mathbb{C}_n \\ m & \mapsto & n - m \end{array}$
Denote by G_n the subsect of $n - \mathbb{C}_n$ consisting of prime numbers and G'_n that of composite numbers we have $n - \mathbb{C}_n = G_n \cup G'_n$ .Let $\mathcal{P}_n$ the set of prime numbers less than or equal to n . Let

$$\delta(n) = card(G_n), \alpha(n) = card(\mathcal{P}_n \backslash G_n), \Pi(n) = card(\mathcal{P}_n)$$

then $\Pi(n) = \delta(n) + \alpha(n)$,obviously $\alpha(n)$ represents the number of ways to write n as the sum of two primes

1.2 Lemma 1

$\forall n \in 2\mathbb{N}^*$, we have $\mathcal{P}_n \backslash G_n \neq \emptyset$
As we said we are going to give later the proof the lemma 1 .Without loss of generality ,suppose that the lemma 1 is true then we have :

1.3 Lemma 2

$\forall p \in \mathcal{P}_n \backslash G_n$,we have $n - p \in \mathcal{P}_n$

1.4 Proof of lemma 2

Let n be an even integer above 20 , and suppose that n-p is not prime, then

$$n - p \in \mathbb{C}_n$$

, as

$$p = n - (n - p)$$

hence

$$p \in G_n$$

.The lemma is thus proven .
Observe that each integer $m \in \mathbb{C}_n$ such that $m \geq 4$ has at least one prime divisor $p \leq \sqrt{n}$.
Let $\mathcal{P}_{\leq\sqrt{n}} = \{p_1, p_2, ..., p_r\}$ where $p_1 = 2$,$p_2 = 3$,...$p_r = \max(\mathcal{P}_{\leq\sqrt{n}})$.
Moreover, remembering that

$$\mathbb{C}_n = \bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}}, p\geq 2} A_{2p} \cup \{1\}$$

where

$$A_{2p} = \{2p, 3p, 4p, ...(\lfloor\frac{n-1}{p}\rfloor)p\}$$

. We notice that A_{2p} is an arithmetic sequence of first term 2p and reason p .
So

$$n - \mathbb{C}_n = f_n(\mathbb{C}_n) = \bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}}, p\geq 2} f_n(A_{2p}) \cup \{n-1\}$$

As

$$f_n(A_{2p}) = \{n-2p, n-3p, n-4p, ...n-\lfloor\frac{n-1}{p}\rfloor p\} = \{n-\lfloor\frac{n-1}{p}\rfloor p, n-(\lfloor\frac{n-1}{p}\rfloor-1)p......, n-3p, n-2p\}$$

Then $f_n(A_{2p})$ is an arithmetic sequence of first term $n - \lfloor\frac{n-1}{p}\rfloor p$ and reason p .
We will evaluate the quantity of prime numbers in $\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}}, p\geq 2} f_n(A_{2p})$
by applying the principle -exclusion of Moivre and Chébotarev -Artin theorem in each $f_n(A_{2p})$ in the case where $p \nmid n$

2 Chebotarev-Artin 's Theorem

Let $a, b > 0$ such that $\gcd(a, b) = 1$,$\Pi(X, a, b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that $\Pi(X, a, b) = \frac{L_i(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$
The prime number theorem states that $\Pi(X) = L_i(X) + \bigcirc(\frac{X}{\ln^2 X})$ so
$\Pi(X, a, b) = \frac{\Pi(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$

3 corollary

Let $a, b > 0$ such that $\gcd(a, b) = 1$,$\Pi(X, a, b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that

$$\frac{\Pi(X, a, b)}{\Pi(X)} = \frac{1}{\phi(b)} + \bigcirc(c \ln Xe^{-\sqrt{\ln X}})$$

.
From probabilistic point of view, the probability of prime numbers less than or equal to X in an arithmetic progression of reason b and of the first term has such that $\gcd(a, b) = 1$ is worth

$\frac{1}{\phi(b)} + \bigcirc(c \ln X e^{-\sqrt{\ln X}})$ for X large enough .In the following we will justify the application of Chebotein-Artin's theorem for sets $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^{k} f_n(A_{2p_{i_j}})$
for $1 \leq i_1 < i_2 < ..... < i_k$

3.1 Remarks

It is obvious to note that for $k > 2$, $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{n}}}^{k} A_{2p_{i_j}}$ is the set of multiples of $\prod_{j=1}^{k} p_{i_j}$ which allows us to write

$$\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{X}}}^{k} f_n(A_{2p_{i_j}}) = \{n - m \prod_{j=2}^{k} p_{i_j} | 1 \leq m \leq \lfloor \frac{n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor \}$$

This set is an arithmetic sequence of reason $\prod_{j=2}^{k} p_{i_j}$ and first term $n - \lfloor \frac{n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor \prod_{j=2}^{k} p_{i_j}$.
The hypothesis of application of Chebotarev-Artin's theorem will be justified if and only if $\gcd(2\prod_{j=2}^{k} p_{i_j}, \prod_{j=2}^{k} p_{i_j} + n) = 1$ which is the case if $\prod_{j=2}^{k} p_{i_j} \nmid n$

4 Demonstration of Goldbach 's conjecture

4.1 Theorem

Let n an even integer be arbitrarily large ,

$$\alpha(n) = card(\mathcal{P}_n \backslash G_n)$$

the numbers of way to write n in sum of two prime numbers ,

$$\beta_n = \prod_{p=3}^{\sqrt{n}} \frac{p(p-2)}{(p-1)^2} \prod_{p=3, p|n}^{\sqrt{n}} \frac{p-1}{p-2}$$

$\exists n_0$ such that $\forall n \geq n_0$

$$\alpha(n) \geq \frac{2\beta_n \Pi(n)}{\ln n}$$

4.2 Useful Lemma

Let $a_1, a_2, ...a_r$ be r numbers then

$$1 - \sum_{i=1}^{r} \frac{1}{a_i} + \sum_{1 \leq i < j \leq r} \frac{1}{a_i a_j} + ... + \frac{(-1)^r}{a_1 a_2 ....a_r} = \prod_{i=1}^{r} \frac{a_i - 1}{a_i}$$

4.3 Proof

Let us consider the polynomial :$P(X) = \prod_{i=1}^{r}(X - \frac{1}{a_i})$ from the coefficient-root relations

$$P(X) = X^r + \sum_{k=1}^{r} \sum_{1 \leq i_1 < i_2 < .... < i_k \leq r} \frac{(-1)^k X^{r-k}}{\prod_{j=1}^{k} a_{i_j}}$$

taking $X = 1$, the lemma is thus proved.

4.4 Proof of Theorem

Let us define ϱ as the function which represents the proportion of prime numbers which appear in a given set over prime numbers less than n . we also define $\psi_{n-1} = 1, 0$ according to n-1 is prime or not With regard to the principle of inclusion -exclusion of Moivre we can write :

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}},p\geq 3,p\nmid n} f_n(A_{2p})) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} \varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{n}},p_{i_j}\nmid n}^{k} f_n(A_{2p_{i_j}}))$$

.Moreover we have

$$\varrho(n-\mathbb{C}_n\backslash n-1) = \varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}},p\geq 3,p\nmid n} f_n(A_{2p})) = \frac{\delta(n)-\psi_{n-1}}{\Pi(n)}$$

. According to Chebotarev's theorem -Artin more precisely the corollary we have : $\forall k \geq 2$

$$\varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{n}},p_{i_j}\nmid n}^{k} f_n(A_{2p_{i_j}})) = \frac{1}{\phi(\prod_{j=2}^{k} p_{i_j})} + h(n)$$

$\forall i \geq 2$

$$\varrho(f_n(A_{2p_i,p_i\nmid n})) = \frac{1}{\phi(p_i)} - \frac{\psi_{n-p_i}}{\Pi(n)} + h(n)$$

, where h(n) represents the error of our estimation Regarding the corollary we $h(n) = \bigcirc(c\ln(n)e^{-\sqrt{\ln(n)}})$ Thus

$$\frac{\delta(n)-\psi_{n-1}}{\Pi(n)} = g(n) - \sum_{k=2}^{r}\frac{\psi_{n-p_k}}{\Pi(n)} + \sum_{k=2}^{r}\sum_{2\leq i_2<i_3<...<i_k\leq r}\frac{(-1)^k}{\prod_{j=2}^{k}(p_{i_j}-1), p_{i_j}\nmid n}$$

where

$$g(n) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} h(n)$$

represents the error of the proportion estimation of prime in $\bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}},p\geq 3,p\nmid n} f_n(A_{2p})$ .Noting that

$$\sum_{k=2}^{r}\psi_{n-p_k} = \sum_{n-p\in\mathcal{P}_n,p\leq p_r} 1 = \sum_{p\in\mathcal{P}_n\backslash G_n,p\leq p_r} 1 = \alpha(p_r)$$

and applying the useful lemma, we have :

$$\frac{\delta(n)-\psi_{n-1}}{\Pi(n)} = g(n) - \frac{\alpha(p_r)}{\Pi(n)} + (1 - \prod_{i=2,p_i\nmid n}^{r}\frac{p_i-2}{p_i-1})$$

As $\delta(n) = \Pi(n) - \alpha(n)$ and $r = \max(i|p_i \leq \sqrt{n})$ so

$$\frac{\alpha(n)-\alpha(\sqrt{n})}{\Pi(n)} = -g(n) + \prod_{p=3,p\nmid n}^{\sqrt{n}}\frac{p-2}{p-1} - \frac{\psi_{n-1}}{\Pi(n)}$$

. The veritable problem of our result is bounded on the error function g . How can we solve it ? . The answer is so simple by noticing that

$$|\frac{g(n)}{h(n)}| = |\sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} 1| = |\sum_{k=2}^{r}(-1)^k\binom{r-1}{k-1}| = |-\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}| = 1$$

Using the previous result our formula becomes :

$$\alpha(n)-\alpha(\sqrt{n}) \sim_{+\infty} \Pi(n)\prod_{p=3,p\nmid n}^{\sqrt{n}}\frac{p-2}{p-1}-\psi_{n-1}$$

In the following we will apply the Mertens' theorem in order to evaluate $c_n=\prod_{p=3,p\nmid n}^{\sqrt{n}}\frac{p-2}{p-1}$. As

$$\prod_{p=3}^{\sqrt{n}}\frac{p-2}{p-1}=\prod_{p=3,p\nmid n}^{\sqrt{n}}\frac{p-2}{p-1}\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-2}{p-1}$$

so we have

$$c_n=\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-2}{p-1}\prod_{p=3}^{\sqrt{n}}\frac{p-1}{p-2}$$

By using the third formula of Mertens we have :

$$\prod_{p\leq\sqrt{n}}(1-\frac{1}{p})=\frac{2e^{-\gamma}}{\ln n}(1+\bigcirc(\frac{1}{\ln n}))$$

Let's put

$$c_2(n)=\prod_{p=3}^{\sqrt{n}}\frac{p(p-2)}{(p-1)^2}=\prod_{p=3}^{\sqrt{n}}\frac{p}{p-1}\prod_{p=3}^{\sqrt{n}}\frac{p-2}{p-1}$$

so

$$c_n=2c_2(n)\prod_{p=2}^{\sqrt{n}}(1-\frac{1}{p})\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-1}{p-2}$$

From the previous part

$$c_n=\frac{4c_2(n)e^{-\gamma}}{\ln n}(1+\bigcirc(\frac{1}{\ln n}))\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-1}{p-2}$$

$$\alpha(n)-\alpha(\sqrt{n}) \sim_{+\infty} \Pi(n)[\frac{4c_2(n)e^{-\gamma}}{\ln n}\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-1}{p-2}]$$

Let

$$\beta_n=c_2(n)\prod_{p=3,p|n}^{\sqrt{n}}\frac{p-1}{p-2}$$

then $\exists n_0\ \forall n\geq n_0$

$$\alpha(n)\geq\alpha(n)-\alpha(\sqrt{n})\geq\frac{2\beta_n\Pi(n)}{\ln n}$$

4.5 proof of lemma 1

Let suppose that $\exists q$ such that $\mathcal{P}_q\backslash G_q=\emptyset$ then $\alpha(q)=card(\mathcal{P}_q\backslash G_q)=0$ .According to the theorem necessarily we have $q\leq n_0$ and we also have

$$\frac{\alpha(q)-\alpha(\sqrt{q})}{\Pi(q)}=-g(q)+\prod_{p=3,p\nmid q}^{\sqrt{q}}\frac{p-2}{p-1}-\frac{\psi_{q-1}}{\Pi(q)}$$

then

$$-g(q) + \prod_{p=3,p\nmid q}^{\sqrt{q}} \frac{p-2}{p-1} - \frac{\psi_{q-1}}{\Pi(q)} = 0$$

more precisely $\prod_{p=3,p\nmid q}^{\sqrt{q}} \frac{p-2}{p-1} = g(q) + \frac{\psi_{q-1}}{\Pi(q)}$ .Which leads us to :

$$\frac{4c_2(q)e^{-\gamma}}{\ln q}(1 + \bigcirc(\frac{1}{\ln q})) \prod_{p=3,p|q}^{\sqrt{q}} \frac{p-1}{p-2} \leq g(q) + \frac{1}{\Pi(q)}$$

Multiplying each member by $\ln(q)$ we have

$$4c_2(q)e^{-\gamma}(1 + \bigcirc(1)) \prod_{p=3,p|q}^{\sqrt{q}} \frac{p-1}{p-2} \leq \ln(q)g(q) + \frac{\ln(q)}{\Pi(q)}$$

.As $\ln(q)g(q) + \frac{\ln(q)}{\Pi(q)} = \bigcirc(c\ln^2(q)e^{-\sqrt{\ln(q)}})$ hence our inequality does not hold . Therefore the lemma 1 is true . The main result is that for any even given integer n the pairwise of Goldbach prime is $(p, n-p)$ where $p \in \mathcal{P}_n \backslash G_n$

4.6 Acknowledgments

The author wish to express their appreciation and sincere thanks to Professor Tanoé François(Université Félix-Houphouet Boigny Ufr de Maths-info) and Professor Pascal Adjamogbo (Université Paris 6) for their encouragements

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