NERVOUS OSCILLATIONS ON A TWISTED CYLINDER

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The Borsuk-Ulam theorem is a general principle able to describe a large amount of brain functions. However, when assessing the neurodata extracted from EEG and fMRI, the BUT-related methods, based on projections and mappings among different functional brain dimensions, are impractical and computationally expensive. Here we show how the BUT's antipodal features with matching description (say, two far apart brain areas that are activated simultaneously and display the same value of entropy) can be described in terms of closed paths on a Möbius strip. This allows to evaluate the nervous system's dynamics in terms of trajectories taking place onto the well-established, easily manageable phase space of a twisted cylinder.

Keywords: Borsuk-Ulam theorem; topology; fMRI; phase space; brain

The Borsuk-Ulam theorem (BUT) has been proven suitable for the description of different nervous functions and activities. The BUT suggests that the neural properties endowed in the physical and biological spaces of the brain can be translated to abstract mathematical ones (Tozzi and Peters, 2016a). To make a few examples, Tozzi and Peters (2016b) assessed logistic maps of neural chaotic activities and were able to describe their nonlinear dynamics in linear terms. Furthermore, the BUT apparatus sheds new light on the puzzling phenomena of (spatial) fractals and (temporal) power laws (Tozzi et al., 2017a), that are ubiquitous during brain oscillations (Friston and Ao, 2012; Beggs and Timme, 2012). Tozzi and Peters (2016b) proposed that a brain symmetry stands for two BUT's features with matching description lying in higher dimensions, while a symmetry break for a single feature lying one dimension lower. These BUT symmetries have been correlated with neural thermodynamic activity and energy requirements/constraints during both spontaneous and evoked nervous function (Tozzi and Peters, 2017b). A BUT framework allows also to analyze how the brain perceives "sharp", non-fuzzy objects in order to tackle the problem of Kullback-Leibler perceptual divergence (Tozzi and Peters, 2016b). Furthermore, a symmetric, topological approach has been provided useful for the evaluation of the multisensory information integration occurring in the cortex during perceptive tasks (Tozzi and Peters, 2017a).

However, a problem arises. The BUT copes with projections and mappings among different functional dimensions, while brain dynamics are experimentally described not in such terms of mappings and projections, rather of paths and trajectories taking place in neural phase spaces. Furthermore, the techniques of algebraic topology that assess the BUT features are quite complex, difficult to approach and quantify. Therefore, a framework is required that allows the description of the BUT's matching features in terms of dynamics taking place in more manageable phase spaces. Here we show that the scenario described by the BUT can be transported to a peculiar phase space, i.e., a Möbius strip, in order that antipodal points can be tackled in terms of trajectories taking place on a rather simple abstract manifold. This means, for example, that the pairwise brain oscillations detected through the currently available neurotechniques (such as EEG and fMRI) can be assessed in terms of couples of matching particles moving along the one-side surface of a twisted cylinder.

MATERIALS AND METHODS

The Borsuk-Ulam theorem and its variants. The Borsuk-Ulam Theorem states that (Borsuk, 1933; Borsuk, 1969):

Every continuous map $f : S^n \to R^n$ *must identify a pair of antipodal points*.

Points on S^n are *antipodal*, provided they are diametrically opposite. An *n*-dimensional Euclidean vector space is denoted by R^n. In terms of brain activity, a feature vector $x \in R^n$ models the description of a brain signal.

In other words, the BUT suggests that a single point on a circumference maps to two points on a sphere. In more technical terms, a point embedded in lower dimensions gives rise to two points with matching description in higher ones, provided that the function under assessment is continuous (Dodson and Parker, 1997; Matoušek 2003). The original formulation of BUT displays versatile ingredients which can be modified, resulting in useful extensions of its rather simple sketch (Tozzi et al., 2017a). For example, antipodal points can be replaced by antipodal regions, or shapes, with matching description (Tozzi and Peters, 2016b). Further, instead of points, novel BUT variants allow the assessment (from one dimension to another) of trajectories, functions, vectors and tensors, particle trajectories in phase spaces, information and activities such as entropies (Tozzi and Peters, 2017c). Also, the two features do not need to be perfectly antipodal: the only requirement is that they must not share points in common and are fully separated in the higher-dimensional manifold (Tozzi et al., 2017a). BUT variants hold not just for concave structures, such as the circumferences and spheres described by the classical BUT, but also for flat and concave structures (Tozzi 2016), such as the rather intricate trajectories detected in several systems' dynamics (Sengupta et al., 2016). Furthermore, the dimensions described by BUT do not stand just for spatial dimensions (such as a circle and a sphere), but also for abstract dimensions (such as for example, fractal measurements and time intervals) (Tozzi and Peters, 2016b).

Möbius strip. The Möbius strip, also called the twisted cylinder, is a one-side surface that displays just one boundary (Möbius, 1858; Starostin and van der Heijden 2007; t'Hooft 2018), when embedded in three-dimensional Euclidean space. A Möbius strip can be built by taking a paper strip and giving it a half-twist, then joining the ends in order to form a loop. This means that a line that starts from the seam down the middle meets back at the seam, but at the other side. If continued, the line meets the starting point, in a point that is double the length of the original strip. This single continuous curve may be described either through a parameterized subset of a three-dimensional Euclidean space, or through cylindrical polar coordinates. Topologically, the Möbius strip can be defined as the square $[0, 1] \times [0, 1]$, with its top and bottom sides identified by the relation $(x, 0) \sim (1 - x, 1)$ for $0 \leq x \leq 1$.
Our aim is to achieve the transport of the BUT's antipodal points to the one-side surface of such twisted cylinder.

RESULTS

If we embed the trajectories of two BUT matching functions x and –x (**Figure A**) on a Möbius strip, we achieve a closed, continuous loop where the two functions are allowed to travel along constrained trajectories. It is easy to see that a piece of strip of a given length, standing for a time interval, may display both x and –x at the same time (**Figure B**). The BUT dictates are preserved because, even if the two matching features are simultaneous, they do not have points in common: indeed, they lie on the opposite surface of the same strip. In nervous dynamics' terms, this means that the oscillations' trajectories of two areas which activate together can be followed in subsequent times, even when their matching activation has disappeared (**Figure C**).

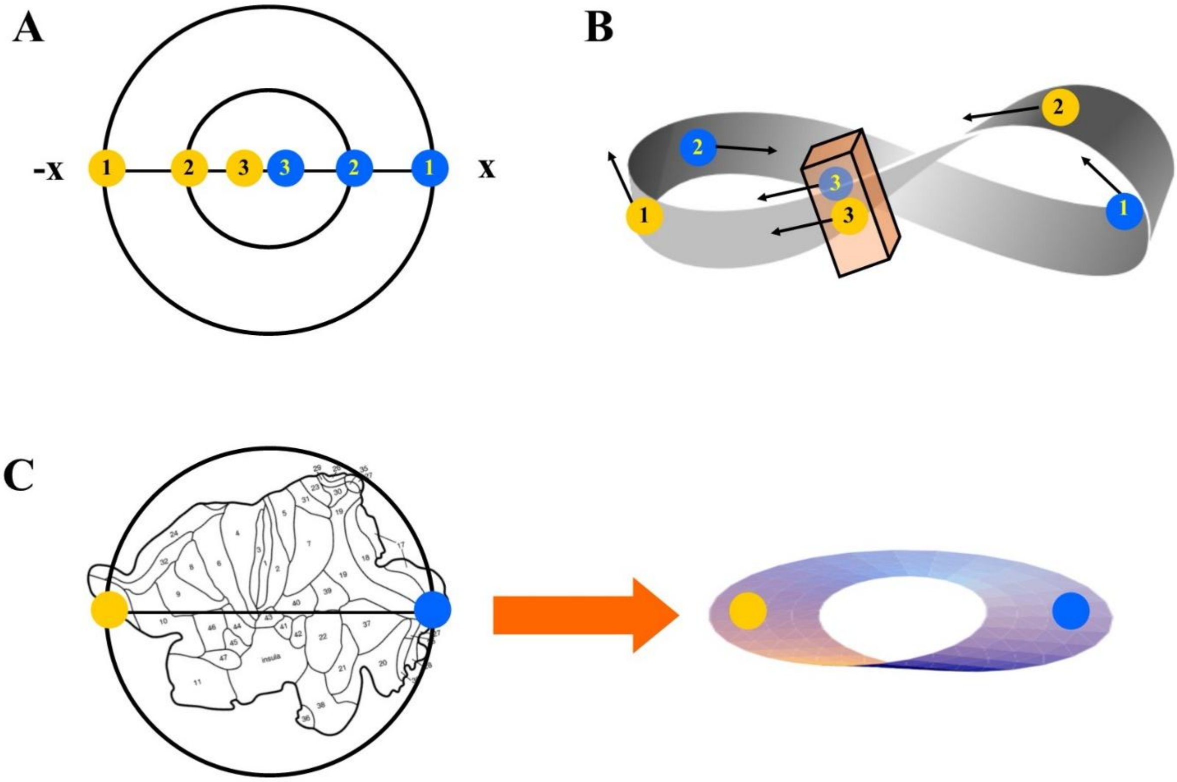

Figure. Transport of the BUT theorem on a Möbius strip. **Figure A**. Changing the radius of the hypersphere makes the antipodal points more or less close. Close to the center, the two points (marked with the number 3) are almost superimposed. **Figure B**. The movements of the antipodal points can be described in terms of trajectories on a Möbius strip. The parallelepiped stands for a slice of time, where both the antipodal features occur simultaneously. **Figure C**. A theoretical example from neuroscience is provided. A cerebral hemisphere is unfolded and flattened into a two-dimensional reconstruction (Van Essen, 2005) that can be embedded into a circular manifold. When two antipodal areas display simultaneously a feature in common, e.g., the same value of pairwise entropy (Watanabe et al. 2014), we achieve a topological description assessable in terms of BUT (**left side**). Such two areas and their subsequent dynamics can be easily visualized and assessed in terms of trajectories taking place on an abstract twisted cylinder (**right side**).

CONCLUSIONS

Results from far-flung scientific disciplines point towards the BUT as a universal principle able to describe and quantitatively assess otherwise elusive biological and physical activities. To make an example, the BUT dictates pave the way to the description of the brain activity as taking place on a multidimensional torus (Tozzi and Peters, 2016a). Approaching novel topological techniques of computational proximity, Peters et al. (2017a) detected a four-dimensional moving hypersphere, located insight the nervous connectome. Their claim has been strengthened by the recent finding that the Rényi informational entropy in primary sensory areas is lower than in associative ones (Peters et al., 2017b): this corroborates the BUT-framed prevision that the brain activity lies in higher dimensions than the three-dimensional (plus time) environment. In such a topological context, systems operations become projections among different levels, giving rise to apparently emergent properties in higher dimensions. Therefore, we are facing a framework based on mappings and projections among different activity levels. However, the three-dimensional data extracted from neural series achieved through, e.g., EEG and fMRI techniques (Van de Ville et al., 2010; Ezaki et al., 2017), are difficult to manage in terms of mapping and projections. Therefore, once established the validity on the BUT framework in the assessment of brain activities, we require a more affordable procedure in order to visualize and process the huge amount of available experimental data. Here we show how the features described by BUT, that occur on an orientable manifold with positive-curvature, can be described in terms of paths on a non-orientable manifold, i.e., a Möbius strip. Indeed, the Möbius strip has the mathematical property of being unorientable, and this allows to elucidate some brain puzzling functions. Indeed, the description of nervous trajectories, detected through experimental observation, on a twisted cylinder instead of on the classical three-dimensional-plus time phase space, allows to notice possible unexpected

correlations. The possibility to locate brain oscillations on a Möbius strip allows the assessment of simultaneous nervous activities that are spatially separated (e.g., two brain areas that display the same value of entropy in fMRI traces). This allows, for example, the evaluation in a single framework of the simultaneous activation of the primary sensitive cortex and the frontal areas during a visual-related task. It must also be taken into account that the BUT requirements, such as two features with no points in common and the proper mappings, are fully preserved when mapped to a Möbius strip. The transport of the BUT apparatus to a Möbius strip displays also another invaluable advantage: because the mapping achieved by making a trip around a twisted cylinder is an inversion, this permits the preservation of the invariance under inversions, therefore obeying to the laws of conservation of energy and information.
In sum, the study of brain oscillations on a twisted cylinder (instead of a three-dimensional Euclidean phase space) is justified by the BUT framework and might pave the way to the detection of unexpected relationships among different synchronous brain activities.

REFERENCES

1) Beggs JM, Timme N. 2012. Being critical of criticality in the brain. Front Physiol. 2012 Jun 7;3:163. doi: 10.3389/fphys.2012.00163.
2) Borsuk K. 1933. DreiSätzeüber die n-dimensionale euklidischeSphäre. FundamentaMathematicae. 20: 177–190.
3) Borsuk K. 1969. Fundamental retracts and extensions of fundamental sequences, 64, no. 1, 55–85.
4) Dodson CTJ, Parker PE. 1997. A user’s guide to algebraic topology, Kluwer, Dordrecht, Netherlands, 1997, xii+405 pp. ISBN: 0-7923-4292-5, MR1430097.
5) Ezaki T, Watanabe T, Ohzeki M, Masuda N. 2017. Energy landscape analysis of neuroimaging data. Philos Trans A Math Phys Eng Sci. 2017 Jun 28;375(2096). pii: 20160287. doi: 10.1098/rsta.2016.0287.
6) Friston K, Ao P. 2012. Free energy, value, and attractors. Comput Math Methods Med; 2012:937860. doi: 10.1155/2012/937860.
7) Matoušek, J. 2003. Using the Borsuk-Ulam Theorem. Lectures on topological methods in combinatorics and geometry. Written in cooperation with Anders Björner and Günter M. Ziegler.*Springer-Verlag, Berlin*, 2003. xii+196 pp. ISBN: 3-540-00362-2.
8) Möbius AF. 1858. Werke, Vol. 2. p. 519.
9) Peters JF, Ramanna S, Tozzi A, İnan E. 2017a. Bold-Independent Computational Entropy Assesses Functional Donut-Like Structures in Brain fMRI Images. Front Hum Neurosci. 2017 Feb 1;11:38. doi: 10.3389/fnhum.2017.00038.
10) Peters JF, Tozzi A, Ramanna S, Inan E. 2017b. The human brain from above: an increase in complexity from environmental stimuli to abstractions. Cognitive Neurodynamics. DOI: 10.1007/s11571-0-17-9428-2.
11) Sengupta B, Tozzi A, Coray GK, Douglas PK, Friston KJ. 2016. Towards a Neuronal Gauge Theory. PLOS Biology 14 (3): e1002400. doi:10.1371/journal.pbio.1002400.
12) Starostin EL, van der Heijden GHM. 2007. The shape of a Möbius strip. Nature Materials. 6 (8): 563–7. doi:10.1038/nmat1929
13) t’Hooft G. 2018. Virtual Black Holes and Space–Time Structure. Foundations of Physics. DOI10.1007/s10701-017-0133-0
14) Tozzi A. 2016. Borsuk-Ulam Theorem Extended to Hyperbolic Spaces. In Computational Proximity. Excursions in the Topology of Digital Images, edited by J F Peters, 169–171. doi:10.1007/978-3-319-30262-1.
15) Tozzi A, Peters JF. 2016a. Towards a Fourth Spatial Dimension of Brain Activity. Cognitive Neurodynamics 10 (3): 189–199. doi:10.1007/s11571-016-9379-z.
16) Tozzi A, Peters JF. 2016b. A Topological Approach Unveils System Invariances and Broken Symmetries in the Brain. Journal of NeuroscienceResearch 94 (5): 351–65. doi:10.1002/jnr.23720.
17) Tozzi A, Peters JF. 2017a. A Symmetric Approach Elucidates Multisensory Information Integration. Information 8,1. doi: 10.3390/info8010004.
18) Tozzi A, Peters JF. 2017b. From abstract topology to real thermodynamic brain activity. Cognitive Neurodynamics. 11: 283. Doi:10.1007/s11571-017-9431-7.
19) Tozzi A, Peters JF. 2017c. What does it mean “the same”? Progress in Biophysics and Molecular Biology. https://doi.org/10.1016/j.pbiomolbio.2017.10.005.
20) Tozzi A, Peters JF, Fingelkurts AA, Fingelkurts AA, Marijuán PC. 2017a. Topodynamics of metastable brains. Physics of Life Reviews. http://dx.doi.org/10.1016/j.plrev.2017.03.001.
21) Van de Ville D, Britz J, Michel CM. 2010. EEG microstate sequences in healthy humans at rest reveal scale-free dynamics. Proc Natl Acad Sci U S A.; 107(42):18179-18184. doi: 10.1073/pnas.1007841107.

22) Van Essen DC. 2005. A Population-Average, Landmark- and Surface-based (PALS) atlas of human cerebral cortex. Neuroimage. 28, 635–666.
23) Watanabe, T., Kan, S., Koike, T., Misaki, M., Konishi, S., Miyauchi, S. Masuda, N. 2014. Network-dependent modulation of brain activity during sleep. NeuroImage, 98, 1–10. http://doi.org/10.1016/j.neuroimage.2014.04.079.