On the division of plane figures in consecutive prime parts

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"Divide et impera" (Cesar, J.)

Abstract

In this paper it is discussed the following problem: “A mathematician wants to divide is garden into consecutive prime parts (first in two parts, after in three parts, and so on), only making straight paths, in a simple way (without retracing his own steps), and without going out of his plot of land. In how many parts can the mathematician divide his garden? ”

Keywords. *plane figure, prime number, vertex, edge, simple path.*

1 Problem solution

The garden can be mathematically described as a plane figure, and the gardener path as a simple path over some graph $G = (V, E)$ inscribed in the plane figure, with each vertex on the plane figure perimeter and straight edges joining the vertices. In how many parts can a plane figure be divided into consecutive prime parts drawing some graph $G = (V, E)$ through a simple path inscribed in the plane figure?

Theorem. *A convex plane figure divided consecutively in prime parts with a simple path inscribed in its perimeter, can at most be divided in seven parts.*

Proof.

Let us call the starting vertex of the graph over which the simple path goes v_0.

To divide some convex plane figure in two parts (the first prime number), v_0 must be at some point of the plane figure's perimeter, and it must be done a path from v_0 to another point v_1 of the plane figure's perimeter, such that $v_0 \neq v_1$. The path goes over the edge $e_1 = (v_0, v_1)$.

Then, starting from v_1, the path must divide the plane figure in three parts (the second prime number), going over some edge $e_2 = (v_1, v_2)$, where v_2 is whichever point of the perimeter such that $v_2 \neq v_1 \neq v_0$.

Note that, at this point, the plane figure's perimeter is divided by the vertices in three different sections, which can be denominated (with the vertices as subindexes) as $s_{0,1}$, $s_{0,2}$, and $s_{1,2}$.

After that, starting from v_2, the path must divide the plane figure in five parts (the third prime number), going over some edge $e_3 = (v_2, v_3)$. We can easily see that, for the plane figure to be divided in five parts, e_3 must cross over e_1 at some cross point c_1; therefore, $v_3 \subset s_{0,1}$, and it transforms $s_{0,1}$ in two different sections, $s_{0,3}$ and $s_{3,1}$. The cross point c_1 divides both e_1 and e_3 in two subedges, which can be denominated with the vertices as subindexes.

From this point v_3, the path must divide the plane figure in seven parts (the fourth prime number), going over some edge $e_4 = (v_3, v_4)$. This can only be achieved if e_4 crosses over the subedge e_{v_0,c_1} at some cross point c_2, so therefore $v_4 \subset s_{0,2}$, and it transforms $s_{0,2}$ in two different sections, $s_{0,4}$ and $s_{4,2}$.

Finally, the path must divide the plane figure in eleven parts (the fifth prime number), going over some edge $e_5 = (v_4, v_5)$. However, this cannot be achieved. To divide the plane figure in eleven parts, the edge e_5 must cross over three subedges, and that could only be done if $v_4 \subset s_{0,4}$ and $v_5 \subset s_{1,2}$. As $v_4 \notin s_{0,4}$, we can not continue dividing the plane figure consecutively in prime parts with a simple path inscribed in its perimeter, and the theorem is demonstrated.

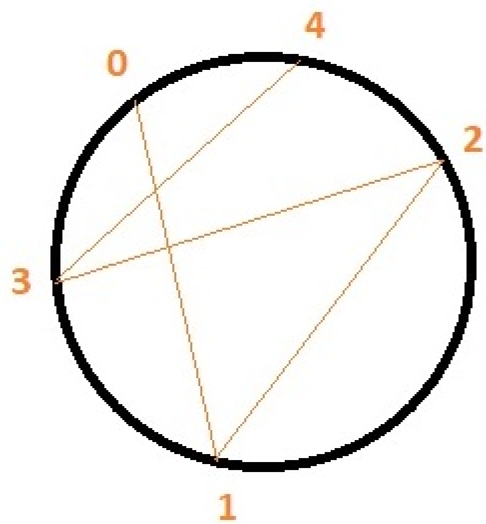

Figure 1:

A circle is a convex plane figure of infinite vertex, and thus can only be divided consecutively in prime parts until it is divided in seven parts.

Note that, if the plane figure is concave (at least has one mouth), the mouth can be used as a tangency point for edge e_5, and then divide the plane figure in eleven parts. An example is showed in Figure 2. In fact, as showed in Figure 3, it seems that the more mouths a concave plane figure has, the more consecutive divisions in prime parts we can achieve.

It is not the purpose of this paper to study this relationship between the mouths of a concave plane figure and the number of consecutive divisions in prime parts we can achieve, but we point it as interesting for further investigation. Some interesting questions arise: ¿Does exist a plane figure which can be divided in consecutive prime parts as many times as wanted? ¿how many mouths must have at least a plane figure to be divided in consecutive prime parts as many times as wanted?

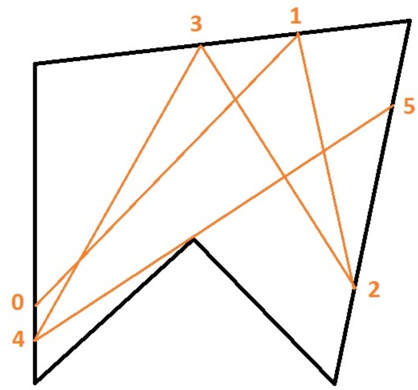

Figure 2:

This concave plane figure has one mouth, and that mouth can be used to trace a path from 4 to 5 that achieves its division from seven parts to eleven parts.

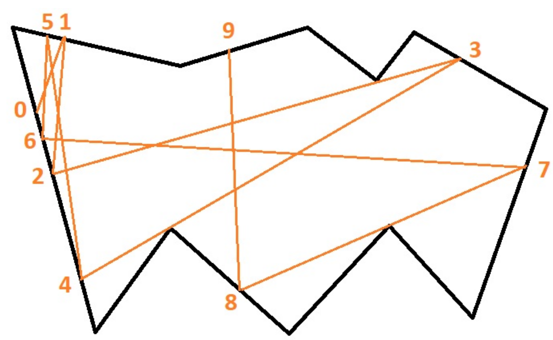

Figure 3:

This concave plane figure has four mouths, and properly used, they can be employed to trace paths that achieve further divisions in consecutive prime parts. In this case, the plane figure has been divided consecutively until twenty three parts.