Question 427 : Some Integrals

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abstract
This note presents a collection of integral formulas

Integrals

1. Let $0 \le u \le \infty, 0 \le v \le 1, \phi = \frac{1+\sqrt{5}}{2}, v = \left(1+u^{\phi}\right)^{-\phi}, u = \left(v^{-1/\phi}-1\right)^{1/\phi}$, then

$$uv + \int_u^{\infty} \left(1+x^{\phi}\right)^{-\phi} dx + \int_v^1 \sqrt[\phi]{\sqrt[\phi]{x^{-1}}-1}\, dx = 1 \qquad (1)$$

2. Let $0 \le u \le \frac{\pi}{3}, 0 \le v \le \infty, v = \left(\ln \frac{\sin u}{\sin\left(u+\pi/3\right)}\right)^2, u = \sin^{-1}\sqrt{\frac{3}{3+\left(1-2e^{\sqrt{v}}\right)^2}}$, then

$$\frac{5\pi^3}{81} = uv + \int_u^{\pi/3} \left(\ln \frac{\sin x}{\sin\left(x+\pi/3\right)}\right)^2 dx + \int_v^{\infty} \sin^{-1}\sqrt{\frac{3}{3+\left(1-2e^{\sqrt{x}}\right)^2}}\, dx \qquad (2)$$

3. Let $0 \le u \le \tan^{-1}\frac{\pi}{2}, \frac{1}{2} \le v \le 1, v = \frac{1}{1+\left(\sin\tan u\right)^2}, u = \tan^{-1}\left(\sin^{-1}\sqrt{v^{-1}-1}\right)$, then

$$\frac{\pi}{2\sqrt{2}}\left(\frac{e^2+3-2\sqrt{2}}{e^2-3+2\sqrt{2}}\right) = uv + \int_u^{\pi/2} \frac{1}{1+\left(\sin\tan x\right)^2}\, dx + \int_v^1 \tan^{-1}\left(\sin^{-1}\sqrt{x^{-1}-1}\right) dx \qquad (3)$$

4. Let $0 \le u \le \pi, e^{-1} \le v \le e, v = e^{\cos u}, u = \cos^{-1}\left(\ln v\right)$, then

$$\pi\sum_{n=0}^{\infty}\frac{2^{-2n}}{(n!)^2}=uv+\int_u^{\pi}e^{\cos x}dx+\int_v^{e}\cos^{-1}(\ln x)\,dx \tag{4}$$

$$\pi\sum_{n=0}^{\infty}\frac{2^{-2n}}{(n!)^2}=\int_0^{\pi}e^{\cos x}dx=\pi e^{-1}+\int_{e^{-1}}^{e}\cos^{-1}(\ln x)\,dx \tag{5}$$

5.

$$\int_0^1 e^{\sqrt{x(1-x)}}\,dx=1+\int_1^{e^{1/2}}\sqrt{1-(2\ln x)^2}\,dx=\pi\sum_{n=0}^{\infty}\frac{2^{-4n-3}}{n!(n+1)!}+\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}^2(n!)^2(2n+1)} \tag{6}$$

6. Let $f(x)=\sqrt[3]{\frac{1}{2x^4}+\sqrt{\frac{1}{4x^8}-\frac{512}{27x^6}}}+\sqrt[3]{\frac{1}{2x^4}-\sqrt{\frac{1}{4x^8}-\frac{512}{27x^6}}}$, then

$$\frac{\pi}{2\sqrt{2}}=\int_0^{3\sqrt{6}/64}\sqrt{2\sqrt{(f(x))^2-\frac{8}{x^2}}-f(x)}\,dx \tag{7}$$

7. Let $0\le u\le\pi,\frac{1}{3}\le v\le\frac{2}{3},v=\left(1+2^{-\cos u}\right)^{-1},u=\cos^{-1}\left(\frac{1}{\ln 2}\ln\frac{v}{1-v}\right)$, then

$$\frac{\pi}{2}=uv+\int_u^{\pi}\frac{1}{1+2^{-\cos x}}\,dx+\int_v^{2/3}\cos^{-1}\left(\frac{1}{\ln 2}\ln\frac{x}{1-x}\right)dx \tag{8}$$

8.

$$\frac{\sqrt{\pi}}{2}=1+\int_0^1\frac{1}{x^2}e^{-\left(x-\frac{1}{x}\right)^2}dx-\int_0^1\frac{2}{\sqrt{-\ln x}+\sqrt{4-\ln x}}\,dx \tag{9}$$

9.

$$2-\frac{\pi}{\sqrt{3}}=-\int_0^1\ln\left(1-x+x^2\right)dx=\int_0^{2\ln 2-\ln 3}\sqrt{4e^{-x}-3}\,dx \tag{10}$$

$$\frac{\pi}{\sqrt{3}}=4-\int_0^{\ln 3}\sqrt{4e^{x}-3}\,dx \tag{11}$$

10.

$$\frac{\pi}{2}=2\sqrt{2}-1-\int_0^1 \frac{1}{2+\sqrt{1-x}+\sqrt{1+x}}\,dx \qquad (12)$$

$$\frac{\pi}{2}=\frac{5}{\sqrt{2}}-2+\left(5\sqrt{2}-7\right)\int_0^1 \frac{\sqrt{x}\left(\left(2\sqrt{2}-3\right)x+1\right)}{\left(\left(2\sqrt{2}-3\right)x-1\right)^2}\,dx \qquad (13)$$

11. Let $0\le u\le 1, 0\le v\le 1, v=\sqrt{\frac{1-u^2}{1+u^2}}$, then

$$\frac{\sqrt{\pi}}{2}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=u\,v+\int_u^1\sqrt{\frac{1-x^2}{1+x^2}}\,dx+\int_v^1\sqrt{\frac{1-x^2}{1+x^2}}\,dx \qquad (14)$$

$$\frac{\sqrt{\pi}}{2}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=-u\,v+\int_0^u\sqrt{\frac{1-x^2}{1+x^2}}\,dx+\int_0^v\sqrt{\frac{1-x^2}{1+x^2}}\,dx \qquad (15)$$

12. Let $0\le u\le 1, 0\le v\le \infty, v=\sqrt{\frac{1-u}{u(1+u)}}, u=\frac{2}{1+v^2+\sqrt{1+6v^2+v^4}}$, then

$$\sqrt{\pi}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=u\,v+\int_u^1\sqrt{\frac{1-x}{x(1+x)}}\,dx+\int_v^\infty\frac{2}{1+x^2+\sqrt{1+6x^2+x^4}}\,dx \qquad (16)$$

$$\sqrt{\pi}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=-u\,v+\int_0^u\sqrt{\frac{1-x}{x(1+x)}}\,dx+\int_0^v\frac{2}{1+x^2+\sqrt{1+6x^2+x^4}}\,dx \qquad (17)$$

$$\sqrt{\pi}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=\int_0^1\sqrt{\frac{1-x}{x(1+x)}}\,dx \qquad (18)$$

$$\sqrt{\pi}\left(\frac{\Gamma(1/4)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\Gamma(1/4)^2}\right)=\int_0^\infty\frac{2}{1+x^2+\sqrt{1+6x^2+x^4}}\,dx \qquad (19)$$

13.

$$\int_0^{\pi}\left(\frac{\sin x}{2+x}+\frac{\cos x}{(2+x)^2}\right)dx=\frac{1}{2}+\frac{1}{2+\pi} \tag{20}$$

14. Let $0\le u\le 1, \sqrt{e}\le v\le e, v=e^{\left(1+u^2\right)^{-1}}, u=\sqrt{\frac{1}{\ln v}-1}$, then

$$\int_0^1 e^{\left(1+x^2\right)^{-1}}dx=uv+\int_u^1 e^{\left(1+x^2\right)^{-1}}dx+\int_v^e\sqrt{\frac{1}{\ln x}-1}\,dx \tag{21}$$

$$\begin{aligned}\int_0^1 e^{\left(1+x^2\right)^{-1}}dx&=\sqrt{e}+\int_{\sqrt{e}}^e\sqrt{\frac{1}{\ln x}-1}\,dx=1+\\&+\sum_{n=2}^{\infty}\frac{2^{-n}}{n!(2n-1)}\sum_{k=1}^{n-1}\prod_{m=1}^{k}\frac{2n-2m+1}{n-m}+\pi\sum_{n=1}^{\infty}\binom{2n-2}{n-1}\frac{2^{-2n}}{n!}\end{aligned} \tag{22}$$

15. Let $1\le u\le\infty, 0\le v\le\infty, v=\frac{1}{\sqrt{u^3-1}}$, then

$$\frac{\Gamma(1/3)^3}{2^{4/3}\pi}=(u-1)v+\int_u^{\infty}\frac{1}{\sqrt{x^3-1}}dx+\int_v^{\infty}\left(\sqrt[3]{1+x^{-2}}-1\right)dx \tag{23}$$

$$\frac{\Gamma(1/3)^3}{2^{4/3}\pi}=\int_1^{\infty}\frac{1}{\sqrt{x^3-1}}dx \tag{24}$$

$$\frac{\Gamma(1/3)^3}{2^{4/3}\pi}=\int_0^{\infty}\left(\sqrt[3]{1+x^{-2}}-1\right)dx \tag{25}$$

16.

$$\int_0^{\infty}f(x)\,dx=\frac{\pi}{\sqrt{15}}F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{5}\right) \tag{26}$$

Remark: F is the hypergeometric function.

$$f\left(x\right)=-\frac{2}{3}+\sqrt[3]{\sqrt{\frac{10000}{19683}+\frac{17}{243x^2}+\frac{1}{324x^4}}+\frac{17}{27}+\frac{1}{18x^2}}-\sqrt[3]{\sqrt{\frac{10000}{19683}+\frac{17}{243x^2}+\frac{1}{324x^4}}-\frac{17}{27}-\frac{1}{18x^2}} \tag{27}$$

$$f\left(x\right)=-\frac{2}{3}+\frac{1}{18x}\sqrt[3]{x\,p\left(x\right)}-\frac{26x}{3\sqrt[3]{x\,p\left(x\right)}} \tag{28}$$

$$p\left(x\right)=324+3672x^2+12\sqrt{3}\sqrt{40000x^4+5508x^2+243} \tag{29}$$

17. Let $0\le u\le 1, 0\le v\le\infty, v=\sqrt{-\ln u}, u=e^{-v^2}$, then

$$\frac{\sqrt{\pi}}{2}=u\,v+\int_u^1\sqrt{-\ln x}\,dx+\int_v^\infty e^{-x^2}dx \tag{30}$$

18.

$$\frac{\sqrt{\pi}}{2}+\ln 2=\int_0^{\ln 2}e^{-\left(\ln\left(e^x-1\right)\right)^2}dx+\int_0^1\ln\left(1+e^{\sqrt{-\ln x}}\right)dx \tag{31}$$

$$\frac{\sqrt{\pi}}{2}-\ln 2=\int_{\ln 2}^{\infty}e^{-\left(\ln\left(e^x-1\right)\right)^2}dx-\int_0^1\ln\left(1+e^{-\sqrt{-\ln x}}\right)dx \tag{32}$$

19.

$$\frac{\pi}{e\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(2n+3\right)2^{-2n-1}}{n!\left(n+1\right)!}=\int_0^1\left(\sqrt{\frac{1+\sqrt{1-x^2}}{x}}-\sqrt{\frac{1-\sqrt{1-x^2}}{x}}\right)e^{-2x}dx \tag{33}$$

20.

$$\frac{\pi}{2}\sqrt{\frac{3}{2}}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n12^{-2n}}{\left(2n+1\right)!}\binom{4n}{2n}=\int_0^1\left(\sqrt{\frac{x}{3-x-\sqrt{9-6x-3x^2}}}-\sqrt{\frac{x}{3-x+\sqrt{9-6x-3x^2}}}\right)\cos\left(\frac{x}{3}\right)dx \tag{34}$$

21.

$$\pi = \frac{8}{3}\int_0^1 \operatorname{sech}^{-1}\left(\sqrt[3]{\sqrt{x^2+\frac{1}{27}}+x}-\sqrt[3]{\sqrt{x^2+\frac{1}{27}}-x}\right)dx \tag{35}$$

$$\pi = \frac{8}{3}\int_0^1 \cosh^{-1}\left(\frac{1}{6x}+\frac{1}{3x\sqrt[3]{4}\sqrt[3]{f(x)}}+\frac{\sqrt[3]{f(x)}}{6\sqrt[3]{2}x}\right)dx \tag{36}$$

$$f(x) = 2+108x^2+3\sqrt{3}\sqrt{16x^2+432x^4} \tag{37}$$

22.

$$\frac{\pi^3}{8}-2\pi G+\frac{7}{2}\zeta(3) = \int_0^{\pi^2/4}\sqrt{x\sin\sqrt{x\sin\sqrt{x\sin\sqrt{x\cdots}}}}\,dx \tag{38}$$

Remark: G is Catalan's constant , $\zeta(3)$ is Apéry's constant.

23.

$$4-e = \int_0^1 e^{x\left(1+\sqrt{1-x}\right)^{-2}}dx = \int_0^1 e^{\left(1-\sqrt{x}\right)/\left(1+\sqrt{x}\right)}dx = 2\int_0^1 xe^{(1-x)/(1+x)}dx \tag{39}$$

24.

$$\pi+2\ln 2-4 = \int_0^1 \tan^{-1}\left(\frac{4\sqrt{1-x}}{4+x}\right)dx = \int_0^1 \tan^{-1}\left(\frac{4\sqrt{x}}{5-x}\right)dx = 2\int_0^1 x\tan^{-1}\left(\frac{4x}{5-x^2}\right)dx \tag{40}$$

$$\pi+2\ln 2-4 = 4\int_0^1 \frac{1-x^2}{\left(1+x^2\right)\left(2+x^2+\sqrt{4+5x^2}\right)}dx \tag{41}$$

25.

$$\int_0^{\sqrt{\pi/2}} \sin x^2\,dx+\int_0^1 \sqrt{\sin^{-1}x}\,dx = \sqrt{\frac{\pi}{2}} \tag{42}$$

$$\int_0^{\sqrt{\pi/2}} \sin x^2\,dx+\int_0^{\pi/2} \sqrt{x}\cos x\,dx = \sqrt{\frac{\pi}{2}} \tag{43}$$

$$\int_0^{\sqrt{\pi}} \sin x^2\,dx = \int_0^1\left(\sqrt{\pi-\sin^{-1}x}-\sqrt{\sin^{-1}x}\right)dx = \int_0^1 \frac{\pi-2\sin^{-1}x}{\sqrt{\pi-\sin^{-1}x}+\sqrt{\sin^{-1}x}}dx \tag{44}$$

$$\int_0^{\sqrt{\pi}} \sin x^2\,dx = \int_0^{\pi/2}\left(\sqrt{\pi-x}-\sqrt{x}\right)\cos x\,dx = \int_0^{\pi/2}\frac{(\pi-2x)\cos x}{\sqrt{\pi-x}+\sqrt{x}}dx \tag{45}$$

$$\int_0^{\pi/2}\left(\frac{\sin x}{\sqrt{x}}+2\sqrt{x}\cos x\right)dx=\sqrt{2\pi} \tag{46}$$

26.

$$\pi=\int_0^{\infty}\frac{\ln\left(1+\sqrt{e^4-1}\,x\right)}{1+x^2}dx+\int_0^{\sqrt{e^4-1}}\frac{\ln x}{1+x^2}dx \tag{47}$$

$$\pi+G=\int_0^{\infty}\frac{\ln\left(1+\sqrt{e^4-1}\,x\right)}{1+x^2}dx+\int_1^{\sqrt{e^4-1}}\frac{\ln x}{1+x^2}dx \tag{48}$$

$$\int_0^{\infty}\frac{\ln\left(1+\sqrt{e^4-1}\,x\right)}{1+x^2}dx=\pi+\tan^{-1}\left(\frac{1}{\sqrt{e^4-1}}\right)\ln\sqrt{e^4-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\frac{1}{\sqrt{e^4-1}}\right)^{2n+1} \tag{49}$$

Reamrk: G is Catalan's constant.

27. Let $0<a<b\le 1$, then

$$\frac{\pi}{2}\ln\frac{b}{a}=\int_a^b\frac{\tan^{-1}x}{x}dx+2\int_{(1-b)/(1+b)}^{(1-a)/(1+a)}\frac{\tan^{-1}x}{1-x^2}dx \tag{50}$$

$$\frac{\pi}{2}\ln\frac{b}{a}-\sum_{n=0}^{\infty}\frac{(-1)^n\left(b^{2n+1}-a^{2n+1}\right)}{(2n+1)^2}=\int_a^b\frac{1}{x}\tan^{-1}\left(\frac{1-x}{1+x}\right)dx=2\int_{(1-b)/(1+b)}^{(1-a)/(1+a)}\frac{\tan^{-1}x}{1-x^2}dx \tag{51}$$

28. Let $0<a\le 1$, then

$$-\frac{\pi}{2}\ln a-G+\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)^2}=\int_a^1\frac{1}{x}\tan^{-1}\left(\frac{1-x}{1+x}\right)dx=2\int_0^{(1-a)/(1+a)}\frac{\tan^{-1}x}{1-x^2}dx \tag{52}$$

Remark: G is Catalan's constant.

References

1. Boros, G. and Moll, V.H. : Irresistible Integrals, Cambridge University Press, 2004.
2. Gradshteyn, I.S. and Ryzhik, I.M. : Table of Integrals , Series, and Products. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition , 2007.