The Dottie Number

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abstract

This note presents some formulas related with Dottie number.

1. INTRODUCTION. Dottie number d is defined by

$$d = \lim_{n\to\infty} \cos^{[n]}(x) = \lim_{n\to\infty} \underbrace{\cos\left(\cos\left(\cos\left(\ldots\left(\cos(x)\right)\right)\right)\right)}_{n} = 0.739085\ldots \tag{1}$$

The limit (1) is independent of x .

Equation for d :

$$\cos d = d \tag{2}$$

Dottie number is the inique real root of $\cos x = x$.

2. RECURRENCES.

Recurrences for d :

$$x_{n+1} = \cos(x_n) \ , x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = d \tag{3}$$

$$x_{n+1} = \frac{1}{2}\sqrt[3]{6x_n + 2\cos(3x_n)} \quad , x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = d \tag{4}$$

$$x_{n+1} = \frac{2}{5}x_n + \frac{3}{5}\cos x_n \quad , x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = d \tag{5}$$

3. NESTED RADICAL.

Nested radical for d :

$$d=\sqrt{\frac{1}{2}+\frac{1}{2}\cos\left(2\sqrt{\frac{1}{2}+\frac{1}{2}\cos\left(2\sqrt{\frac{1}{2}+\ldots}\right)}\right)} \tag{6}$$

$$d=\frac{1}{2}\sqrt{2+2\cos\sqrt{2+2\cos\sqrt{2+\ldots}}} \tag{7}$$

4. INTEGRALS.

$$\begin{aligned}\frac{2\pi}{1+\sin d}=-\int_0^1\frac{i}{x-i-\cos(x-i)}dx+\int_{-1}^1\frac{1}{1+ix-\cos(1+ix)}dx+\\+\int_0^1\frac{i}{x+i-\cos(x+i)}dx-\int_{-1}^1\frac{1}{ix-\cos(ix)}dx\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\pi}{1+\sin d}=\int_0^1\frac{1+\sinh 1\sin x}{(x-\cosh 1\cos x)^2+(1+\sinh 1\sin x)^2}dx+\\+\int_0^1\frac{1-\cos 1\cosh x}{(x+\sin 1\sinh x)^2+(1-\cos 1\cosh x)^2}dx+\int_0^1\frac{\cosh x}{(\cosh x)^2+x^2}dx\end{aligned} \tag{9}$$

$$\frac{\pi}{d(1+\sin d)}=\int_{-\pi/2}^{\pi/2}\frac{e^{ix}}{(\sin e^{ix})^2+e^{2ix}-1}dx+\int_0^1\frac{2}{1+x^2+(\sinh x)^2}dx \tag{10}$$

$$\frac{2\pi}{1+\sin d}=\int_0^{2\pi}\frac{e^{ix}}{e^{ix}-\cos e^{ix}}dx \tag{11}$$

$$1+2\sin d-d^2=2\int_0^1\cos\left(x-\cos\left(x-\cos\left(x-\ldots\right)\right)\right)dx \tag{12}$$

$$d=\int_0^1\frac{\left(1+\sin e^{2\pi ix}\right)e^{4\pi ix}}{e^{2\pi ix}-\cos e^{2\pi ix}}dx \tag{13}$$

Remark: $i=\sqrt{-1}$.

5. SERIES.

Dottie number via inversion of series:

$$d+\sin^{-1} d=\frac{\pi}{2} \Rightarrow d=\sum_{n=0}^{\infty} a_n\left(\frac{\pi}{2}\right)^{2n+1} \quad ,a_n \in \mathbb{Q} \tag{14}$$

$$d=\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{96}\left(\frac{\pi}{2}\right)^3-\frac{1}{1920}\left(\frac{\pi}{2}\right)^5-\frac{43}{1290240}\left(\frac{\pi}{2}\right)^7-\frac{223}{92897280}\left(\frac{\pi}{2}\right)^9-\ldots \tag{15}$$

$$d=\frac{\pi}{4}-y\,,\ \cos y+\sin y=\frac{\pi}{2\sqrt{2}}-\sqrt{2}\,y \Rightarrow y=\sum_{n=1}^{\infty} b_n\left(\frac{\pi}{2\sqrt{2}}-1\right)^n \tag{16}$$

$$b_n=\left\{\sqrt{2}-1,\frac{5\sqrt{2}-7}{2},\frac{25\sqrt{2}}{2}-\frac{53}{3},\frac{301\sqrt{2}}{4}-\frac{1277}{12},\frac{5981\sqrt{2}}{12}-\frac{10573}{15},\frac{630233\sqrt{2}}{180}-\frac{222821}{45},\ldots\right\} \tag{17}$$

Series via iterator function:

$$d=\cos 1-2\sum_{n=0}^{\infty}\sin\left(\frac{f(n+1)-f(n)}{2}\right)\sin\left(\frac{f(n+1)+f(n)}{2}\right) \tag{18}$$

where

$$f(n)=\cos^{[n]}(1)=\underbrace{\cos\cos\cos\ldots\cos 1}_{n},\ n\in\mathbb{N},\ f(0)=1 \tag{19}$$

6. PI FORMULAS.

$$\pi=8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{d+\sqrt{1+d^2}}\right)^{2n-1} c_n \tag{20}$$

$$\pi=12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{d\sqrt{3}+\sqrt{1+3d^2}}\right)^{2n-1} c_n \tag{21}$$

where

$$c_n = \cos((2n-1)d) = \sum_{k=0}^{\left[\frac{2n-1}{2}\right]} \binom{2n-1}{2k} (-1)^k d^{2n-2k-1} (1-d^2)^k \quad , n \in \mathbb{N} \tag{22}$$

$$\pi = 4\sqrt{1-d^2} \sum_{n=0}^{\infty} \frac{u_n}{n+1} \left(\frac{1}{d+\sqrt{1-d^2}} \right)^{n+1} \tag{23}$$

where

$$u_{n+1} = 2du_n - u_{n-1} \quad , u_0 = 1, u_1 = 2d \tag{24}$$

$$\pi = \frac{2}{1-d^2} \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \right)^2 \left(\frac{2n-1-d^2}{2n+1-d^2} \right) \tag{25}$$

7. RELATIONS WITH GAMMA FUNCTION.

$$\Gamma\left(\frac{1}{2}+\frac{d}{\pi}\right)\Gamma\left(\frac{1}{2}-\frac{d}{\pi}\right) = \frac{\pi}{d} \tag{26}$$

$$\sqrt{\frac{1}{\pi^2} - \left(\frac{d}{\pi}\right)^2}\, \Gamma\left(1-\frac{d}{\pi}\right)\Gamma\left(\frac{d}{\pi}\right) = 1 \tag{27}$$

$$x_{n+1} = \left(\Gamma\left(\frac{1}{2}+x_n\right)\Gamma\left(\frac{1}{2}-x_n\right) \right)^{-1} \quad , x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = \frac{d}{\pi} \tag{28}$$

Remark: Gamma function: $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad , x > 0$.

References

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