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A survey on Smarandache notions in number theory I: Smarandache function

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Abstract In this paper we give a survey on recent results on Smarandache function.

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§1. Definition and simple properties

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is,

$$S(n) = \min\{m : \ n \mid m!, \ n \in \mathbb{N}\}. \tag{1.1}$$

Many people studied the lower bound of $S(n)$.

M. Le [16]. *Let $p > 2$ be a prime. Then $S\left(2^{p-1}(2^p - 1)\right) \geq 2p + 1$.*

J. Su [35]. *Let $p \geq 5$ be a prime. Then $S\left(2^{p-1}(2^p - 1)\right) \geq 6p + 1$.*

J. Su and S. Shang [36]. *Let $p \geq 7$ be a prime. Then $S(2^p + 1) \geq 6p + 1$.*

M. Liang [23]. *Let $p > 7$ be a prime. Then $S(2^p \pm 1) \geq 8p + 1$.*

T. Wen [40]. *Let $p \geq 17$ be a prime. Then $S(2^p \pm 1) \geq 10p + 1$.*

C. Shi [33]. *Let $p \geq 17$ be a prime. Then $S(2^p \pm 1) \geq 14p + 1$.*

X. Wang [38]. *For any $m \in \mathbb{N}$, let $p \geq 9m^2(\log m + 1)^3$ be a prime. Then $S(2^p - 1) \geq 2mp + 1$.*

F. Li and C. Yang [20]. *Let a and b be distinct positive integers, and let $p \geq 17$ be a prime. Then $S\left(a^p + b^p\right) \geq 8p + 1$.*

P. Shi and Z. Liu [34]. *Let a and b be distinct positive integers, and let $p \geq 17$ be a prime. Then $S\left(a^p + b^p\right) \geq 10p + 1$.*

L. Gao, H. Hao and W. Lu [6]. *Let a and b be positive integers with $a > b$, and let $p \geq 17$ be a prime. Then $S\left(a^p - b^p\right) \geq 8p + 1$.*

J. Wang [37]. *Let $F_n = 2^{2^n} + 1$ be the Fermat number. Then $S(F_n) \geq 8 \cdot 2^n + 1$ for $n \geq 3$.*

M. Zhu [55]. *Let $F_n = 2^{2^n} + 1$ be the Fermat number. Then $S(F_n) \geq 12 \cdot 2^n + 1$ for $n \geq 3$.*

M. Liu and Y. Jin [26]. *Let $F_n = 2^{2^n} + 1$ be the Fermat number. Then $S(F_n) \geq 4(4n + 9) \cdot 2^n + 1$ for $n \geq 4$.*

M. Bencze [2]. *For positive integer sequences* $m_1, \cdots, m_n$*, we have*

$$S\left(\prod_{k=1}^{n} m_k\right) \leq \sum_{k=1}^{n} S\left(m_k\right).$$

M. Le [17]. *There are infinite many* $n \in \mathbb{N}$ *such that* $S(n) \leq S(n - S(n))$.

The distribution properties have also been studied.

W. Zhu [56]. *Let* $m = p_1^{T_1} p_2^{T_2} \cdots p_k^{T_k}$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. For any* $n \in \mathbb{N}$*, we have*

$$S\left(m^n\right) = n \cdot \max_{1 \leq i \leq k} \{(p_i - 1)T_i\} + O\left(\frac{m}{\ln m} \ln n\right).$$

M. Le [15]. *For any distinct positive integers* k *and* n*,* $\log_{k^n} S\left(n^k\right)$ *is never a positive integer.*

F. Du [4]. 1. *Assume that* $n = p_1 p_2 \cdots p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. Then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

2. *Suppose that* $n = p^T$*, where* $p > 2$ *is a prime and* $T \leq p$*. Then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

3. *Let* $n = p_1^{T_1} p_2^{T_2} \cdots p_{k-1}^{T_{k-1}} \cdot p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. If* $S(n) = p_k$*, then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

L. Huan [9]. 1. *Assume that* $n = p_1 p_2 \cdots p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. Then we have*

$$\prod_{d|n} S(d) = p_1 \cdot p_2^2 \cdots p_{k-1}^{2^{k-2}} p_k^{2^{k-1}}.$$

B. Liu and X. Pan [25]. *For any positive integer* n*, the formula*

$$\frac{S(2)S(4)\cdots S(2n)}{S(1)S(3)\cdots S(2n-1)}$$

is an integer if and only if $n = 1$.

A. Zhang [49]. *For integer* $n > 1$*, we have*

$$\frac{1}{n}\left|\{m: \ 1 \leq m \leq n, S(m) \text{ is a prime}\}\right| = 1 + O\left(\frac{1}{\ln n}\right).$$

W. Xiong [43]. *Define*

$$ES(n) = \left|\{a : 1 \leq a \leq n, 2 \mid S(a)\}\right|, \qquad OS(n) = \left|\{a : 1 \leq a \leq n, 2 \nmid S(a)\}\right|.$$

Then for integer $n > 1$*, we have*

$$\frac{ES(n)}{OS(n)} = O\left(\frac{1}{\ln n}\right).$$

Q. Liao and W. Luo [24]. *Let p be a prime and α be a positive integer.*

1) *For any positive integer r and $\alpha = p^r$, we have*

$$S(p^\alpha) = p^{r+1} - p^r + p.$$

2) *For any positive integer r, $t \in [1, r]$ and $\alpha = p^r - t$, we have*

$$S(p^\alpha) = p^{r+1} - p^r.$$

3) *For any positive integer r, $t \in [r+1, p^r - p^{r-1}]$ and $\alpha = p^r - t$.*

(I) *If*

$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n p^{k_n}$$

with

$$k_i < p^{k_i - 1}(p-1) - 1, \qquad 1 \le i \le n-1,$$

then we have

$$S(p^\alpha) = (p-1)\left(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\right) + (-1)^n p.$$

(II) *If*

$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n \left(p^{k_n} - t\right)$$

with $t \in [1, k_n]$ and

$$k_i < p^{k_i - 1}(p-1) - 1, \qquad 1 \le i \le n-1,$$

then

$$S(p^\alpha) = (p-1)\left(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\right).$$

Q. Liao and W. Luo [24]. *Let $\phi(n)$ be the Euler function and let $\sigma(n)$ be the sum of the different positive factors for n.*

1) *For any positive integer k, there are no any prime p and positive integer m coprime with p, such that $\phi(pm) = S(p^k)$ and $S(p^k) \ge S(m^k)$.*

2) *For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\phi(p^2 m) = S(p^{2k})$ and $S(p^{2k}) \ge S(m^k)$, then $p = 2k+1$ or $2 \ne p \le k$. Furthermore,*

(I) *If $2k + 1 = p$, then*

$$(p, m) = (2k+1, 1), \quad (2k+1, 2), \quad (2, 3).$$

(II) *If $2 \le p \le k$, then $k \ge 3$ and*

$$\begin{cases} 2 \le \phi(m) \le \frac{2k^2+k-1}{3}, & k \equiv 2(\text{mod}3), \\ 2 \le \phi(m) \le \frac{2k^2+k}{3}, & \textit{otherwise}. \end{cases}$$

3) *For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\phi(p^\alpha m) = S(p^{\alpha k})$ and $S(p^{\alpha k}) \geq S(m^k)$. Then $\alpha k + 1 > p^{\alpha-3}(p^2-1)$ and $1 \leq \phi(m) \leq q$, where*

$$\alpha k + 1 = q p^{\alpha-3}(p^2-1) + r, \qquad 0 \leq r < p^{\alpha-3}(p^2-1).$$

4) *For any positive integer k, there exist some prime p and positive integer m coprime with p, such that $\phi(p^3 m) = S(p^{3k})$ and $S(p^{3k}) \geq S(m^k)$, $m = 1, 2$.*

Q. Liao and W. Luo [24]. 1) *For any prime p, there is no any positive integer α such that $\dfrac{\sigma(p^\alpha)}{S(p^\alpha)}$ is a positive integer.*

2) *Let p be an odd prime, $\alpha \geq 1$ and $n = 2^\alpha p$.*

(I) *If $\displaystyle\sum_{i=1}^{\infty}\left[\frac{p}{2^i}\right] \geq \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then $2^{\alpha+1} \equiv 1 (\mathrm{mod}\ p)$.*

(II) *If $\displaystyle\sum_{i=1}^{\infty}\left[\frac{p}{2^i}\right] < \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then*

$$\frac{\sigma(n)}{S(n)} = m\frac{2^{\alpha+1}-1}{d} \qquad \text{and} \qquad p = m\frac{S(2^\alpha)}{d} - 1,$$

where $d = \left(2^{\alpha+1}-1, S(2^\alpha)\right)$ and $0 < m \leq d$.

§2. Mean values of the Smarandache function

C. Yang and D. Liu [45]. *Define $\sigma(n) = \sum_{d|n} d$. For any real $x \geq 3$ we have*

$$\sum_{n\leq x}\sigma\left(S(n)\right) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Y. Wang [39]. *For any real $x \geq 2$ we have the asymptotic formula*

$$\sum_{n\leq x}\frac{S(n)}{n} = \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

W. Yao [48]. *Let $\Lambda(n)$ be the Mangoldt function. For any real $x \geq 1$ we have*

$$\sum_{n\leq x}\Lambda(n)S(n) = \frac{x^2}{4} + O\left(\frac{x^2\log\log x}{\log x}\right).$$

B. Shi [31]. *Let k be any fixed positive integer. For any real $x \geq 1$ we have*

$$\sum_{n\leq x}\Lambda(n)S(n) = x^2\sum_{i=0}^{k}\frac{c_i}{\log^i x} + O\left(\frac{x^2}{\log^{k+1} x}\right),$$

where c_i $(i = 0, 1, \cdots, k)$ are constants, and $c_0 = 1$.

Z. Lv [28]. *Let k be any fixed positive integer. For any real $x > 2$ we have the asymptotic formula*

$$\sum_{n\leq x}\left(S(n)-S\left(S(n)\right)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}\sum_{i=1}^{k}\frac{c_i}{\log^i x}+O\left(\frac{x^{\frac{3}{2}}}{\log^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, c_i $(i=1,2,\cdots,k)$ are computable constants, and $c_1=1$.

J. Ge [7]. *The Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,2,\cdots,k]$, where $[1,2,\cdots,k]$ denotes the least common multiple of $1,2,\cdots,k$. Let k be any fixed positive integer. For any real $x > 2$ we have the asymptotic formula*

$$\sum_{n\leq x}\left(SL(n)-S(n)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}\sum_{i=1}^{k}\frac{c_i}{\log^i x}+O\left(\frac{x^{\frac{3}{2}}}{\log^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, c_i $(i=1,2,\cdots,k)$ are computable constants.

X. Fan and C. Zhao [5]. *Let $d(n)$ be the divisor function. For any real $x \geq 2$ we have*

$$\sum_{n\leq x}S(n)d(n)=\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right).$$

Z. Lv [29]. *Let $k \geq 2$ be any fixed positive integer. For any real $x > 1$ we have*

$$\sum_{n\leq x}S(n)d(n)=\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i=2,3,\cdots,k)$ are computable constants.

M. Zhu [54]. *Define $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$, $\alpha\geq 1$. Let $k\geq 2$ be any fixed positive integer. For any real $x > 1$ we have*

$$\sum_{n\leq x}S(n)\sigma_\alpha(n)=\frac{\zeta(\alpha+2)\zeta(2)}{2+\alpha}\cdot\frac{x^{\alpha+2}}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^{\alpha+2}}{\ln^i x}+O\left(\frac{x^{\alpha+2}}{\ln^{k+1}x}\right),$$

where c_i $(i=2,3,\cdots,k)$ are computable constants.

H. Zhou [53]. *Let $k\geq 1$ be any fixed positive integer. For any complex s with* Re $s > 1$ *we have*

$$\sum_{n=1}^{\infty}\frac{\Lambda(n^k)}{S^s(n^k)}=-\zeta(s)\frac{\zeta'(ks)}{\zeta(ks)}.$$

Y. Guo [8]. *Define a function $F(n)$ as follows:*

$$F(n)=\begin{cases}0, & \text{if } n=1,\\ \alpha_1p_1+\alpha_2p_2+\cdots+\alpha_rp_r, & \text{if } n>1 \text{ and } n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.\end{cases}$$

Let $k\geq 1$ be any fixed positive integer. For any real $x > 1$ we have

$$\sum_{n\leq x}\left(F(n)-S(n)\right)^2=\sum_{i=1}^{k}\frac{c_i\cdot x^2}{\ln^{i+1}x}+O\left(\frac{x^2}{\ln^{k+2}x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ *are computable constants, and* $c_1 = \frac{\pi^2}{6}$.

C. Shi [32]. *For any positive integer* k*, the Smarandache* kn*-digital sequence* $a(k, n)$ *is defined as all positive integers which can be partitioned into two groups such that the second part is* k *times bigger that the first. For* $1 \le k \le 9$ *and real* $x > 1$ *we have*

$$\sum_{n \le x} \frac{S(n)}{a(k,n)} = \frac{3\pi^2}{20k} \ln\ln x + O(1).$$

C. Yang, C. Li and D. Liu [44]. *For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} S^2(n) = \frac{\zeta(3)x^3}{3\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

$$\sum_{n \le x} \frac{S^2(n)}{n} = \frac{\zeta(3)x^2}{3\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

W. Huang [11]. *Let* $k \ge 1$ *be any fixed integer. For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} S^k(n) = \frac{\zeta(k+1)}{k+1} \cdot \frac{x^{k+1}}{\ln x} + O\left(\frac{x^{k+1}}{\ln^2 x}\right),$$

$$\sum_{n \le x} \frac{S^k(n)}{n} = \frac{2\zeta(k+1)}{k+1} \cdot \frac{x^k}{\ln x} + O\left(\frac{x^k}{\ln^2 x}\right).$$

C. Li, C. Yang and D. Liu [19]. *Let* $P(n)$ *denote the largest prime factor of* n*. For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} \frac{S(n)}{P(n)} = x\ln 2 + \frac{6x^{\frac{2}{3}}}{\ln x} + O\left(\frac{x^{\frac{2}{3}}}{\ln^2 x}\right).$$

M. Yang [46]. *For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} \frac{S(n)}{SL(n)} = x + O\left(\frac{x\ln\ln x}{\ln x}\right),$$

$$\sum_{n \le x} \frac{P(n)}{SL(n)} = x + O\left(\frac{x\ln\ln x}{\ln x}\right).$$

L. Li, J. Hao and R. Duan [22]. *For any real* $x \ge 1$ *we have*

$$\sum_{n \le x} \ln S(n) = x\ln x + O(x).$$

Z. Liu and P. Shi [27]. *For any real* $x \ge 3$ *and* $\beta > 1$ *we have*

$$\sum_{n \le x} (S(n) - P(n))^\beta = \frac{2\zeta\left(\frac{\beta+1}{2}\right) x^{\frac{\beta+1}{2}}}{(\beta+1)\ln x} + O\left(\frac{x^{\frac{\beta+1}{2}}}{\ln^2 x}\right).$$

W. Huang [12]. *For* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$*, we define* $\varpi(n) = p_1 + p_2 + \cdots + p_k$*. For any real* $x \geq 2$ *we have*

$$\sum_{n \leq x} S(n)\varpi(n) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where B *is computable constant.*

G. Chen [3]. *Define* $H(n) = \sum_{[r,s]=n} S(r)S(s)$*. Let* $k \geq 1$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} H(n) = \sum_{i=1}^{k} \frac{d_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where d_i $(i = 1, 2, \cdots, k)$ *are computable constants, and* $d_1 = \frac{1}{3} \cdot \frac{\zeta^3(3)}{\zeta(6)}$.

Q. Yang [47]. *For any real* $\delta \leq 1$*, the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$$

diverges.

For any real $\epsilon > 0$*, the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$$

converges.

§3. Mean values of the Smarandache function over sequences

W. Zhang and Z. Xu [50]. *Let* $a(n)$ *denote the square complements of* n*. For any real* $x \geq 3$ *we have the asymptotic formula*

$$\sum_{n \leq x} S(a(n)) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

H. Li and X. Zhao [21]. *Let* $r_k(n)$ *denote the integer part of* k*-th root of* n*. For any real* $x \geq 3$ *we have*

$$\sum_{n \leq x} S(r_k(n)) = \frac{\pi^2}{6(k+1)} \cdot \frac{x^{1+\frac{1}{k}}}{\ln x} + O\left(\frac{x^{1+\frac{1}{k}}}{\ln^2 x}\right).$$

J. Ma [30]. *Define* $L(n) = [1, 2, \cdots, n]$*. For any real* $x \geq 1$ *we have*

$$\sum_{n \leq x} S\left(L(n)\right) = \frac{1}{2}x^2 + O\left(x^{\frac{23}{18}+\epsilon}\right).$$

Q. Wu [41]. *Define* $Z(n) = \min\left\{k : n \leq \frac{k(k+1)}{2}\right\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(Z(n)\right) = \frac{\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln 2x} + \sum_{i=2}^{k}\frac{c_i(2x)^{\frac{3}{2}}}{\ln^i 2x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

H. Zhao [51]. *Let* $a_k(n)$ *denote the* k*-th power complements of* n. *For any real* $x \geq 3$ *we have*

$$\sum_{n\leq x}\left(S(a_k(n)) - (k-1)P(n)\right)^2 = \frac{2\zeta\left(\frac{3}{2}\right)}{3}\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

W. Huang [10]. *Define* $u(n) = \min\{k : n \leq k(2k-1)\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(u(n)\right) = \frac{\pi^2}{144}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{c_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

Q. Zhao and L. Gao [52]. *Define* $W(n) = \min\{k : n \leq k(3k+1)\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(W(n)\right) = \frac{\pi^2}{486}\cdot\frac{(3x)^{\frac{3}{2}}}{\ln\sqrt{3x}} + \sum_{i=2}^{k}\frac{b_i(3x)^{\frac{3}{2}}}{\ln^i\sqrt{3x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

W. Huang and J. Zhao [14]. *Define*

$$\begin{aligned} u_r(n) &= \min\left\{m + \frac{1}{2}m(m-1)(r-2) : n \leq m + \frac{1}{2}m(m-1)(r-2), r \in \mathbb{N}, r \geq 3\right\}, \\ v_r(n) &= \max\left\{m + \frac{1}{2}m(m-1)(r-2) : n \geq m + \frac{1}{2}m(m-1)(r-2), r \in \mathbb{N}, r \geq 3\right\}. \end{aligned}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\begin{aligned} \sum_{n\leq x} S(u_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{2(r-2)x}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{2(r-2)x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right), \\ \sum_{n\leq x} S(v_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{2(r-2)x}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{2(r-2)x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right), \end{aligned}$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

W. Huang [13]. *Define* $a(n) = n - u_r(n)$ *and* $b(n) = v_r(n) - n$. *Let* $k \geq 1$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S(n)a(n) = \frac{8\sqrt[4]{2}\pi^2}{63(r-2)^{\frac{5}{4}}}\cdot\frac{x^{\frac{7}{4}}}{\ln 2x} + O\left(\frac{x^{\frac{7}{4}}}{\ln^2 2x}\right),$$

$$\sum_{n\le x} S(n)b(n) = \frac{8\sqrt[4]{2}\pi^2}{63(r-2)^{\frac{5}{4}}}\cdot\frac{x^{\frac{7}{4}}}{\ln 2x}+O\left(\frac{x^{\frac{7}{4}}}{\ln^2 2x}\right).$$

R. Xie, L. Gao and Q. Zhao [42]. *Define* $q_d(n)=\prod_{\substack{d|n\\ d<n}}$. *Let* $k\ge 1$ *be any fixed positive integer. For any real* $x>1$ *we have*

$$\sum_{n\le x}\left(S\left(q_d(n)\right)-\left(\frac{1}{2}d(n)-1\right)P(n)\right)^2=\sum_{i=1}^{k}c_i\frac{x^{\frac{3}{2}}}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,2,\cdots,k)$ *are computable constants, and*

$$c_1=\frac{3}{2}\cdot\frac{\zeta^4\left(\frac{3}{2}\right)}{\zeta(3)}-2\zeta^2\left(\frac{3}{2}\right)+\frac{2}{3}\zeta\left(\frac{3}{2}\right).$$

B. Li, J. Guo and H. Dong [18]. *Define*

$$U(n)=\begin{cases}1, & \text{if } n=1,\\ \max\limits_{1\le i\le r}\{\alpha_1p_1,\alpha_2p_2,\cdots,\alpha_rp_r\}, & \text{if } n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.\end{cases}$$

Let $k\ge 2$ *be any fixed positive integer. For any real* $x\ge 3$ *we have*

$$\sum_{n\le x}\left(S\left(a_k(n)\right)-(k-1)\,U(n)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)k^2\cdot\frac{x^{\frac{3}{2}}}{\ln x}+O\left(\frac{x^{\frac{11}{6}}}{\ln^2 x}\right).$$

J. Bai and W. Huang [1]. *Let* $\mathcal{A}$ *denote the set of the simple numbers. Let* $k\ge 2$ *be any fixed positive integer. For any real* $x\ge 2$ *we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} S^k(n)=\frac{Bx^{k+1}}{(k+1)\ln x}+\sum_{i=2}^{k}\frac{C_ix^{k+1}}{\ln^i x}+O\left(\frac{x^{k+1}}{\ln^{k+1}x}\right),$$

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{1}{S(n)}=D\ln\ln x+\frac{E\sqrt{x}\ln\ln x}{\ln x}+O\left(\frac{\sqrt{x}}{\ln x}\right),$$

where $B,D,E,$ C_i $(i=2,3,\cdots,k)$ *are computable constants.*

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