Using "Enhanced Quantization" to bound the Cosmological constant, (for a bound-on graviton mass), by comparing two action integrals(one being from general relativity) at the start of inflation

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Abstract

We are looking at comparison of two action integrals and we identify the Lagrangian multiplier as setting up a constraint equation (on cosmological expansion). This is a direct result of the fourth equation of our manuscript which unconventionally compares the action integral of General relativity with the second derived action integral, which then permits equation 5, which is a bound on the Cosmological constant. What we have done is to replace the Hamber Quantum gravity reference-based action integral with a result from John Klauder's "Enhanced Quantization" . In doing so, with Padamabhan's treatment of the inflaton, we then initiate an explicit bound upon the cosmological constant. The other approximation is to use the inflaton results and conflate them with John Klauder's Action principle for a way to, if we have the idea of a potential well, generalized by Klauder, with a wall of space time in the Pre Planckian-regime to ask what bounds the Cosmological constant prior to inflation. And, get an upper bound on the mass of a graviton. We conclude with a redo of a multiverse version of the Penrose cyclic conformal cosmology to show how this mass of a heavy graviton is consistent from cycle to cycle. All this is possible due to equation 4. And we compare all this with results of reference [1] in the conclusion.

Key words : Inflaton, action integral, Cosmological Constant, Penrose cyclic cosmology

1. Basic idea, can two First Integrals give equivalent information?

We admit this paper has some similarity to [1], what we will do is instead of using the Hamber result of [2] as to a first integral we are instead using what John Klauder wrote in [3]as to form a first integral in order to make a 1 to 1 equivalence with the first integral associated with general relativity [4], [5] As what was done in [1] we have a 1 to 1 relationship between two first action integrals, i.e. and the idea is to avoid a point cosmic singularity, but to instead have a regime of space-time incorporating the idea of a cosmic bounce, as given in [6] with interior and exterior regimes, i.e. this also over laps with work done by the author in [7] with the caveat that there is a barrier between interior and exterior regimes of space-time and that we are evaluating the space in the interior of a space-time bubble. Having said that. The Integrands in the two integrals are assumed to have a 1-1 and onto relationship to one another. And we will in the next section identify the two first integrals.

2. **Now for the General Relativity First integral. From [1]**

We use the Padmanabhan 1st integral [8] of the form , with the third entry of Eq. (1) having a Ricci scalar defined via [9] and usually the curvature $\aleph$ set as extremely small, with the general relativity version of , from [1]

$$
\begin{aligned}
& S_1 = \frac{1}{2\kappa}\int\sqrt{-g}\cdot d^4x\cdot\left(\Re - 2\Lambda\right) \\
& \& - g = -\det g_{uv} \\
& \& \Re = 6\cdot\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\aleph}{a^2}\right)
\end{aligned}
\qquad (1)
$$

Also, the variation of $\delta g_{tt} \approx a_{\min}^2 \phi$ as given by [10, 11] will have an inflaton, ϕ given by [9]

3.

$$
\begin{aligned}
& a \approx a_{\min} t^{\gamma} \\
& \Leftrightarrow \phi \approx \sqrt{\frac{\gamma}{4\pi G}}\cdot\ln\left\{\sqrt{\frac{8\pi G V_0}{\gamma\cdot(3\gamma-1)}}\cdot t\right\} \\
& \Leftrightarrow V \approx V_0\cdot\exp\left\{-\sqrt{\frac{16\pi G}{\gamma}}\cdot\phi(t)\right\}
\end{aligned}
\qquad (2)
$$

Leading to [1, 9] to the inflaton which is combined into other procedures for a solution to the cosmological constant problem.

$$
\phi \approx \sqrt{\frac{\gamma}{4\pi G}}\cdot\ln\left\{\sqrt{\frac{8\pi G V_0}{\gamma\cdot(3\gamma-1)}}\cdot t\right\} \qquad (3)
$$

Here, we have that $a_{\min}$ is a minimum value of the scale factor presumably given by [12] as a tiny but non-zero value. Or at least a quantum bounce as given by [1]

3. Next for the idea from Klauder

We are going to go to page 78 by Klauder [3] as to his idea of what he calls on page 78 a restricted Quantum action principle which he writes as: S_2 where we then write a 1-1 equivalence as in [1] so that

$$
\begin{aligned}
& S_2 = \int_0^T dt\cdot\left[p(t)\dot{q}(t) - H_N\left(p(t), q(t)\right)\right] \\
& \approx S_1 = \frac{1}{2\kappa}\int\sqrt{-g}\cdot d^4x\cdot\left(\Re - 2\Lambda\right)
\end{aligned}
\qquad (4)
$$

Our assumption is that Λ is a constant, hence we assume then the following, i.e. a Pre Planckian-instant of time, say some power of Planck Time length, hence getting the following approximation

$$\Lambda \approx \frac{-\left[p(\tilde{t})\dot{q}(\tilde{t}) - H_N\left(p(\tilde{t}), q(\tilde{t}) \right) \right]}{\frac{1}{\kappa}\int \sqrt{-g} \cdot d^3x} + \frac{\frac{1}{2\kappa}\int \sqrt{-g} \cdot d^3x \cdot \left(\Re = 6 \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{\aleph}{a^2} \right) \right) \Big|_{t=\tilde{t}}}{\frac{1}{2\kappa}\int \sqrt{-g} \cdot d^3x} \qquad (5)$$

4. **Filling in the details of the above using details from [3] with explanations**

To do this, we are making several assumptions.

a. That the two mentioned integrals are evaluated from a Pre Planckian to Planckian space-time domain. i.e. in the same specified integral of space-time. $S_2 \approx S_1$

b. That in doing so, the Universe is assumed to avoid the so called cosmic singularity. In doing so assuming a finite "Pre Planckian to Planckian" regime of space time like that given in [1]. With reference also, to the cosmic bounce given in [7]

c. assuming that even in the Pre Planck-Planck regime that curvature $\aleph$ will be a very small part of Ricci scalar $\Re$ and that to first approximation even in the Plank time regime, that to first order [13] has a value altered to be

$$\Re = 6 \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{\aleph}{a^2} \right) \sim 6 \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right) \qquad (6)$$

Furthermore, we can make assumptions as to the nature of the cosmic bubble, in assuming that there is a barrier between the Pre-Planckian to Planckian physics regimes so that we have a quantum mechanical style potential well, so to speak in evaluation of the [7] reference which has then if we use Klauder's [3] notation that N represents the strength of the wall, i.e. the Pre Planckian to Planckian bubble boundary

$$\frac{p_0^{\ 2}}{2} = \frac{p_0^{\ 2}(N)}{2} + N; \quad for \quad 0 < N \le \infty$$
$$q = q_0 \pm p_0 t$$
$$V_N(x) = 0; \quad for \quad 0 < x < 1 \qquad (7)$$
$$V_N(x) = N; \quad otherwise$$
$$H_N\left(p(t), q(t)\right) = \frac{p_0^2}{2} + \frac{(\hbar \cdot \pi)^2}{2} + N; \quad for \quad 0 < N \le \infty$$

Our innovation is to then equate $q = q_0 \pm p_0 t \sim \phi$ and to assume small time step values. Then

$$\Lambda \approx \frac{-\left[\frac{V_0}{3\gamma - 1} + 2N + \frac{\gamma \cdot (3\gamma - 1)}{8\pi G \cdot \tilde{t}^2}\right]}{\frac{1}{\kappa}\int \sqrt{-g} \cdot d^3x} + \frac{\int \sqrt{-g} \cdot d^3x \cdot \left(6 \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right)\Big|_{t=\tilde{t}}}{\int \sqrt{-g} \cdot d^3x}$$
$$\sim \frac{-\left[\frac{V_0}{3\gamma - 1} + 2N + \frac{\gamma \cdot (3\gamma - 1)}{8\pi G \cdot \tilde{t}^2}\right]}{\frac{1}{\kappa}\int \sqrt{-g} \cdot d^3x} + \left(6 \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right)\Big|_{t=\tilde{t}} \qquad (8)$$

These are terms within the bubble of space-time given in [7] using the same inflaton potential. The scale factor is presumed here to obey the value of the scale factor given in [12]

5. Why this is linked to gravity/massive gravitons

Klauder's program is to isolate a regime of space time for a proper canonical quantization of a classical system. i.e. what we did is to utilize the ideas of [3] to make the identification of Eq. (7) which when combined with inflaton physics to have enhanced quantization of the often assumed to be classical inflaton, as given in Eq.(3). I.e. to embed via Eq.(7) as a quantum mechanical well for a Pre Planckian-system for inflaton physics as given by Eq. (3). In short, the scaling of our problem for a bound as to the cosmological constant, in Pre Planckian-space-time, as given in Klauder's treatment of the action integral as of page 87 of [3] where Klauder talks of the weak correspondence principle, where an enhanced classical Hamiltonian, is given 1-1 correspondence with quantum effects, in a non-vanishing fashion.

I.e. for the sake of Argument we will make the following assumptions which may be debatable, i.e.

$\sqrt{-g}$ is approximately a constant (9)

For extremely small-time intervals (in the boundary between Pre Planckian to Planckian physical boundary regime). As given in [11]. This approximation is why the author assumes Eq. (9).

$$g_{tt} \sim \delta g_{tt} \approx a_{\min}^2 \phi \qquad (10)$$

If so, if we through this procedure, make a linkage directly to the mass of a graviton, as given by Novello, [13]

$$m_g = \frac{\hbar \cdot \sqrt{\Lambda}}{c} \qquad (11)$$

This is a way, then to ascertain a bound, based upon the early universe conditions so set forth, as a way to ascertain a bound to the effective heavy graviton

6. Conclusion, reviewing multiverse generalization of the CCC of Penrose, and suggestions as to a uniform bound to the Graviton, per cyclic conformal cosmology cycle, and how this relates to reference [1]s conclusions

We are extending Penrose's suggestion of cyclic universes, black hole evaporation, and the embedding structure our universe is contained within, This multiverse embeds BHs and may resolve what appears to be an impossible dichotomy. The following is largely taken from [14[and has serious relevance to the final part of the conclusion .That there are no fewer than N universes undergoing Penrose 'infinite expansion' (Penrose) [15] contained in a mega universe structure. Furthermore, each of the N universes has black hole evaporation, with the Hawking radiation from decaying black holes. If each of the N universes is defined by a partition function, called $\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}$, then there exist an information ensemble of mixed minimum information correlated as about $10^7 - 10^8$ bits of information per partition function in the set $\left.\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}\right|_{before}$, so minimum information is conserved between a set of partition functions per universe

$$\left.\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}\right|_{before} \equiv \left.\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}\right|_{after} \qquad (`12)$$

However, there is non-uniqueness of information put into each partition function $\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}$.

Furthermore Hawking radiation from the black holes is collated via a strange attractor collection in the mega universe structure to form a new big bang for each of the N universes represented by $\left\{\Xi_i\right\}_{i\equiv N}^{i\equiv 1}$. Verification of this mega structure compression and expansion of information with a non-uniqueness of information placed in each of the N universes favors ergodic mixing treatments of initial values for each of N universes expanding from a singularity beginning. The n_f value, will be using (Ng, 2008) $S_{entropy} \sim n_f$. [16] . How to tie in this energy expression, as in Eq.(12) (30) will be to look at the formation of a nontrivial gravitational measure as a new big bang for each of the N universes as by $n(E_i)$. the density of states at a given energy E_i for a partition function. **(**Poplawski**, 2011) [17]**

$$\left\{ \Xi_i \right\}_{i \equiv 1}^{i \equiv N} \propto \left\{ \int_0^\infty dE_i \cdot n(E_i) \cdot e^{-E_i} \right\}_{i \equiv 1}^{i \equiv N} . \tag{13}$$

Each of E_i identified with Eq.(13) above, are with the iteration for N universes (Penrose, 2006)[15] Then the following holds, namely, this is taking a nod to the unpredictability of black hole physics, as given in [18] by Hawkings, by asserting the following claim to the universe, as a mixed state, with black holes playing a major part, due to the CCC cosmological picture, by starting off with

Claim 1,

$$\frac{1}{N} \cdot \sum_{j=1}^{N} \Xi_j \Big|_{j-before-nucleation-regime} \xrightarrow[vacuum-nucleation-tranfer]{} \Xi_i \Big|_{i-fixed-after-nucleation-regime} \tag{14}$$

For N number of universes, with each $\Xi_j \Big|_{j-before-nucleation-regime}$ for j = 1 to N being the partition function of each universe just before the blend into the RHS of Eq. (14) above for our present universe. Also, each of the independent universes given by $\Xi_j \Big|_{j-before-nucleation-regime}$ are constructed by the absorption of one to ten million black holes taking in energy**. I.e. (Penrose)** [14,15]. Furthermore, the main point is similar to what was done in [19] in terms of general ergodic mixing

Claim 2

$$\Xi_j \Big|_{j-before-nucleation-regime} \approx \sum_{k=1}^{Max} \tilde{\tilde{\Xi}}_k \Big|_{black-holes-jth-universe} \tag{15}$$

What is done in **Claim 1 and Claim 2** is to come up with a protocol as to how a multi dimensional representation of black hole physics enables continual mixing of spacetime [19] largely as a way to avoid the Anthropic principle, as to a preferred set of initial conditions.

Claim 2 is particularly important. The idea here is to use what is known as CCC cosmology, which can be thought of as the following. First. Have a big bang (initial expansion) for the universe. After redshift z = 10, a billion years ago, SMBH formation starts. Matter- energy is vacuumed up by the SMBHs, which at a much later date than today (present era) gather up all the matter-energy of the universe and recycles it in a cyclic conformal translation, as follows, namely

$$E = 8\pi \cdot T + \Lambda \cdot g$$
$$E = source \quad for \quad gravitational \quad field$$
$$T = mass \quad energy \quad density \tag{16}$$
$$g = gravitational \quad metric$$

$$\Lambda = vacuum \quad energy, rescaled \quad as \quad follows$$

$$\Lambda = c_1 \cdot [Temp]^{\beta} \tag{17}$$

C1 is , here a constant. Then

The main methodology in the Penrose proposal has been in Eq. (17) evaluating a change in the metric g_{ab} by a conformal mapping $\hat{\Omega}$ to

$$\hat{g}_{ab} = \hat{\Omega}^2 g_{ab} \tag{18}$$

Penrose's suggestion has been to utilize the following[18], [20]

$$\hat{\Omega} \xrightarrow[ccc]{} \hat{\Omega}^{-1} \tag{19}$$

In fall into cosmic black hopes has been the main mechanism which the author asserts would be useful for the recycling apparent in Eq(19) above with the caveat that $\hbar$ is kept constant from cycle to cycle as represented by

$$\hbar_{old-\text{cosmology}-cycle} = \hbar_{present-\text{cosmology}-cycle} \tag{20}$$

We claim that Eq. (20) combined with Eq. (11) above gives a good indication of a uniform mass to a graviton, per cycle, as far as heavy gravity, provided that Eq. (20) holds'

Note that all these above results should be compared with the initial Hamber based results [2] which lead to an initial idea we give as given in [1] which we duplicate below, i.e. we claim we have kept full fidelity with this program and improved on it. Quoting from [1] :

First of all, we have what is known as a scale factor $a(t)$. Which is nearly zero, in the Pre Planckian regime of space-time. And equal to 1 in the present era. A good reference as to the physics behind how we set up $a(t)$ is [20,21] . In addition we will define, for the purpose of analysis, of the integrals, the following symbols as given in [2], for the Quantum paths sensitive first integral, with

$$\int dt\sqrt{g_{tt}}V_3(t) = V_4(t) \sim 8\pi^2 r^4 / 3$$
$$\& V_3(t) = 2\pi^2 a(t)^3 / 3 \qquad (21)$$
$$\& k_2 = 9\left(2\pi^2\right)^{2/3}$$

These are the purported volume elements of the [2] first integral. The second first integral is using the usual GR inputs as defined by Padmanbhan in [8,9]. To review what is meant by first integrals we refer the readers to [22, 23, 24]. Roughly put, a Lagrangian multiplier invokes a constraint of how a "minimal surface" is obtained by constraining a physical process so as to use the idea of [22, 23, 24] which invokes the idea of minimization of a physical processes. In the case of [23], the minimization process is implicitly that, if $a(t)$ were a scale factor as defined by Roos, [20] and if g_{tt} were a time component of a metric tensor, which we will later define . Here, the subscripts 3 and 4 in the volume refer to 3 and 4 dimensional spatial dimensions, and this will lead to us writing, via [2] a 1st integral as defined by [1, 2], in the form , if G is the gravitational constant, that if we have following [1, 2], a first integral defined by

$$S_1 = \frac{1}{24\pi G}\cdot\left(\int dt\sqrt{g_{tt}}\left(\frac{g_{tt}\dot{V}_3^2}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t)\right)\right) \qquad (22)$$

This should be compared against the Padmabhan 1st integral [8, 9] of the form , with the third entry of Eq. (3) having a Ricci scalar defined via [5] and usually the curvature $\aleph$ [5] set as extremely small, with the general relativity version of

$$S_2 = \frac{1}{2\kappa}\int\sqrt{-g}\cdot d^4x\cdot(\Re - 2\Lambda)$$
$$\& -g = -\det g_{uv} \qquad (23)$$
$$\& \Re = 6\cdot\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\aleph}{a^2}\right)$$

End of quote , from [1]

Our presentation uses all this, and aligns it with the ideas of the Klauder Enhanced quantization [3] for what we think is a better extension of the same idea. We claim that what we have done improves upon this idea, and is in full fidelity with the FFP 15 presentation, with an additional refinement added in. In [1], we make the following argument, namely. We reference from [1], Quote

In order to obtain maximum results, we will be stating that the following will be assumed to be equivalent.

$$\sqrt{g_{tt}}\left(\frac{g_{tt}\dot{V}_3^2}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t)\right) \sim \frac{1}{2\kappa}\int\sqrt{-g}\cdot d^3x\cdot\left(6\cdot\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) - 2\Lambda\right) \qquad (24)$$

i.e.

$$\sqrt{g_{tt}}\left(\frac{g_{tt}\dot{V}_3^2}{V_3(t)}+k_2V_3^{1/3}(t)\right)\sim\frac{1}{2\kappa}\int\sqrt{-g}\cdot d^3x\cdot\left(6\cdot\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right)\right) \qquad (25)$$

And

$$\sqrt{g_{tt}}\left(\lambda V_3(t)\right)\sim\frac{1}{2\kappa}\int\sqrt{-g}\cdot d^3x\cdot(2\Lambda) \qquad (26)$$

End of quote from [1]. So, we argue that we are , as given in [1] where we have, from [1] the following : Quote again from [1],

Simply put a relationship of the Lagrangian multiplier giving us the following:

$$\lambda\sim\frac{1}{\kappa}\sqrt{\frac{-g}{\left(\delta g_{tt}\approx a_{\min}^2\phi\right)}}\cdot\Lambda \qquad (27)$$

End of quote from [1]. We are obtaining the exact same physics, as in [1] for when we appeal to Eq. (8) as a bound to the enhanced quantization, hence we have extended our basic idea via use of [1] and [3]

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