Riemann's Analytic Continuation of $\zeta(s)$ Contradicts the Law of the Excluded Middle, and is Derived by Using Cauchy's Integral Theorem While Contradicting the Theorem's Prerequisites

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Abstract

The Law of the Excluded Middle holds that either a statement X or its opposite $\overline{X}$ is true, thereby excluding the state in which both are true. In Boolean algebra form, $Y = X \,\mathrm{XOR}\, \overline{X}$, wherein $X, \overline{X}, Y \in \{0, 1\}$.

Riemann's analytic continuation of $\zeta(s)$ contradicts the Law of the Excluded Middle. The Dirichlet series $\zeta(s)$ is proven divergent ("X") in the half-plane $\mathrm{Re}(s) \leq 1$. Riemann's $\zeta(s)$ claims to be convergent ("$\overline{X}$") at all s, including at $\mathrm{Re}(s) \leq 1$ (except $s = 1$). Since convergence and divergence are opposites ("X" XOR "$\overline{X}$"), therefore the Law of the Excluded Middle, when combined with the proven divergence of the Dirichlet series at $\mathrm{Re}(s) \leq 1$, disproves Riemann's claim.

Further inspection of the derivation of Riemann's analytic continuation of $\zeta(s)$ shows that it is false. The derivation uses a corollary of the Cauchy integral theorem to equate a function comprising a logarithm (and having its branch cut as the domain), to the same function (but having a Hankel-type contour surrounding the branch cut as the domain). But neither path satisfies the prerequisites of the corollary. So the equality is false, and thus the derivation is false.

Another example of analytic continuation does not contradict the Law of the Excluded Middle. When the "unit disc" method is applied to the Taylor series of $f(s) = 1/(1 - s)$, it is not used to claim convergence of $f(s)$ where $f(s)$ is known to be divergent (i.e. at $s = 1$, where divergence is easily proven due to division by zero).

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1 Introduction

The Law of the Excluded Middle holds that either a statement X or its opposite $\overline{X}$ is true, thereby excluding the possibility that both are true. In Boolean algebra form, $Y = X \,\mathrm{XOR}\, \overline{X}$, wherein $X, \overline{X}, Y \in \{0, 1\}$.

Riemann's analytic continuation of $\zeta(s)$ contradicts the Law of the Excluded Middle. The Dirichlet series $\zeta(s)$ is proven divergent (statement "X") in the half-plane $\mathrm{Re}(s) \leq 1$. Riemann's $\zeta(s)$ claims to be convergent (statement "$\overline{X}$") at all s, including at $\mathrm{Re}(s) \leq 1$ (except $s = 1$). Since convergence and divergence are two mutually exclusive states ("X" XOR "$\overline{X}$"), thus the Law of the Excluded Middle, when combined with the proven divergence of the Dirichlet series at $\mathrm{Re}(s) \leq 1$, disproves Riemann's claim.

Further inspection of the derivation of Riemann's analytic continuation of $\zeta(s)$ shows that it is false. The derivation uses a corollary of the Cauchy integral theorem to equate a function comprising a logarithm (and having its branch cut as the domain), to the same function (but having a Hankel-type contour surrounding the branch cut as the domain). But neither path satisfies the prerequisites of the corollary. So the equality is false, and thus the derivation is false.

Also, if Riemann's claim were true, the result would contradict a basic law of arithmetic: that a sum of positive real numbers is greater than each summand. If a and b are positive real numbers, then $a + b > a$ and $a + b > b$. This is because as positive real numbers, $a > 0$ and $b > 0$. So if we add b to both sides of $a > 0$, we get $a + b > b$. Likewise, if we add a to both sides of $b > 0$, we get $a + b > b$. In contrast, Riemann's analytic continuation holds that $\zeta(0) = -1/2$ at $s = 0$. (See Tao[Tao]). This result contradicts the rule that a sum of positive real numbers is greater than each summand.

Finally, another example of analytic continuation does not contradict the Law of the Excluded Middle. When the "unit disc" method is applied to the Taylor series of $f(s) = 1/(1 - s)$, it is not used to claim convergence of $f(s)$ where $f(s)$ is known to be divergent (i.e. at $s = 1$, where divergence is easily proven due to division by zero).

2 The Law of the Excluded Middle

According to the Law of the Excluded Middle, one of the two statements "X" and "not X" ("$\overline{X}$") is true. (See Plisko [Pli], and Stanford [Dep]). In Boolean algebra form, $Y = X \,\mathrm{XOR}\, \overline{X}$, wherein $X, \overline{X}, Y \in \{0, 1\}$. This "law" of classical logic dates back to Aristotle, and is accepted by most mathematicians.

A minority of mathematicians, who belong to the Intuitionist movement founded by L.E.J. Brouwer, is critical of the use of the Law of the Excluded Middle. According to Plisko [Pli], the Intuitionists accept the use of the Law of the Excluded Middle when one of the statements "X" or "not X" has been proved. But according to Stanford [Dep] "[I]ntuitionistic logic is logic without the law of excluded middle."

This article attempts to placate the former group of Intuitionists by first proving that the Dirichlet series of $\zeta(s)$ is divergent in the half-plane $\mathrm{Re}(s) \leq 1$. (The latter group will reject this paper outright regardless).

The first proofs show that the Dirichlet series is divergent along the half-line $s \leq 1$, wherein s is a real number ($s \in \mathbb{R}$). This is then extended to the half-plane $\mathrm{Re}(s) \leq 1$ wherein s is a complex number ($s \in \mathbb{C}$). So the Dirichlet series $\zeta(s)$ is proven divergent (statement "X") in the half-plane $\mathrm{Re}(s) \leq 1$.

In contrast, Riemann's analytic continuation of $\zeta(s)$ claims to be a version of $\zeta(s)$ that is convergent (statement "$\overline{X}$") at all s, except at $s = 1$. The Law of the Excluded Middle holds that $\zeta(s)$ is either X or $\overline{X}$ at $\mathrm{Re}(s) \leq 1$, except at $s = 1$. Because the Dirichlet series $\zeta(s)$ is proven divergent at $\mathrm{Re}(s) \leq 1$, Riemann's analytic continuation of $\zeta(s)$ is false there, except at $s = 1$.

3 Divergence of the Dirichlet series Along Half-Line s =< 1, for Real values of "s"

The following are simple well-known proofs that the Dirichlet series $\zeta(s)$ is divergent (X) at values of $s \leq 1$, wherein $s \in \mathbb{R}$. When graphed on the complex plane, this region corresponds to the half line $(\sigma \leq 1, t = 0)$. At $s = 0$, the Dirichlet series is

$$\zeta(0) = \sum_{n=1}^{M} n^{-0}$$

and thus $\zeta(0) = 1^0 + 2^0 + 3^0 + \ldots + M^0$. So $\zeta(0) = M$, and as $M \to \infty$, thus $\zeta(0) \to \infty$. Conceptually, at $s = 0$ the series is an infinite sum of ones. At $s = -1.5$, the Dirichlet series is

$$\zeta(-1.5) = \sum_{n=1}^{M} n^{1.5}$$

and thus $\zeta(-1.5) = 1^{1.5} + 2^{1.5} + 3^{1.5} + \ldots + M^{1.5}$. Because $1^{1.5} < 2^{1.5} < 3^{1.5} < \ldots < M^{1.5}$, as $M \to \infty$, thus $\zeta(-1.5) \to \infty$. This proof holds true for all values of s wherein $s \in \mathbb{R}$ and $s < 0$. Conceptually, at $s < 0$ the series is an infinite sum of successively larger terms, with each term being larger than all of its predecessors.

In regards to the Dirichlet series at values of s in the "critical strip" $0 < s \leq 1$, it is divergent as shown by the integral test for convergence (a.k.a. the Maclaurin–Cauchy test). For example, at $s = 0.5$, according to the integral test of $\zeta(0.5)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.5}} \geq \int_{1}^{M} \frac{1}{n^{0.5}} = \int_{1}^{M} n^{-0.5} = \left(2 \cdot n^{0.5} + C\right)\Big|_{1}^{M} = \left(2 \cdot M^{0.5} - 2\right)$$

and as $M \to \infty$, thus $(2 \cdot M^{0.5} - 2) \to \infty$. Conceptually, at $0 < s \leq 1$ the Dirichlet series is an infinite sum of successively smaller terms, with each term being smaller than its predecessors, but the sum continues to grow with each new term, never approaching a limit. The examples above, in combination, prove that $\zeta(s)$ is divergent at $s \leq 1$, wherein $s \in \mathbb{R}$.

4 Extending the Divergence of the Dirichlet Series from Real Values of "s" to Complex Values

The divergence of the Dirichlet series in $s \leq 1$, $s \in \mathbb{R}$, can be extended to all values in $\mathrm{Re}(s) \leq 1$, $s \in \mathbb{C}$. This is shown by deriving a trigonometric version of the Dirichlet series $\zeta(s)$, by substituting Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ into the formula, using the definition $s = \sigma + it$. Since $\zeta(s) = \sum n^{-s}$, therefore $\zeta(s) = \sum n^{-\sigma} n^{-it}$, and $n^{-it} = \exp(\ln(n^{-it}))$. The laws of exponents and logarithms say that $x = \exp(\ln(x))$ and that $\ln(x^y) = y\ln(x)$, so $n^{-it} = \exp(-it \cdot \ln(n))$, the result is:

$$n^{-it} = e^{-it\cdot\ln(n)} = \cos\Big(-t\cdot\ln(n)\Big) + i\sin\Big(-t\cdot\ln(n)\Big) \tag{4.1}$$

and then $\zeta(s) = \sum n^{-s}$ can be rewritten as

$$\zeta(s) = \sum_{n=1}^{\infty}\Big[n^{-\sigma}\cdot\Big(\cos\big(t\cdot\ln(n)\big) - i\sin\big(t\cdot\ln(n)\big)\Big)\Big] \tag{4.2}$$

This trigonometric version of the Dirichlet series can be separated into real and imaginary components:

$$\mathrm{Re}\Big[\zeta(\mathrm{s})\Big] = \sum_{n=1}^{\infty}\Big[n^{-\sigma}\cdot\cos\Big(t\cdot\ln(n)\Big)\Big] \tag{4.3}$$

$$\mathrm{Im}\Big[\zeta(\mathrm{s})\Big] = -i\cdot\sum_{n=1}^{\infty}\Big[n^{-\sigma}\cdot\sin\Big(t\cdot\ln(n)\Big)\Big] \tag{4.4}$$

The real and imaginary portions of trigonometric $\zeta(s)$ are divergent if and only if $\sigma \leq 1$. If $\sigma > 1$, then $\sum_{n=1}^{\infty} n^{-\sigma}$ is convergent. The sine and cosine factors, with their oscillating positive and negative half-cycles, make it converge faster. However, if $\sigma \leq 1$, then $\sum n^{-\sigma}$ is divergent, and the sine and cosine functions do not result in convergence.

More specifically, in the strip $0 < \sigma \leq 1$, wherein $t \neq 0$, both Abel's lemma and Dirichlet's test for convergence incorrectly determine that this trigonometric version of the Dirichlet series is convergent. This is because $\sum n^{-\sigma}$ has monotonically decreasing terms there, and both sine and cosine are bounded functions. However, both Abel's lemma and Dirichlet's test fail to envision a divergent series with monotonically decreasing terms, multiplied by a bounded sine or cosine function having a logarithm function nested therein. The nested logarithm function results in half-cycles with ever-increasing half-period durations, so the end result of the product with the terms of the divergent series $\sum n^{-\sigma}$ is a series that oscillates between diverging to $+\infty$ and to $-\infty$.

5 Riemann's Analytic Continuation of Zeta(s) Contradicts the Law of the Excluded Middle

Contradicting the Dirichlet series, Riemann's analytic continuation of $\zeta(s)$ claims to be convergent at all $s \in \mathbb{C}$, except $s = 1$. (See Edwards, [Edw01], page 11; and Riemann [Rie59], page 1). This claim includes all real values of s, except $s = 1$. (Note that both the Dirichlet series $\zeta(s)$ and Riemann's analytic continuation of $\zeta(s)$ agree on divergence at $s = 1$, and on convergence at $s > 1$).

This claim contradicts the above proofs of the Dirichlet series $\zeta(s)$ at $s < 1$, wherein $s \in \mathbb{R}$, and wherein $s \in \mathbb{C}$. So can $\zeta(s)$ be <u>both</u> convergent <u>and</u> divergent at these values of s? Or is Dirichlet series $\zeta(s)$ true at these values of s, and Riemann's analytic continuation is false? Or vice versa?

According to the Law of the Excluded Middle, only one of the two statements "X" and "not X" can be true at any value of s. In Boolean notation: $Y = X \text{ XOR } \overline{X}$, wherein $X, \overline{X}, Y \in \{0, 1\}$). Since convergence and divergence are opposite states, $\zeta(s)$ cannot simultaneously be both. At each value of s, the function $\zeta(s)$ is either one or the other. So throughout $s < 1$, regardless of whether $s \in \mathbb{R}$ or $s \in \mathbb{C}$, either the proofs of the divergence of Dirichlet series $\zeta(s)$ are true, or Riemann's analytic continuation is true and $\zeta(s)$ is convergent.

Since the proofs of the divergence of Dirichlet series $\zeta(s)$ are true throughout half-plane Re(s) ≤ 1, Riemann's analytic continuation is false and $\zeta(s)$ cannot be convergent there.

6 The Error in the Derivation of Riemann's Zeta Function

Riemann's analytic continuation of the Dirichlet series $\zeta(s) = \sum n^{-s}$ is derived by determining the value of the following Hankel-type contour, wherein $s, z \in \mathbb{C}$. (See Whittaker [WW20], page 266):

$$\zeta(s,a) = \int_{+\infty}^{\lambda} \frac{(-z)^{s-1} \cdot e^{-az}}{(1-e^{-z})} \cdot dz + \int_{|z|=\lambda} \frac{(-z)^{s-1} \cdot e^{-az}}{(1-e^{-z})} \cdot dz + \int_{\lambda}^{+\infty} \frac{(-z)^{s-1} \cdot e^{-az}}{(1-e^{-z})} \cdot dz$$

However, the factor $f_1(s,z) = (-z)^{s-1}$ in each of the three integrands can be rewritten, using the laws of logarithms, as $f_1(s,z) = \exp[(s-1) \cdot \log(-z)]$. The logarithmic function $f_2(s,z) = \log(-z)$, which is a factor of $f_1(s,z)$, is undefined on the nonnegative real axis (the "branch cut" of $\log(-z)$). Therefore, "strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from $+\infty$ to 0 and slightly below the real axis as it goes from 0 back to $+\infty$". (See Edwards [Edw01], page 10). This solution to the path of integration issue is copied from Hankel's derivation of $\Gamma(s)$. (See Whittaker [WW20], pages 244-245 and 266).

But what is the logical basis for replacing the branch cut with the Hankel-type contour as the domain of the function? On what basis can we assume that these two inputs result in the same output, especially given that the logarithm function is <u>undefined</u> at all values of s on the branch cut?

Whittaker [WW20] (see pages 87 and 244) states that the Cauchy integral theorem's path equivalence corollary is the basis for equating (as domains of a logarithm function) the Hankel-type contour to the branch cut. However, both the Hankel-type contour and the branch cut contradict the prerequisites of the Cauchy integral theorem's corollary.

More specifically, Cauchy's integral theorem states that if $f(z)$ is a function of complex variable z, if $f(z)$ is holomorphic at all points on a simple closed curve ("contour") C, and if $f(z)$ is holomorphic at all points inside the contour, then the contour integral of $f(z)$ is equal to zero:

$$\int_{(C)} f(z) \cdot dz = 0 \tag{6.1}$$

(See Whittaker [WW20], page 85).

The path equivalence corollary of Cauchy's integral theorem states that if there exist two points z_0 and Z in the complex domain, connected by two distinct paths z_0AZ and z_0BZ, and if $f(z)$ is a function of complex variable z that is holomorphic at all points on these two paths, and holomorphic at all points enclosed by these two paths, then the line integral between the two points z_0 and Z in this region has the same value, regardless of whether the path of integration is z_0AZ, or z_0BZ, or any path between z_0AZ and z_0BZ. (See Whittaker [WW20], page 87, Corollary 1).

However, the example of the Hankel-type integral falls on the branch cut of the function $f(z) = \ln(-z)$. Therefore $f(z)$ is undefined at all of the non-positive values of z on the branch cut. It is not possible to calculate a derivative at a point on a curve where the curve is undefined, so no point on the branch cut is holomorphic. Therefore the path equivalence corollary of Cauchy's integral theorem cannot be applied to the branch cut, and thus cannot be used to equate the branch cut to any other path.

Moreover, the Hankel-type contour is open (not closed), and thus inapplicable for the Cauchy integral theorem. Even if we assume that the Hankel-type contour is indeed closed at $+\infty$ on the branch cut (as described in Whittaker [WW20], page 245), then it would enclose the entire branch cut (which consists of non-holomorphic points). There would also be a non-holomorphic point on the Hankel-type contour, at the point where it connects to the real axis at $+\infty$. For this additional reason it is improper to use the Cauchy integral theorem's path equivalence corollary to equate the branch cut to the Hankel-type open contour. So Riemann's analytic continuation of the $\zeta(s)$ is not valid.

7 Analytic Continuation of Zeta(s) Contradicts Basic Law of Arithmetic

All of the evidence indicates that Riemann's analytic continuation is false. The divergence proofs of the Dirichlet series $\zeta(s)$ at $s < 1$, $s \in \mathbb{R}$ are consistent with other principles of mathematics. But Riemann's analytic continuation is not. For example, as discussed above, at $s = 0$, the Dirichlet series is $\zeta(0) = \sum_{n=1}^{M} n^{-0}$, and thus $\zeta(0) = 1^0 + 2^0 + 3^0 + \ldots + M^0$. So $\zeta(0) = M$, and as $M \to \infty$, thus $\zeta(0) \to \infty$.

This result is consistent with the rule of arithmetic that a sum of positive real numbers is greater than each summand. If a, b, c, ... n are positive real numbers, then $a > 0$, $b > 0$, $c > 0$, ...$n > 0$. So if we add b to both sides of $a > 0$, we get $a + b > b$. Likewise, if we add a to both sides of $b > 0$, we get $a + b > a$. If we add c to both sides of $a + b > a$, we get $a + b + c > a + c$. Also, if we add c to both sides of $a > 0$, we get $a + c > c$. So $(a + b + c) > (a + c) > c$

In contrast, Riemann's analytic continuation determines that $\zeta(0) = -1/2$ at $s = 0$. (See Tao[Tao]). This result contradicts the above-stated rule of arithmetic,[1] because the result is

$$-1/2 < (1^0) < (1^0 + 2^0) < (1^0 + 2^0 + 3^0) < \ldots < (1^0 + 2^0 + 3^0 + \ldots + M^0)$$

In other words, according to Riemann's analytic continuation, the sum is <u>less</u> than <u>every</u> summand. So, Riemann's analytic continuation of $\zeta(s)$ is false at $s = 0$.

Riemann's analytic continuation of $\zeta(s)$ is also false at real values of s in $s < 0$, where the Dirichlet series $\zeta(s)$ is a series of increasing terms. If we assume that analytic continuation's claim of convergence at s in $s < 1$ is true, then $\zeta(s)$ has a <u>finite limit</u> at each s in $s < 0$, as a direct consequence of this convergence. However, each finite limit in $s < 0$ contradicts the proofs that the Dirichlet series is divergent at all s in $s \leq 1$, and also contradicts the rule that a sum of positive real values is greater than

[1] Also, Occam's razor favors the simpler proof for divergence of $\zeta(s)$ at $s = 0$ over the complex arguments in favor of convergence, whose complexity makes them difficult to refute. (See e.g. Tao[Tao]).

each summand. These results are inherent when a finite limit is equated to an infinite series of increasing terms.

At real values of s in $0 < s < 1$, where the Dirichlet series $\zeta(s)$ is divergent but has decreasing terms, again Riemann's analytic continuation of $\zeta(s)$ is false. The reasons are the same as for s in $s < 0$. In regards to a finite limit in $0 < s < 1$ contradicting the rule that a sum of positive real values is greater than each summand, this is proven by repartitioning the terms in the divergent infinite series.

8 Another Example of Analytic Continuation Does Not Contradict the Law of the Excluded Middle

Another example of analytic continuation does not contradict the Law of the Excluded Middle. When the "unit disc" method is applied to the Taylor series of $f(s) = 1/(1-s)$, it is not used to claim convergence of $f(s)$ where $f(s)$ is known to be divergent (i.e. at $s = 1$, where divergence is easily proven due to division by zero).

9 Conclusion

Riemann's analytic continuation of $\zeta(s)$ not valid. So $\zeta(s)$ is exclusively defined by the Dirichlet series, which has no zeros anywhere, and is divergent in the half plane $\mathrm{Re}(s) \leq 1$. Therefore the Riemann Hypothesis is false, because it erroneously assumes that Riemann's analytic continuation of $\zeta(s)$ is true. (The absence of zeros is a situation analogous to that in Bertrand Russell's famous example: "The present King of France is bald." See Stanford [Dep]).

Regarding the Birch and Swinnerton-Dyer Conjecture [2], $\zeta(1)$ is divergent (it is the harmonic series) and so $\zeta(1) \neq 0$. This result is consistent with the Hasse–Weil zeta function, which when rearranged is $L(E, s) = [\zeta(s) \cdot \zeta(s-1)]/Z_{E,\mathbf{Q}}(s)$. Because the analytic continuation of $\zeta(s)$ is false, neither $\zeta(s)$ nor $\zeta(s-1)$ can equal zero. So $L(E, s) \neq 0$ at all values of s. So at $s = 1$, the function $L(E, 1) \neq 0$.

Totaro [Tota][Totb] has excellent advice as to how to proceed from here.

[2] According to the Clay Mathematics Institute [Ins] description of the Birch and Swinnerton-Dyer (BSD) conjecture: "this amazing conjecture asserts that if $\zeta(1)$ is equal to 0, then there are an infinite number of rational points (solutions), and conversely, if $\zeta(1)$ is not equal to 0, then there is only a finite number of such points." Given that both the Dirichlet series $\zeta(s)$ and Riemann's analytic continuation of $\zeta(s)$ agree on divergence at $s = 1$, $\zeta(1)$ is divergent (and thus not equal to 0) regardless of whether Riemann's analytic continuation of $\zeta(s)$ is true or false.

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