A Derivation of the Kerr Metric by Ellipsoid Coordinate Transformation

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Abstract

Einstein's general relativistic field equation is a nonlinear partial differential equation that lacks an easy way to obtain exact solutions. The most famous examples are Schwarzschild and Kerr's black hole solutions. The Kerr metric has astrophysical meaning because most of cosmic celestial bodies are rotating. The Kerr metric is even more difficult to derive than the Schwarzschild metric specifically due to off-diagonal term of metric tensor. In this paper, a derivation of Kerr metric was obtained by ellipsoid coordinate transformation, which causes elimination a large amount of tedious derivation. This derivation is not only physics enlightening, but also further deducing some characteristics of the rotating black hole.

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I. INTRODUCTION

The theory of general relativity proposed by Albert Einstein in 1915 was one of the greatest advances in modern physics. It describes the distribution of matter to determine the space-time curvature, and the curvature determines how the matter moves. Einstein's field equation is very simple and elegant, but because Einstein's field equation is a set of nonlinear differential equations, it has proven difficult to find the exact analytic solution. The exact solution has physical meanings, only in some simplified assumptions, the most famous of which include Schwarzschild and Kerr's black hole solution, and Friedman's solution to cosmology. One year after Einstein published his equation, Schwarzschild discovered the spherical symmetry, static vacuum solution with center singularity.[1] Nearly 50 years later, Kerr solved the fixed axis symmetric rotating black hole in 1963.[2] Some of these exact solutions have been used to explain topics related to the gravity in cosmology, such as Mercury's precession of the perihelion, gravitational lens, black hole, expansion of the universe, and gravitational waves.

Today, many solving methods of Einstein field equations have been proposed. For example: Pensose-Newman's method,[3] or Bcklund transformations.[4] Despite their great success in dealing with the Einstein equation, these methods are technically complex and expert-oriented.

The Kerr solution is important in astrophysics because most cosmic celestial bodies are rotating and rarely completely at rest. Traditionally, the general method of the Kerr solution can be found in *The Mathematical Theory of Black Holes* by the classical works of S.Chandrasekhar.[5] However, the calculation is so lengthy and complicated that college or institute students find it difficult to understand. Recent literature review showed that it is possible to obtain Kerr metric through the oblate spheroidal coordinates transformation.[6] This encourage me to look for a more concise way to solve the vacuum solution of Einstein's field equation through coordinate transformation.

The motivation of this derivation simply came from my desire to use a relatively simple way of Schwarzschild method to derive the Kerr metric, which can enable more students interested in the general relativity to self-deduce the exact solution. In this paper, I will introduce a more enlighted way to find this solution. It is not only a new try, but also the derivation is further linked to some important features of the rotating black hole.

II. SCHWARZSCHILD AND KERR SOLUTIONS

The exact solution of the Einstein field equation is usually expressed in metric. For example, Minkowski space-time is four-dimension coordinates combining three-dimensional Euclidean space and one-dimension time can be expressed in Cartesian form in Eq. (1):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

and in polar coordinate form in Eq. (2):

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2}$$

Schwarzschild employed a non-rotational sphere-symmetric object with polar coordinate in Eq. (2) with two variables from functions $\nu(r), \lambda(r)$, which was shown in Eq. (3):

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{3}$$

In order to solve the Einstein field equation, Schwarzschild used a vacuum condition, let $R_{\mu\nu} = 0$, calculating Ricci tensor from Eq. (3), and get the first exact solution of the Einstein field equation, Schwarzschild metric, which was shown in Eq. (4).[1]

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{4}$$

However, Schwarzschild metric cannot be used to describe rotation, axial- symmetry, and charged heavenly bodies. From the examination of the metric tensor $g_{\mu\nu}$ in the Schwarzschild metric, one can obtain the components:

$$\begin{aligned} &g_{00} = 1 - \frac{2M}{r},\ g_{11} = -(1 - \frac{2M}{r})^{-1}, \\ &g_{22} = -r^2,\ g_{33} = -r^2 sin^2\theta \end{aligned}$$

Which can also be presented as:

$$\begin{aligned} &g_{tt} = 1 - \frac{2M}{r},\ g_{rr} = -(1 - \frac{2M}{r})^{-1}, \\ &g_{\theta\theta} = -r^2,\ g_{\phi\phi} = -r^2 sin^2\theta \end{aligned} \tag{5}$$

Differences of metric tensor $g_{\mu\nu}$ between the Schwarzschild metric in Equation (4) and Minkowski space-time in Equation (2) are only in time-time terms (g_{tt}) and radial-radial terms (g_{rr}).

Kerr metric is the second exact solution of the Einstein field equation, which can be used to describe space-time geometry in the vacuum area near a rotational, axial-symmetric heavenly body.[2] It is a generalized form of Schwarzschild metric. Kerr metric in Boyer-Lindquist coordinate system can be expressed in Equation (6):

$$\begin{aligned} ds^2 = &\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4Mra\sin^2\theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2 \\ &-\rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \end{aligned} \tag{6}$$

Where define $\rho^2 \equiv r^2 + a^2 cos^2\theta$ and $\Delta \equiv r^2 - 2Mr + a^2$, M is the mass of the rotational material, a is the spin parameter or specific angular momentum and is related to the angular momentum J by $a = J/M$. In all physical quality, we adopt $c = G = 1$.

By examining the components of metric tensor $g_{\mu\nu}$ in Equation (6), one can obtain:

$$\begin{aligned} &g_{00} = 1 - \frac{2Mr}{\rho^2}, g_{11} = -\frac{\rho^2}{\Delta}, g_{22} = -\rho^2, \\ &g_{03} = g_{30} = \frac{2Mra\sin^2\theta}{\rho^2} \\ &g_{33} = -\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right) sin^2\theta \end{aligned} \tag{7}$$

Comparison the components of Schwarzschild metric Equation (4) with Kerr metric Equation (6):

1. Both $g_{03}(g_{t\phi})$ and $g_{30}(g_{\phi t})$ off-diagonal terms in Kerr metric are not present in Schwarzschild metric, apparently due to rotation. If the rotation parameter $a = 0$, these two terms vanish.

2. $g_{00}g_{11} = g_{tt}g_{rr} = -1$ in Schwarzschild metric, but not in Kerr metric.

3. When spin parameter $a = 0$, Kerr metric turns into Schwarzschild metric and therefore is a generalized form of Schwarzschild metric.

III. TRANSFORMATION OF ELLIPSOID SYMMETRIC ORTHOGONAL COORDINATE

To derive Kerr metric, if we start from the initial assumptions, we must introduce $g_{00}, g_{11}, g_{22}, g_{03}, g_{33}$ five variables, all are a function of (r, θ), and finally we will get monster-like complex equations. Apparently, due to the off-diagonal term, Kerr metric cannot be solved by the spherical symmetry method used in Schwarzschild metric. Besides, Previous study showed that the space-time of Kerr metric is ellipsoidal.[7].

Different from the derivation methods used in classical works of Chandrasekhar (1983), the author used the changes in coordinate of Kerr metric into ellipsoid symmetry firstly to get a simplified form, and then used Schwarzschild's method to solve Kerr metric. First of all, the following ellipsoid coordinate changes were apply to Equation (1)[8]:

$$\begin{aligned} x &\rightarrow (r^2+a^2)^{1/2} sin\theta cos\phi \\ y &\rightarrow (r^2+a^2)^{1/2} sin\theta sin\phi \\ z &\rightarrow r cos\theta \\ t &\rightarrow t \end{aligned} \tag{8}$$

Where a is the coordinate transformation parameter. The metric under the new coordinate system becomes Equation (9):

$$ds^2 = dt^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \left(r^2+a^2\right) sin^2\theta d\phi^2 \tag{9}$$

Equation (9) has physics significance, which represents the coordinate with ellipsoid symmetry in vacuum; it can also be obtained by assigning mass $M = 0$ to the Kerr metric in Equation (6). Due to the fact that most of the celestial bodies, stars and galaxy for instance, are ellipsoid symmetric, Bijan started from this vacuum ellipsoid coordinate and derived a Schwarzschild-like solution for ellipsoidal celestial objects following Equation (10)[9]:

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}\frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \left(r^2+a^2\right) sin^2\theta d\phi^2 \tag{10}$$

Equation (10) morphs into the Schwarzschild's solution in Equation (4) when the coordinate transformation parameter $a = 0$ and therefore Equation (10) is also a generalization of Schwarzschild's solution.

In order to eliminate the difference between Kerr metric and Schwarzschild metric that is described earlier, we can assume to rewrite the Kerr metric in the following coordinates:

$$ds^2 = G'_{00}dT^2 + G'_{11}dr^2 + G'_{22}d\theta^2 + G'_{33}d\Phi^2 \tag{11}$$

To eliminate the off-diagonal term:

$$dT \equiv dt - pd\phi,\ d\Phi \equiv d\phi - qdt \tag{12}$$

to obtain

$$G'_{00}G'_{11} = -1 \tag{13}$$

By comparing the coefficient, Equations (14) to (18) were obtained.

$$G'_{00}p + G'_{33}q = \frac{-2Mrasin^2\theta}{\rho^2} \tag{14}$$

$$G'_{00} + G'_{33}q^2 = 1 - \frac{2Mr}{\rho^2} \tag{15}$$

$$G'_{00}p^2 + G'_{33} = -\left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right) sin^2\theta \tag{16}$$

$$G'_{22} = -\rho^2 \tag{17}$$

$$G'_{11} = \frac{-\rho^2}{\Delta} \tag{18}$$

By solving six variables $G'_{00}, G'_{11}, G'_{22}, G'_{33}, p, q$ in the six dependent Equations (13) to (18), the results shown in Equation (19) were obtained:

$$\begin{aligned}
&p = \pm asin^2\theta, take\ positive\ result \\
&q = \pm \frac{a}{r^2 + a^2}, take\ positive\ result \\
&G'_{00} = \frac{\Delta}{\rho^2} \\
&G'_{11} = -\frac{\rho^2}{\Delta} \\
&G'_{22} = -\rho^2 \\
&G'_{33} = -\frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}
\end{aligned} \tag{19}$$

Put them into Equation (8) and obtain Equation (20):

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - asin^2\theta d\phi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{r^2 + a^2}dt\right)^2 \tag{20}$$

Equation (20) can be found in the literature and also textbook by O'Neil. It is also called the Kerr metric with Boyer-Lindquist in orthonormal frame.[10] There is no off-diagonal terms, and $g_{00}g_{11} = -1$ after the coordinate transformation.

IV. CALCULATING THE RICCI TENSOR

From previous discussion, Equation (9) can be recognized as the coordinate under the ellipsoid symmetry in vacuum. Therefore, when the mass M approached 0, Kerr metric

Equation (20) will also be transformed into Equation (21), which equals Equation (9). The differences of metric tensor components are in time-time and radial-radial terms, just the same as between Schwarzschild metric (Equation (4)) and Minkowski space-time (Equation (2)). dT and $d\Phi$ defined in Equation (22) are ellipsoid coordinate transformation functions.

$$ds^2 = \frac{r^2 + a^2}{\rho^2} dT^2 - \frac{\rho^2}{r^2 + a^2} dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2} d\Phi^2 \tag{21}$$

$$\begin{aligned} dT &\equiv dt - asin^2\theta d\phi, \\ d\Phi &\equiv d\phi - \frac{a}{r^2 + a^2} dt \end{aligned} \tag{22}$$

In this paper, Schwarzschild method was used to solve Kerr metric from Equations (21) to (22) by introducing two new functions $e^{2\nu(r,\theta)}, e^{2\lambda(r,\theta)}$:

$$ds^2 = e^{2\nu(r,\theta)} dT^2 - e^{2\lambda(r,\theta)} dr^2 - \rho^2 d\theta^2 - \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2} d\Phi^2 \tag{23}$$

Define the parameters ρ^2 and h in Equation (24):

$$\begin{aligned} \rho^2 &\equiv r^2 + a^2 cos^2\theta \\ h &\equiv r^2 + a^2 \end{aligned} \tag{24}$$

Metric tensor in the matrix form shown in Equations (25) to (26):

$$g_{\mu\nu} = \begin{pmatrix} e^{2\nu(r,\theta)} & 0 & 0 & 0 \\ 0 & -e^{2\lambda(r,\theta)} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -\frac{h^2 sin^2\theta}{\rho^2} \end{pmatrix} \tag{25}$$

$$g^{\mu\nu} = \begin{pmatrix} e^{-2\nu(r,\theta)} & 0 & 0 & 0 \\ 0 & -e^{-2\lambda(r,\theta)} & 0 & 0 \\ 0 & 0 & -\rho^{-2} & 0 \\ 0 & 0 & 0 & -\frac{\rho^2}{h^2 sin^2\theta} \end{pmatrix} \tag{26}$$

Chrostoffel Symbols can be obtained by the following steps in Equation (27):

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu_\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{27}$$

Non-zero Chrostoffel symbols are listed in Equations (28) to (37):

$$\Gamma^1_{00} = e^{2(\nu-\lambda)}\partial_1\nu \tag{28}$$

$$\Gamma^1_{11} = \partial_1\lambda \tag{29}$$

$$\Gamma^0_{10} = \Gamma^0_{01} = \partial_1\nu \tag{30}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{r}{\rho^2} \tag{31}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{2r}{h} - \frac{r}{\rho^2} \tag{32}$$

$$\Gamma^1_{22} = -re^{-2\lambda} \tag{33}$$

$$\Gamma^3_{32} = \Gamma^3_{23} = cot\theta\left(\frac{h}{\rho^2}\right) \tag{34}$$

$$\Gamma^1_{33} = -re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right) \tag{35}$$

$$\Gamma^2_{22} = -\frac{h^2 sin\theta cos\theta}{\rho^2} \tag{36}$$

$$\Gamma^2_{33} = -sin\theta cos\theta\left(\frac{h^3}{\rho^6}\right) \tag{37}$$

The calculation of Ricci curvature tensor can be derived by the following Equation (38):

$$R_{\alpha\beta} = R^\rho_{\alpha\rho\beta} = \partial_\rho\Gamma^\rho_{\beta\alpha} - \partial_\beta\Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\rho\alpha} \tag{38}$$

and the results are listed in Equations. (39) to (50):

$$R^0_{101} = \partial_1\nu\partial_1\lambda - (\partial_1\nu)^2 - \partial_1^2\nu \tag{39}$$

$$R^0_{202} = -re^{-2\lambda}\partial_1\nu \tag{40}$$

$$R^0_{303} = -re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)\partial_1\nu \tag{41}$$

$$R^1_{212} = e^{-2\lambda}\left(r\partial_1\lambda - 1 + \frac{r^2}{\rho^2}\right) \tag{42}$$

$$R^1_{313} = re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)\partial_1\lambda \tag{43}$$

$$R^2_{323} = sin^2\theta\left[\frac{h^4}{\rho^8}\left(\frac{5r^2 - 4\rho^2}{h}\right) - \frac{r^2 h}{\rho^4}\left(2 - \frac{h}{\rho^2}\right)e^{-2\lambda}\right] \tag{44}$$

$$R^1_{010} = g^{11} g_{00} R^0_{101} = e^{2(\nu-\lambda)} \left[-\partial_1 \nu \partial_1 \lambda + (\partial_1 \nu)^2 + \partial_1^2 \nu \right] \tag{45}$$

$$R^2_{020} = g^{22} g_{00} R^0_{202} = e^{2(\nu-\lambda)} \frac{r}{\rho^2} \partial_1 \nu \tag{46}$$

$$R^3_{030} = g^{33} g_{00} R^0_{303} = e^{2(\nu-\lambda)} \left(\frac{2r}{h} - \frac{r}{\rho^2} \right) \partial_1 \nu \tag{47}$$

$$R^2_{121} = g^{22} g_{11} R^1_{212} = \frac{1}{\rho^2} \left(r \partial_1 \lambda + \frac{r^2}{\rho^2} - 1 \right) \tag{48}$$

$$R^3_{131} = g^{33} g_{11} R^1_{313} = \left(\frac{2}{h} - \frac{1}{\rho^2} \right) \left(r \partial_1 \lambda + \frac{2r^2}{\rho^2} - \frac{2r^2}{h} \right) \tag{49}$$

$$R^3_{232} = g^{33} g_{22} R^2_{323} = \left[\frac{h^2}{\rho^4} \left(\frac{5r^2 - 4\rho^2}{h} \right) - \frac{r^2}{h} \left(2 - \frac{h}{\rho^2} \right) e^{-2\lambda} \right] \tag{50}$$

$R_{\mu\nu}$ can be calculated by the Equations (51) to (54):

$$R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = e^{2(\nu-\lambda)} \left[-\partial_1 \nu \partial_1 \lambda + (\partial_1 \nu)^2 + \partial_1^2 \nu + \frac{2r}{h} \partial_1 \nu \right] \tag{51}$$

$$R_{11} = R^0_{101} + R^2_{121} + R^3_{131} = \partial_1 \nu \partial_1 \lambda - (\partial_1 \nu)^2 - \partial_1^2 \nu + \frac{2r}{h} \partial_1 \lambda \tag{52}$$

$$R_{22} = R^0_{202} + R^1_{212} + R^3_{232} = e^{-2\lambda} \left(r \left(\partial_1 \lambda - \partial_1 \nu \right) - 1 + \frac{2r^2}{\rho^2} - \frac{2r^2}{h} \right) + \frac{h^2}{\rho^4} \left(\frac{5r^2 - 4\rho^2}{h} \right) \tag{53}$$

$$\begin{aligned} R_{33} &= R^0_{303} + R^1_{313} + R^2_{323} \\ &= sin^2\theta \left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4} \right) \left[e^{-2\lambda} \left(r \left(\partial_1 \lambda - \partial_1 \nu \right) - \frac{r^2}{\rho^2} \right) + \frac{h^2}{\rho^4} \left(\frac{5r^2 - 4\rho^2}{h} \right) \left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4} \right)^{-1} \right] \end{aligned} \tag{54}$$

V. FINDING A SOLUTION OF THE VACUUM EINSTEIN FIELD EQUATIONS

To solve vacuum Einstein's field equations, first we set the Ricci tensor as zero, which means: $R_{\mu\nu} = 0\, R = 0$, in empty space, θ is approximately constant. Then combine with R_{00} and R_{11} to get Equation (55), and solve the equation, Equations (56) to (58) were obtained:

$$e^{-2(\nu-\lambda)} R_{00} + R_{11} = \frac{2r}{h} \left(\partial_1 \nu + \partial_1 \lambda \right) = 0 \tag{55}$$

$$\partial_1 \nu + \partial_1 \lambda = \partial_1 \left(\nu + \lambda \right) = 0 \tag{56}$$

$$\nu = -\lambda + c,\;\; \nu \left(r, \theta \right) = -\lambda \left(r, \theta \right) + c \tag{57}$$

$$e^{\nu} = e^{-\lambda} \tag{58}$$

To solve this partial differential equation, one has to remember that when the angular momentum approaches zero ($a \to 0$), Kerr metric (Equation (6)) turns into Schwarzschild metric (Equation (4)). Then Equations (59) to (63) were obtained:

$$\lim_{a\to 0} h = r^2,\ \lim_{a\to 0} \rho = r \tag{59}$$

$$\lim_{a\to 0} R_{22} = e^{-2\lambda}\left(r\left(\partial_1\lambda - \partial_1\nu\right) - 1\right) + 1 \tag{60}$$

$$\lim_{a\to 0} R_{33} = sin^2\theta\left[e^{-2\lambda}\left(r\left(\partial_1\lambda - \partial_1\nu\right) - 1\right) + 1\right] = sin^2\theta R_{22} \tag{61}$$

$$\lim_{a\to 0} R_{22} = 0 \tag{62}$$

$$e^{2\nu} = 1 + \frac{C}{r},\ let\ C = -2M \tag{63}$$

So, under the limit condition when angular momentum approaches zero ($a \to 0$), the equations could be solved as shown in Equation (64):

$$\begin{aligned}\lim_{a\to 0} e^{2\nu} &= 1 - \frac{2M}{r} = \frac{r^2 - 2Mr}{r^2}\\ \lim_{a\to 0} e^{2\lambda} &= \left(1 - \frac{2M}{r}\right)^{-1} = \frac{r^2}{r^2 - 2Mr}\end{aligned} \tag{64}$$

One could also demand another limit condition of flat space-time, where the mass approaches zero ($M \to 0$) in the Equation (18), which could be represented as in Equation (65):

$$\begin{aligned}\lim_{M\to 0} e^{2\nu} &= \frac{r^2 + a^2}{\rho^2} = \frac{r^2 + a^2}{r^2 + a^2cos^2\theta}\\ \lim_{M\to 0} e^{2\lambda} &= \frac{\rho^2}{r^2 + a^2} = \frac{r^2 + a^2cos^2\theta}{r^2 + a^2}\end{aligned} \tag{65}$$

Deduced from the above conditions in Equations (64) to (65), the equations of Ricci tensor could be solved as in Equation (66):

$$\begin{aligned}e^{2\nu} &= \frac{r^2 - 2Mr + a^2}{r^2 + a^2cos^2\theta}\\ e^{2\lambda} &= \frac{r^2 + a^2cos^2\theta}{r^2 - 2Mr + a^2}\end{aligned} \tag{66}$$

Finally, the Kerr metric was gotten as shown in Equation (67):

$$\begin{aligned}ds^2 = &\frac{r^2 - 2Mr + a^2}{\rho^2}\left(dt - asin^2\theta d\phi\right)^2 - \frac{\rho^2}{r^2 - 2Mr + a^2}dr^2 - \rho^2 d\theta^2\\ &- \frac{\left(r^2 + a^2\right)^2 sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{r^2 + a^2}dt\right)^2\end{aligned} \tag{67}$$

VI. DISCUSSION

It is proven that the Kerr metric (Equation (67)) can be obtained by combining the ellipsoid coordinate transformation and the assumptions listed in Equations (21) to (23) following these steps: transforming the Euclidian four-dimention space-time in Equation (1) to vacuum Minkowski space-time with ellipsoid symmetry in Equation 9; transforming from (t, r, θ, ϕ) to (T, r, θ, Φ) under the new coordinate system to eliminate the major difference in metric tensor components between the Kerr metric and the Schwarzschild metric: there are no off-diagonal terms and the product of $g_{00}g_{11}$ becomes -1; solving vacuum Einstein's equation by using the Schwarzschild method from Equation (23); applying limit method to calculate Ricci curvature tensor; and finally deducting the Kerr metric.

Table I shows the list of the metric tensor components discussed in previous sections, including the Minkowski space-time, the Schwarzschild solution, empty ellipsoid, a Schwarzschild-like ellipsoid solution, and the Kerr solution. The Minkowski space-time and the Schwarzschild solution have spherical symmetry, and the others have ellipsoid symmetry.

Further, some of the characteristics with deeper physics meaning of ellipsoid symmetry, Kerr metric, and rotating black hole can be obtained from this new coordinate function $dT, d\Phi$. Remember, when a approaches to zero $(a \to 0), dT, d\Phi$ degenerates to $dt, d\phi$.

A. Ellipsoid symmetry and the Kerr metric

While metric with spherical symmetry in vacuum has the following expression:

$$- r^2 d\theta^2 - r^2 sin^2\theta d\theta^2 \tag{68}$$

And metric of ellipsoid symmetric in vacuum has the following expression in Equation (69), where $\frac{a}{r^2+a^2}$ and a $sin^2\theta$ term can be seen in multiply and divide combination:

$$\begin{aligned}
& - \rho^2 d\theta^2 - \left(r^2 + a^2\right) sin^2\theta d\phi^2 \\
&= -\rho^2 d\theta^2 - \left(\frac{r^2 + a^2}{a}\right) \left(asin^2\theta\right) d\phi^2
\end{aligned} \tag{69}$$

Terms of $d\theta^2, d\phi^2$ in the Kerr metric is showed in Equation (70), where $\frac{a}{r^2+a^2}$ and $asin^2\theta$ term can also be seen in linear combination:

$$\begin{aligned}
& - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right) sin^2\theta d\phi^2 \\
&= -\rho^2 d\theta^2 - \left(\frac{r^2 + a^2}{a} + \frac{2Mrasin^2\theta}{\rho^2}\right) \left(asin^2\theta\right) d\phi^2
\end{aligned} \tag{70}$$

The a in Equation (69) represents a parameter in the ellipsoid symmetric coordinate transformation, however,a in Kerr metric (Equation (70)) represents a spin parameter, which is proportional to angular momentum. Both a 's are equivalent in mathematic perspective and used to transform the space-time into ellipsoid symmetry with a rotational symmetric z-axis. As $a \to 0$, both Equation (69) and Equation (70) degenerate into spherical symmetry Equation (68).

B. Frame-dragging angular momentum

In physics, a spinning heavenly body with a non-zero mass will generate a frame-dragging phenomenon along the equators direction, which has been proven by Gravity Probe B experiment.[11] Therefore, an extra term $\frac{2Mra^2sin^2\theta}{\rho^2}$ in the Kerr metric is found in Equation (70) compared to the vacuum ellipsoid symmetry in Equation (69). As the mass approaches zero $M \to 0$, Equation (70) degenerates into Equation (69).

To order to describe frame-dragging, Kerr metric can be re-written as Equation (71):

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\ &= \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right)dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}\left(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 \end{aligned} \tag{71}$$

The definition of angular momentum (Ω) in frame-dragging:

$$\begin{aligned} \Omega = -\frac{g_{t\phi}}{g_{\phi}\phi} &= \frac{\frac{2Mrasin^2\theta}{\rho^2}}{\left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right)sin^2\theta} \\ &= \frac{2Mra}{\rho^2\left(r^2+a^2\right) + 2Mra^2sin^2\theta} \\ &= \frac{2Mr}{\rho^2\left(\frac{r^2+a^2}{a}\right) + 2Mr\left(asin^2\theta\right)} \end{aligned} \tag{72}$$

So, we see both the $asin^2\theta$ and $\frac{a}{r^2+a^2}$ term in Ω , which means $dT, d\Phi$ would have some relation with frame-dragging angular momentum.

C. Black hole angular velocity

Its close relationship to the black hole angular velocity (Ω_H) can be easily identified by examining $d\Phi$ term in Equation (73).

$$
\begin{aligned}
& d\Phi = d\phi - \frac{a}{r^2 + a^2} dt \\
& \Omega_H = \frac{a}{r_+^2 + a^2} \\
& from\ \Delta = 0,\ solve\ r_\pm = M \pm \sqrt{M^2 - a^2}
\end{aligned} \tag{73}
$$

Based on this derivation, in the future we will further study whether the method mentioned in this paper can be extended to other more general cases. For example, suppose we start with three functions $e^{2\nu(r,\theta)}, e^{-2\nu(r,\theta)}, e^{2\lambda(r,\theta)}, e^{2\mu(r,\theta)}$ as shown in Equation (74):

$$ds^2 = e^{2\nu(r,\theta)} dT^2 - e^{-2\nu(r,\theta)} dr^2 - e^{2\lambda(r,\theta)} d\theta^2 - e^{2\mu(r,\theta)} d\Phi^2 \tag{74}$$

Besides, as $dT, d\Phi$ is shown to be related with ellipsoid symmetry, frame-dragging angular momentum, and black hole angular velocity, which are all rotation parameters, it deserves further study to determine if this method could be extended to solve the other axial-symmetry exact solutions of vacuum Einstein's field equation.

VII. CONCLUSION

In this paper, we derive the Kerr metric from the coordinate transformation method. First, we obtain the Kerr Metric with Boyer-Lindquist in orthonormal frame, and then we prove that it is possible to derive the Kerr metric from the vacuum ellipsoid symmetry, and this derivation allows us to better understand the physical properties of the rotating black hole, such as the frame-dragging effect, and the angular velocity. This deduction method is different from classical papers written by Kerr and Chandrasekhar, and is expected to encourage future study in this subject.

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TABLE I. Metric Tensor Components and Symmetry

Metric Tensor	$dt^2(dT^2)$	dr^2	$d\theta^2$	$d\phi^2(d\Phi^2)$	Symmetry and State
Minkowski	1	-1	$-r^2$	$-r^2 sin^2\theta$	Spherical, Empty
Schwarzschild	$\frac{r^2-2Mr}{r^2}$	$-\frac{r^2}{r^2-2Mr}$	$-r^2$	$-r^2 sin^2\theta$	Spherical, Static, Mass
Ellipsoid	$\frac{r^2+a^2}{\rho^2}$	$-\frac{\rho^2}{r^2+a^2}$	$-\rho^2$	$-\frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}$	Ellipsoid, Empty
Schwarzschild-like	$\frac{r^2-2Mr}{r^2}$	$-\frac{r^2}{r^2-2Mr}\frac{\rho^2}{r^2+a^2}$	$-\rho^2$	$-\left(r^2+a^2\right) sin^2\theta$	Ellipsoid, Static, Mass
Kerr	$\frac{r^2-2Mr+a^2}{\rho^2}$	$-\frac{\rho^2}{r^2-2Mr+a^2}$	$-\rho^2$	$-\frac{\left(r^2+a^2\right)^2 sin^2\theta}{\rho^2}$	Ellipsoid,Axisymmetric, Mass

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[12] *National Taiwan University OpenCourseWare: general relativity”* Web Site, `<http://ocw.aca.ntu.edu.tw/ntu-ocw/index.php/ocw/cou/099S134>`.