Platonic Solids and Elementary Particles

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Abstract

The groups of symmetry of regular polyhedra are considered. It is shown that a total number and types of gauge bosons in the Grand Unified Theory with the group SU(5) can be deduced from the structure of the cube rotation group. Possible connections of fundamental fermions with the icosahedral symmetry are discussed.

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Introduction

A well-known expert in high-energy physics Lev B. Okun wrote: "Physicists can be called hunters for symmetries: in a sense, they differ from other people in that they search in nature for ever more hidden and increasingly fundamental types of symmetries".

This is clearly seen in the field of elementary particles, where it is the identification of symmetries that serves as an instrument that allows us to reduce the entire diversity of observed entities to the few underlying structures. On the other hand, in mathematics there are selected objects which possess a nontrivial, surprising symmetry. First of all, these are regular polyhedra, or Platonic solids, whose possible role in physics is yet to be revealed.

As we know, in a three-dimensional space there are only five such polyhedra (Fig. 1): a tetrahedron (it has four faces), a cube (six), an octahedron (eight), a dodecahedron (twelve) and an icosahedron (twenty). They are called Platonic bodies, since mathematicians, close to Plato's Academy, first studied their entire set.

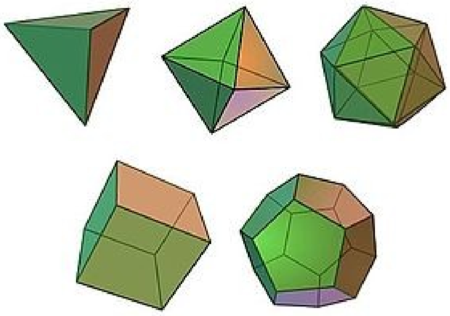

Figure 1

Ancient Greek natural philosophers connected the first four bodies to four elements (fire, air, earth, water), and the fifth (dodecahedron) – to quintessence in the form of the entire universe. Then interest in such figures arose in the Renaissance -- they were viewed by geometers and astronomers, architects and artists. Johannes Kepler tried to discover on their basis the principles of the structure of the solar system (the ratio of the distances of the planets from the Sun). Studies of Platonic solids by rigorous methods of mathematics began in the XIX century, when the theory of groups and other important branches of this science were developed.

With such figures one can perform many different operations of rotation and reflection, in which they remain unchanged. At this moment we are interested only in rotations that form discrete finite groups. Note that by joining the centers of adjacent faces of any Platonic body, we get another, dual to it body from the same list -- the tetrahedron will go back to the tetrahedron, the cube to the octahedron (and back), the icosahedron to the dodecahedron (and back). Possible rotations of dual figures coincide, so there are only three different rotation groups: 1) of a tetrahedron; 2) of a cube (and octahedron); 3) of a icosahedron (and dodecahedron).

Let's see what are the orders of these groups. A case of the tetrahedron is the simplest. How much rotation does it allow? This is easy to calculate, considering that any symmetry axis of the *n*-th order makes possible *(n - 1)* different rotations. In a tetrahedron there are four axes of the third order that pass through its vertices and the centers of the faces opposite to them, and also three axes of the second order connecting the midpoints of opposite (not having common vertices) edges. One should add the identical turn. Total: 4x2 + 3x1 + 1 = 12. Similarly, considering the possible rotations of cube and icosahedron, we find that the orders of their groups are 24 and 60, respectively.

Rearrangement of items

Now for a time let's forget about the Platonic bodies and we'll do all kinds of arrangements for *N* objects. The operations of their permutations also specify a group, which is denoted by S_n. How many will there be different situations? In the first place can be any of the *N* objects, on the second -- any of the remaining, that is *(N-1),* on the third -- any of *(N-2)* and so on. Multiplying, we obtain: *Nx(N - 1) ..... 3x2x1*, or *N! (N*-factorial).
Note: 3! = 6; 4! = 24; 5! = 120; these are the orders of S_3, S_4 and S_5.

In addition to the groups S_n of all permutations of *N* objects, there are also important groups of even permutations (they are denoted by A_n). The fact is that from one arrangement of objects to some other one, we can go not by one, so to speak, a large jump, but by several small steps -- successively changing only two elements each time (such substitutions are called transpositions). For example, if we have two sets of four digits 2-4-3-1 and 1-4-2-3, then from one to another we can go with two transpositions: 2-4-3-1 -> 3-4-2-1 -> 1-4-2-3.

So the set of all arrangements of objects is divided into two equal classes: one is obtained from some initial by even number of transpositions, others -- by odd. Since the number of all the permutations *N!*, then of even and odd will be *N!/2.*
Hence, the order of the group A_3 is 3, A_4 -- 12, A_5 -- 60.

The groups of all permutations S_n are called symmetric (the notation S), and the groups A_n are alternating (or alternative, whence the symbol A). Why such names? This is due to the symmetry of the polynomials that these groups reflect. For example, let's take two functions from three variables:

1) $X_1X_2 + X_2X_3 + X_1X_3$;

2) $(X_1 - X_2)(X_2 - X_3)(X_1 - X_3)$.

Let's give some concrete values to all variables X_i and see how the functions behave when pairwise X-renames (with transpositions). The first of them will retain its value for any permutations of the arguments, so the group S_3 will describe it. And the second function changes sign at each transposition, that is, it is alternating, and in order for this polynomial to remain unchanged, we must make an even number of such substitutions; hence, there is a group of even permutations of three elements A_3.

The famous mathematician Felix Klein established an important fact: the rotation groups of Platonic solids are mathematically indistinguishable from certain groups of permutations --symmetric or alternative:

1) The tetrahedron. For rotations about third-order axes, one of its vertices remains stationary, and the three remaining ones are cyclically rearranged. And when you rotate around axes of the second order, the vertices change in pairs. All together they give 12 even permutations of four vertices (group A_4).

2) The cube (and the octahedron). For any rotation of the cube, its four large diagonals change places, and its rotation group coincides with S_4, the group of all permutations of four elements. Its order is 24.

3) The icosahedron (and the dodecahedron). The group of even permutations A_5 (60 operations). What are the five elements that change places? In this case, they are more difficult to see: it turns out that these are five octahedra inscribed in an icosahedron (or five cubes into a dodecahedron). In Fig. 2 shows a cube inscribed in the dodecahedron; it is clear that there can actually be five such cubes, because in each pentagonal face there are five diagonals that serve as ribs.

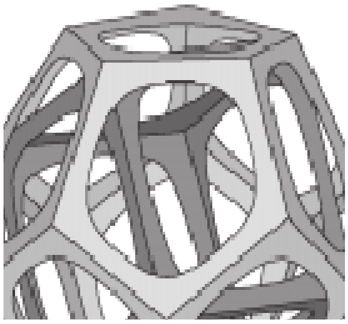

Figure 2

On the path to unity

Now it's time to look at the connection between the world of polyhedra and the world of elementary particles. In fact, such a relationship has long been established, but it remains somewhat unnoticed: the essence lies in the groups S_n and A_n, which describe, on the one hand, the rotation of the Platonic solids, and on the other, the systems consisted of micro-particles. When quantum chemistry arose in the 1930s, it began widely to use the theory of groups; even talked about the "group plague". And a lot of attention was paid to the groups of permutations.

The cause is clear: in quantum mechanics a system of many particles is described by a single wave function, and the probability of an event is determined by the square of this wave function. The particles of the same name are indistinguishable, so if two of them are interchanged (transpose -- remember our polynomials), then the square of the wave function will not change. And this is possible in two cases: if the wave function retains its sign under transposition, that is, it is symmetric, or changes sign (antisymmetric). Depending on which option is implemented, all the particles are divided into two classes -- bosons and fermions (they differ in the value of the spin). Hence, bosons will be described by S_n groups, and fermions A_n (therefore such groups are used in nuclear and atomic physics, quantum chemistry and spectroscopy).

Of course, the theory of groups plays a leading role in the study of the elementary particles themselves, describing the different symmetries in them. First of all, in accordance with this principle, it divides them into fermions and bosons. Fermions serve as the building blocks of the physical world, and bosons transfer interactions.

Now we know 24 fundamental (to date -- structureless) fermions, which are three similar sets of eight particles in each -- three generations of them, differing by masses. Each generation includes two leptons (an electron or its heavier analog, as well as a neutrino of one of three possible types) and two quarks, each of which is of three colors. Can in the future other such generations (the fourth, the fifth) be open? - this question is still open.

On the agenda is the task of combining different physical forces, in other words, finding a group that would encompass all particles and their interactions. Electric and weak forces are already unified (Salam-Weinberg theory), and now physicists are fighting over the "Great Unification", which should include strong interaction.

Here the model of Howard Georgie and Sheldon Glashow is popular. They formally, mechanically combined groups that correspond to separate known interactions (electroweak and strong), and included them in the most economical way in a wider, so-called SU(5) group. It describes fermions as a matrix five by five, then the number of bosons -- carriers of interactions is equal to the number of matrix elements minus one. Thus, the model explains the total number of bosons, namely, 24. It is important that this set of bosons realizing all fermions transformations is not subject to change (even if the number of fermions generations increases).

What is this set? First, four particles carrying electroweak forces: photon, W^+, W^-, Z^0. Secondly, there are eight gluons that ensure the color interaction of quarks. To these, we need to add two more sixes of hypothetical particles called X- and Y-particles, or leptoquarks. They, according to the Georgie--Glashow model, can transform quarks into leptons (X-particles – into an electron or its analogies, Y-particles – into different neutrinos) and vice versa at the very high energy (the possible instability of the proton is associated with them).

The approach of Georgie and Glashow is just one of many possible. The main problem lies in finding the underlying physical reality group. Usually it is searched among various continuous groups (Lie groups) like SU(5).

But maybe is it worth to look at the finite groups? And will not be the very fact of the presence in nature of two types of particles -- fermions and bosons -- to serve a clue to the problem? In accordance with the behavior of the wave function under transpositions, the fundamental fermions could be the realization of some alternating group, and the bosons -- of symmetric. As we have seen, in the cases of four or five permutable elements, these groups are isomorphic to the rotation groups of Platonic solids; what is more, the number of operations in them and the number of fundamental particles lie in the same boundaries.

Perhaps a geometric principle that will allow us to reveal the inner logic of the world of elementary particles, to predict their composition and basic properties is hidden in the right polyhedra.

Cube of bosons

First, let's look from this point of view at bosons. As we said, there are 24 of them, but this is the order of the symmetric group S_4 (it is also the group of rotation of the cube). We just wanted to compare the bosons to one of these groups and immediately rightly got their total number. But that's not all. For a mathematician to understand the structure of a group means to identify its subgroups, to determine their type and the relationships between them (a special role is played by normal subgroups). Let's see the internal arrangement of the cube`s group.

In it, there is a normal subgroup of the tetrahedron. After all, two tetrahedra can be inscribed into a cube (Fig. 3), which, intersecting, form a non-convex, star-shaped polyhedron, which Kepler named *stella octangula* (eight-pointed star). And each of the tetrahedra defines its normal subgroup. And the tetrahedron group, in turn, also has a normal subgroup. It consists of the identical rotation and three rotations on 180° around three axes of the second order -- the lines connecting the midpoints of the opposite edges (above we considered possible rotations of a tetrahedron); it is called quadratic, or Klein's group.

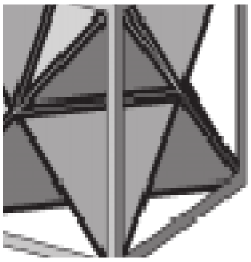

Figure 3

I want to put forward the hypothesis: operations of the Klein group correspond to four carriers of electroweak interaction -- the photon, W^+, W^-, Z^0. But the tetrahedron can still be rotated around the four axes of the third order, and there are eight such operations -- these will be eight gluons. Finally, there are 12 rotations of the cube that rearrange two tetrahedra in the *stella octangula*, and geometrically (in the type of rotation axes) such rotations are divided into two sixes, which it is logical to associate with six X- and six Y-particles.

Then we see that the weak forces correspond to the Klein group, in which four elements are pairwise rearranged -- it is known that four fermions always participate in weak interactions. And the color forces between the quarks correspond to the rotations of the tetrahedron around the third-order axes when three elements are cyclically rearranged. One can fantasize that quarks and gluons are not observed in a free form because they are generated by subgroups that are not normal.

As the result, the decomposition of the cube group (24 = 4 + 8 + 6 + 6) completely coincides with the sets of bosons carrying different physical forces in the SU(5) model. But here they arose directly from the structure of the group S_4, that is, from the group of rotation of the cube.

Fermions and icosahedron

It seems that the basic fermions are related to the icosahedron-dodecahedron. As Georgie and Glashow have discovered, symmetries of the fifth order manifest themselves in a multitude of these particles, and this is a characteristic feature of precisely such figures (see Glashow's book `The Charm of Physics`).

We said that the final number of fermions generations is not yet known. Up to date we have 24 particles, but in order to obtain their total number, one must also take into account the antiparticles and different polarizations (two possible spin values). For one generation, 8x2x2 = 32, but neutrinos are of only left polarization, and antineutrino are of the right polarization. Hence, only 30, and for three generations 90.

However, if there are actually four generations (which is quite possible), there will be 120 of them, and this quantity fits better in the "Platonic bed". This, as we recall, is the order of the group S_5 (5-factorial), but now we are dealing with fermions, and therefore we are searching not a symmetric group, but an alternating one.

The rotation group of the icosahedron A_5 is of order 60, but it can be doubled: if we add reflection to one of the planes of symmetry of this polyhedron to admissible operations, then we go to the extended group $\underline{A}_5$. (Although it contains as many elements as S_5, these are two completely different groups.)

The tetrahedron also has an extended $\underline{A}_4$ group. And what an interesting effect: the A_5 group (the icosahedron rotation) has the subgroup A_4 (the tetrahedron rotations), but in the extended group $\underline{A}_5$ the extended $\underline{A}_4$ group no longer serves as a subgroup. Instead of it, $\underline{A}_5$ has another subgroup (we denote it by $*A_4$), consisting of a group of tetrahedron supplemented by a reflection operation with respect to its center. This subgroup corresponds to nothing else than *stella octangula*.

Let's try to give the newly introduced operations a physical meaning. Reflection from the plane could correspond to a transition from a particle to an antiparticle, and reflection from the center to a different polarization. Let us also assume that the extended group $\underline{A}_5$ acts at the Grand Unification energies, and at lower ones its subgroup $*A_4$ (as the theory of phase transitions teaches, it becomes less symmetrical upon cooling of the system).

Then, with a drop in temperature, the equality of particles and antiparticles should have been violated. Hence, the properties of finite groups can give the solution of the observed asymmetry of nature -- the presence of the world, but not the anti-world.

It was noticed (Georgie wrote about this) that every generation of fermions is well modeled by a cube. In fact, arrange the cube on a horizontal plane so that it stands on top (Fig. 4). We draw on the plane three axes A, B, C at 120° angles to each other -- they will represent the color charges; the value of the electric charge is plotted vertically. Then the lower and upper vertices of the cube correspond two leptons of one generation, say, neutrinos and positrons (their electrical charges are 0 and 1, and color charges are absent). The remaining six vertices form regular triangles in two horizontal planes, displaying all the color states of the u-quark and the d-antiquark. Their electric charges, as it should be, are fractional: 2/3 and 1/3.

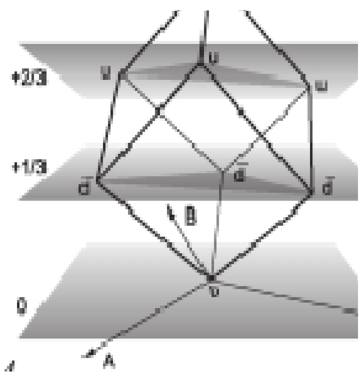

Figure 4

But in fact such a cube can be replaced with an eight-pointed star inscribed in it, the ends of which are the same eight points as the cube's vertices. Two of its crossed tetrahedra that form it reflect the symmetry between two quadruples of particles in each fermions generation. Apparently, it corresponds to *stella octangula* -- a subgroup $^*A_4$ of the extended icosahedron group $\underline{A}_5$.

Forward to Plato

Of course, all of the above is not so much a solution as posing a problem. Fermions and bosons are not independent, and therefore both classes of particles must be interconnected. In addition, we need to reconcile these considerations with modern field theory. And yet it is difficult to get rid of the impression that the right polyhedra are really capable of shedding new light on the structure of matter.

Among the Platonic solids, the icosahedron is most interesting, and it is encountered, sometimes quite unexpectedly, in the most diverse areas of mathematics (see a known book about icosahedron by F. Klein). This fact should serve as a heuristic when working on a unified theory of elementary particles -- indeed, in nature the most sophisticated abstract structure is certainly embodied. Its search is the Promethean task of our days.

As Werner Heisenberg wrote, "the development of physics looks as if in the end very simple laws of nature will be found, such as Plato hoped to see them". It is not ruled out that these laws will be connected with regular polyhedra. Even when knowledge of physical reality was still very scarce, there were thinkers (Plato, Kepler) who saw in these bodies the key to its understanding. They probably make up the rearguard, which is always ahead.