# A note on a problem invoving a square in a curvilinear triangle

Hiroshi Okumura

**Abstract.** A problem involving a square in the curvilinear triangle made by two touching congruent circles and their common tangent is generalized.

**Keywords.** square in a curvilinear triangle

**Mathematics Subject Classification (2010).** 01A27, 51M04

Let $\alpha_1$ and $\alpha_2$ be touching circles of radius $a$ with external common tangent $t$. In this note we consider the following problem [1, 4, 5] (see Figure 1).

**Problem 1.** $ABCD$ is a square such that the side $DA$ lies on $t$ and the points $C$ and $B$ lie on $\alpha_1$ and $\alpha_2$, respectively. Show that $2a = 5|AB|$.

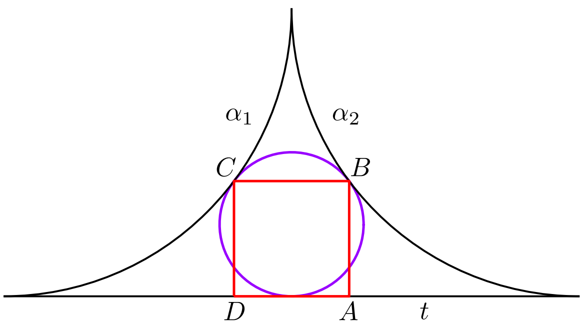


Figure 1.

If $\gamma_1$, $\gamma_2$, $\cdots$, $\gamma_n$ are congruent circles touching a line $s$ from the same side such that $\gamma_1$ and $\gamma_2$ touch and $\gamma_i$ $(i = 3, 4, \cdots, n)$ touches $\gamma_{i-1}$ from the side opposite to $\gamma_1$, then $\gamma_1$, $\gamma_2$, $\cdots$, $\gamma_n$ are called congruent circles on $s$. The curvilinear triangle made by $\alpha_1$, $\alpha_2$ and $t$ is denoted by $\Delta$. The incircle of $\Delta$ touches $\alpha_1$ and $\alpha_2$ at $C$ and $B$, respectively as in Figure 1. Indeed the problem is generalized as follows (see Figure 2):

**Theorem 1.** *If $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ are congruent circles on $t$ lying in $\Delta$ such that $\beta_1$ touches $\alpha_1$ at a point $C$ and $\beta_n$ touches $\alpha_2$ at a point $B$ and $A$ is the foot of perpendicular from $B$ to $t$, then the following relations hold.*
(i) $n|AB| = |BC|$.
(ii) $2a = \left((\sqrt{n}+1)^2 + 1\right)|AB|$.

*Proof.* Let $b$ be the radius of $\beta_1$. By Theorem 5.1 in [2] we have

$$a = (\sqrt{n}+1)^2 b. \tag{1}$$

Let $d = |AB|$. Since $C$ divides the segment joining the centers of $\alpha_1$ and $\beta_1$ in the ratio $a : b$ internally, we have

$$\frac{d-b}{b} = \frac{a-b}{a+b}. \tag{2}$$

Eliminating $b$ from (1) and (2), and solving the resulting equation for $d$, we get $d = 2a/(1 + (1 \pm \sqrt{n})^2)$. But in the minus sign case we get $2b - d = 2a(1 - 4\sqrt{n})/(n^2 - n + 2\sqrt{n} + 2) < 0$ by (1). Hence $d = 2a/(1 + (1 + \sqrt{n})^2)$. This proves (ii). Let $|BC| = 2h$. Then from the right triangle formed by the line $BC$, the segment joining the centers of $\alpha_1$ and $\beta_1$, and the perpendicular from the center of $\alpha_1$ to $BC$, we get $(a - h)^2 + (a - d)^2 = a^2$. Solving the equation for $h$, we have $h = a - \sqrt{(2a - d)d} = an/(1 + (1 + \sqrt{n})^2)$. This proves (i). □

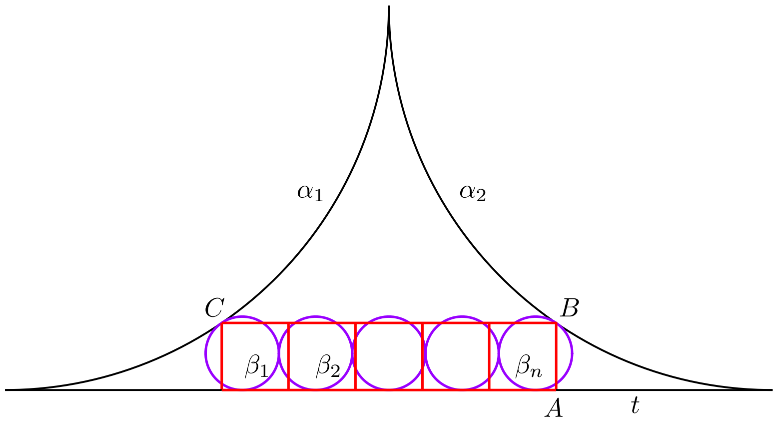


Figure 2: $n = 5$

The figure consisting of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ and $t$ is denoted by $\mathcal{B}(n)$ and considered in [2]. The next theorem also shows that the points $B$ and $C$ lies on the incircle of $\Delta$ in Figure 1 (see Figure 3).

**Theorem 2.** *Let $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ be congruent circles on a line $s$. If a circle $\alpha$ touches $s$ and $\beta_1$ and $\beta_n$ externally at points $C$ and $B$, respectively, $A$ is the foot of perpendicular from $B$ to $s$, then the following relations hold.*
(i) $(n - 1)|AB| = |BC|$.
(ii) $2a = ((n - 1)^2 + 4)|AB|/4$.

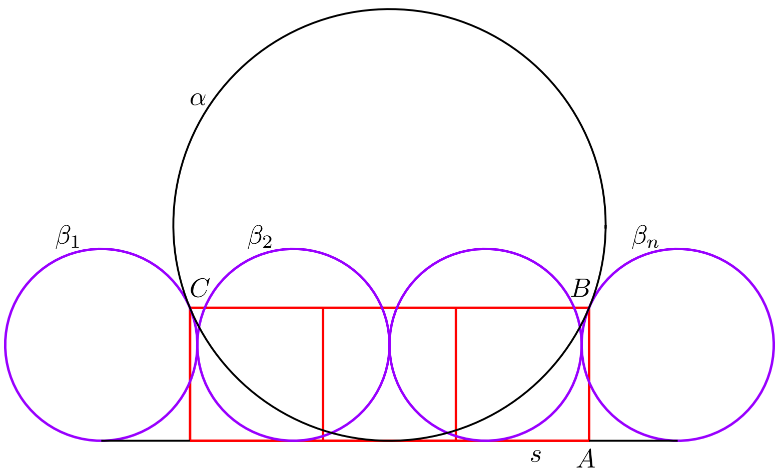


Figure 3: $n = 4$

Theorem 2 is proved in a similar way as Theorem 1 using the fact that the ratio of the radii of $\alpha$ and $\beta_1$ equals $(n - 1)^2 : 4$ [3]. The figure consisting of $\alpha$, $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ and $s$ is denoted by $\mathcal{A}(n)$ and considered in [2].

## References


[1] Aida ed., Sampō Tenshōhō, 1788, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100002292.

[2] H. Okumura, Configurations of congruent circles on a line, Sangaku J. Math., **1** (2017) 24–34.

[3] H. Okumura, Variatios of the ratio 1 : 4, Math. and Informatics Quarterly, **3**(4) (1993) 162–166.

[4] Enrui Tekitō Shū, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003918.

[5] Yōjutsu Kugōhyū, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007186.


Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.