Diophantine Quintuples over Quadratic Rings

Philip Gibbs

Abstract: A Diophantine m-tuple is a set of m distinct non-zero integers such that the product of any two elements of the set is one less than a square. The definition can be generalised to any commutative ring. A computational search is undertaken to find Diophantine 5-tuples (quintuples) over the ring of quadratic integers $\mathbb{Z}[\sqrt{D}]$ for small positive and negative D. Examples are found for all positive square-free D up to 22, but none are found for the complex rings including the Gaussian integers.

Introduction

A Diophantine m-tuple with the property $D(n)$ over a commutative ring is a set of m distinct non-zero elements of the ring with the property that the product of any two distinct elements of the m-tuple increased by n is a squared element in the ring

$$a_i a_j + n = {x_{ij}}^2, \qquad 1 \le i < j \le m$$

n itself can be an integer or some other element of the ring.

Diophantus looked for m-tuples with the property $D(1)$ over the rational numbers finding 3-tuples (rational Diophantine triples) and 4-tuples (rational Diophantine quadruples) [1]. Fermat considered Diophantine quadruples over the integers finding examples including the set {1,3,8,120} [2]. Euler showed that Fermat's Diophantine quadruples could be extended to rational Diophantine quintuples [3]. In recent times rational Diophantine sextuples were found [4,5,6,7]. It is now known that there are infinitely many [8,9]. However, by a result still subject to peer review there is known to be no Diophantine quintuples in integers [10].

Diophantine m-tuples with the property $D(n)$ have been studied over various other commutative rings including integer polynomials [11-13], real polynomials [14], finite fields [14,15], Gaussian integers [16-20] and other quadratic rings [21-25], and algebraic rings [26-29]. These results have mostly been limited to quadruples. The focus here is on the existence of Diophantine quintuples with the property $D(1)$ over quadratic rings.

Generalities

A Diophantine m-tuple with property $D(n)$ over a commutative ring $\mathbb{X}$ (such as the integers) provides an m-tuple with property $D(n^2)$ in the quadratic ring extension $\mathbb{X}[\sqrt{n}]$. Simply multiply each element and each root by $\sqrt{n}$. Furthermore, if $\mathbb{X}$ is also a field (such as the rationals) there is an m-tuple with property $D(1)$ in $\mathbb{X}[\sqrt{n}]$ obtained by dividing through by $\sqrt{n}$.

For the case of property $D(1)$ well-known extensions methods can be used. In any ring $\mathbb{X}$ a Diophantine pair can usually be extended to a regular triple and a triple can usually be extended to a regular quadruple using the method of Arkin, Hoggatt, Strauss [30]. If $\mathbb{X}$ is also a field then a quadruple can normally be extended to a regular quintuple using the method of Dujella [31]. These methods will only fail in exceptional circumstances where the formula returns zero, or duplicates of existing elements of the set, or (in the case for extension of quadruples) when the denominator is zero. See [4] for the meaning of "regular". By these methods an infinite number of quadruples can normally be constructed for any non-finite ring and an infinite number of quintuples for any non-finite field.

Furthermore, the extension methods usually provide two alternative extension elements $a_{m+1}, a_{m+2} \in \mathbb{X}$ when the m-tuple being extended is not itself regular. If both these elements can be added to the set, this then provides an almost-$(m+2)$-tuple where only the product $(a_{m+1}, a_{m+2} + 1)$ is usually not a square. This is therefore a full $(m+2)$-tuple over the ring $\mathbb{X}[\sqrt{a_{m+1}, a_{m+2} + 1}]$. This means for example that there are Diophantine quintuples over the quadratic ring $\mathbb{Z}[\sqrt{D}]$ for some positive D and there are Diophantine sextuples over the quadratic field $\mathbb{Q}[\sqrt{D}]$ for some positive D. An alternative method for constructing such sextuples is to use the formula for regular sextuples [32] to extend a Diophantine quintuple over a field $\mathbb{X}$ with a new element a_6. This does not normally provide a sextuple over $\mathbb{X}$ but the products $(a_i a_6 + 1)(a_j a_6 + 1)$, $i < j$ will all be squares. This gives the sextuple property $D(1)$ over the field extension $\mathbb{X}[\sqrt{a_1, a_6 + 1}]$. Finally, in a survey of rational Diophantine sextuples three examples of almost Diophantine septuples were found where only one product fails to be one less than a square [33]. This provides three examples of Diophantine septuples over $\mathbb{Q}[\sqrt{D}]$ for three different positive integers D.

Diophantine quadruples over $\mathbb{Z}[\sqrt{D}]$

The extension formula provide an infinite number of Diophantine quadruples with property $D(1)$ over the quadratic ring $\mathbb{Z}[\sqrt{D}]$ for any square free integer $D \notin \{0,1\}$, positive or negative.

Irregular $D(1)$ quadruples that include two regular triples are easy to check for. This requires two factorisation steps starting with the factorisation of the number 3 [4]. For $D < 0$ only a few factorisations are possible and it can be checked that all cases lead to invalid quadruples where two elements are equal, or one element is zero. In the remaining cases where $D > 0$ and square free, every number has an infinite number of factorisations from solutions of Pell's equations. These provide an infinite number of irregular quadruples.

A computational search easily finds many other irregular quadruples for $D > 0$ which do not contain two regular triples. No irregular quadruples have been found for $D < 0$ and it seems reasonable to conjecture that they do not exist.

Diophantine quintuples over $\mathbb{Z}[\sqrt{D}]$

A search for Diophantine quintuples with property $D(1)$ over $\mathbb{Z}[\sqrt{D}]$ was conducted for square free D with $|D| < 50$, including the Gaussian integers. 160 examples were found. There were solutions for all positive square-free values for D between 2 and 22. All of them were irregular but included two regular quadruples. No Diophantine quintuples were found for negative D.

The smallest example was the quintuple

$$\{4, \quad 7+3\sqrt{5}, \quad 7-3\sqrt{5}, \quad 50+22\sqrt{5}, \quad 50-22\sqrt{5}\}$$

A full list is given below. Each is shown in the form $D, (a\ b), (c\ d), \ldots$ to indicate $a+\sqrt{D}b, \quad c+\sqrt{D}d, \quad \ldots$

2, (0 2), (4 8), (42 30), (7416 5244), (-28 20)
2, (0 21), (88 148), (426 309), (15240708 10776808), (-79360 56116)
2, (1 0), (7 0), (24 0), (368 260), (368 -260)
2, (3 0), (7 4), (7 -4), (119 -84), (119 84)
2, (3 0), (39 20), (39 -20), (4407 3116), (4407 -3116)
2, (3 12), (-5 4), (-39 36), (345 244), (-26887 19012)
2, (4 0), (272 170), (2454 1530), (18934804 13388928), (73664 -52088)
2, (4 1), (30 9), (872 552), (766340 541884), (8272 -5848)
2, (4 10), (18 42), (-116 92), (39728 28092), (-206956 146340)
2, (6 0), (31 15), (31 -15), (6200 -4384), (6200 4384)
2, (6 0), (403 279), (403 -279), (81536 -57652), (81536 57652)
2, (6 2), (8 -4), (31 20), (255 -180), (2523 1784)
2, (7 5), (-21 63), (-146 112), (-5176 3660), (23840 16860)
2, (10 0), (13 9), (13 -9), (176 124), (176 -124)
2, (10 10), (-12 12), (-16 52), (-21612 15282), (13896 9826)
2, (12 10), (-60 318), (-170 172), (-903360 638772), (1492940 1055668)
2, (17 12), (17 -12), (21 0), (97 68), (97 -68)
2, (17 36), (96 72), (-197 140), (26864 18996), (-156440 110620)
2, (20 16), (-30 25), (-56 49), (-43224 30564), (6132 4336)
2, (21 0), (97 68), (97 -68), (6977 4932), (6977 -4932)
2, (21 19), (-48 56), (-105 75), (3264 2308), (-315528 223112)
2, (30 18), (160 112), (1767 1240), (124329075 87913932), (215 -152)
2, (32 22), (246 -126), (1300 690), (19481564 13775546), (244792 -173094)
2, (39 36), (407 292), (-429 304), (136695 96660), (-122633 86716)
2, (56 40), (-65 46), (528 496), (16221 11470), (-25935 18366)
2, (65 36), (319 -220), (593 228), (1671855 1182180), (4804655 -3397404)
3, (0 1), (0 21), (0 65), (14336 8277), (-14336 8277)
3, (0 1), (0 56), (0 96), (56134 32409), (-56134 32409)
3, (0 1), (60 44), (-60 44), (-7802 4505), (7802 4505)
3, (0 2), (0 28), (0 60), (35074 20250), (-35074 20250)
3, (0 7), (0 40), (0 8), (23374 13495), (-23374 13495)
3, (0 8), (0 40), (0 7), (23374 13495), (-23374 13495)
3, (0 8), (0 100), (0 210), (1746458 1008318), (-1746458 1008318)
3, (0 11), (0 95), (0 31), (336896 194507), (-336896 194507)
3, (0 17), (70 41), (-70 41), (8592 4961), (-8592 4961)

3, (0 23), (40 35), (-144 87), (53768 31043), (-484536 279747)
3, (0 24), (0 217), (0 1160), (62785250 36249081), (-62785250 36249081)
3, (0 28), (0 836), (0 390), (94874770 54775974), (-94874770 54775974)
3, (2 2), (-6 4), (12 8), (-86 50), (294 170)
3, (6 2), (18 4), (28 -16), (170 98), (3198 -1846)
3, (6 2), (280 -44), (4302 2440), (32905214 18997834), (137370 -79310)
3, (6 3), (50 -21), (120 35), (8392 -4845), (55320 31939)
3, (8 6), (-20 12), (-60 41), (348 201), (-25740 14861)
3, (9 3), (23 -13), (60 -20), (32916 -19004), (388 224)
3, (9 5), (15 -7), (36 -20), (160 92), (1400 -808)
3, (12 7), (-20 12), (-36 28), (510 295), (-982 567)
3, (12 7), (-20 12), (48 42), (-316 183), (4716 2723)
3, (34 20), (-48 28), (50 98), (-15030 8678), (15294 8830)
3, (42 22), (90 40), (246 -142), (79912 -46136), (1484 856)
3, (54 50), (-156 316), (-672 406), (3435308 1983376), (-63070136 36413560)
5, (1 1), (5 3), (-52 24), (138 66), (-550 246)
5, (1 1), (43 29), (138 66), (199730 89322), (-550 246)
5, (2 0), (14 6), (20 -6), (760 338), (112 -50)
5, (2 1), (-56 36), (90 51), (42580 19044), (-1708 764)
5, (2 4), (-58 26), (-516 234), (40 18), (-1677440 750174)
5, (3 3), (-15 7), (46 22), (1310 586), (-762 342)
5, (4 0), (7 3), (7 -3), (50 -22), (50 22)
5, (6 0), (34 10), (110 -48), (1870 836), (30590 -13680)
5, (6 0), (590 168), (1954 -850), (9918590 -4435728), (619150 276892)
5, (6 2), (60 -26), (228 -64), (3408 1524), (134460 -60132)
5, (8 4), (41 20), (-147 66), (-2360 1056), (1780 804)
5, (8 4), (-144 72), (-170 114), (49028 21926), (-245480 109782)
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5, (18 6), (25 -9), (41 17), (1288 -576), (24320 10876)
5, (18 8), (20 8), (76 -34), (198 -88), (30 -12)
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5, (30 8), (32 14), (54 -24), (2140 956), (1932 -864)
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6, (3 2), (303 374), (-993 422), (-4721905 1927710), (784831 320406)

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6, (6 3), (-12 7), (36 16), (10424 4256), (-284 116)
6, (10 3), (36 -13), (44 12), (5340 -2180), (10680 4360)
6, (42 18), (-57 38), (232 136), (3511515 1433570), (-63721 26014)
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7, (0 3), (232 88), (416 166), (6311712 2385603), (-336 127)
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7, (32 12), (114 -42), (126 24), (69802 26382), (7350 -2778)
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37, (4 0), (43 5), (43 -5), (7482 -1230), (7482 1230)
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41, (69 3), (125 17), (366 -54), (1145250 -178858), (835006 130406)
46, (16 4), (-32 8), (81 12), (312011 46004), (-841 124)

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