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Investigation of the Characteristics of the Zeros of the Riemann Zeta Function in the Critical Strip Using the Characteristics of the Real Component of the Dirichlet eta Function.

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Abstract: This paper investigates the characteristics of the zeros of the real component of the Riemann zeta function (of s) in the critical strip by using the real component of the Dirichlet eta function, which has the same zeros (A necessary condition for a zero of the complete function is a zero of the real component). The derivative of the real component for a fixed imaginary part of s is shown to be always positive for negative or zero values of the real component of the function, meaning that each value of the imaginary part of s produces at most one zero. Combined with the fact that the zeros of the Riemann xi function are also the zeros of the zeta function and xi(s) = xi(1-s), this leads to the conclusion that the Riemann Hypothesis is true.

Keywords: Riemann xi; Riemann zeta; Zeros; Dirichlet eta; Critical Strip; Analysis; partial sums; real component; derivative with respect to single variable

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1. Introduction

This paper investigates one of the key unresolved questions arising from Riemann's original 1859 paper regarding the distribution of prime numbers ('Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse'[1, p. 145] - translation in Edwards [2, p. 299]) - the nature of the roots of the Riemann xi function ('One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real' - referring to the roots of the Riemann Xi function).

This paper starts in section 2.1 from Riemann's original definition of ζ(s) and ξ(s) and notes the implications of ξ(s) in power series form for the roots of ξ(s) and therefore of ζ(s).

Section 2.1 also highlights the characteristics of the real and imaginary components of ζ(s) and investigates the behaviour of the function re(ζ(s))=im(ζ(s)) for a specific example, showing the unlikely nature of there being two zeros of the entire function for a fixed value of the imaginary part of s.

Section 2.2 looks more formally at the Dirichlet eta function (η(s)) which has the same zeros as ζ(s). The behaviour of the real component of η(s) (recognising that a necessary condition for a zero of η(s) is a zero of the real component of η(s)) is investigated, together with its derivative, using a series representation. The derivative of the real component of η(s) with a fixed imaginary component of s is shown to be always positive for negative or zero values of the real component of η(s). This leads to the conclusion that any fixed imaginary component of s can produce at most one zero for the real component of η(s).

Section 3 develops the implications of the earlier investigations, leading to the conclusion that the Riemann Hypothesis is true.

2. Results

2.1. Observations of the characteristics of the real and imaginary components of the Riemann Zeta function highlighting when they have the same value.

2.1.1. Riemann zeta Function and Riemann xi function definitions

Riemann's paper starts from the definition (Riemann)[1, p. 145]:

$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$ (Convergent for Re(s)>1).

Riemann then extends the zeta function analytically for all s and defines the xi function (which has the same zeros as the zeta function) and shows that it can be written as a power series (Edwards)[2, p. 17]:

$\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-\frac{1}{2})^{2n}$ where a_{2n}=$4\int_1^{\infty}(d/dx(x^{3/2}\psi'(x))x^{-1/4}\frac{logx^{2n}}{2^{2n}(2n)!})dx$

Now, using Riemann's s = 1/2+it and defining t=(a+bi), then (s-$\frac{1}{2}$) = it = (ai-b), and:

ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai-b)^{2n}$

Note that the functional equation of the zeta function is equivalent to ξ(s)=ξ(1-s) (Edwards)[2, p. 16]. This, combined with the fact that any complex root of the power series will also have the complex conjugate of that root as a root, means that if (b+ai) is a root of ξ(s), then so are all of (b-ai), (-b+ai) and (-b-ai). This, in turn, means that (1/2+b +ai), (1/2+b -ai), (1/2-b +ai) and (1/2-b -ai) are all roots of ζ(s).

For convenience, the real part of s (equivalent to (1/2 +/- b)) will be referred to as σ in the rest of this paper.

2.1.2. Riemann zeta Function real and imaginary component characteristics observations.

Analytic extensions of the function valid for all s are well documented and have been used to make useful (numerical) applications for calculating ζ(s). One of these numerical applications (from matlab) was used to create the 2 figures following, before we look at a more formal approach.

Observing the characteristics of the real and imaginary parts of the ζ(s) for various values of σ and a in figure 1 below, it is useful to note the following:

Firstly, the real component of ζ(s) is reflected across the vertical axis, while the imaginary component is rotated by π around the origin, highlighting the fact that in general, ζ(s) does not necessarily equal ζ(1-s) (contrasting with the Riemann xi function, where ξ(s)=ξ(1-s)).

Secondly, looking carefully at the points of intersection of the real and imaginary parts of ζ(s) (ie where the real part of ζ(s) is equal to the imaginary part of ζ(s)), we can start to see the path that the implicit function described by Re(ζ(s))=Im(ζ(s)) traces.

Focussing on the points where the real and imaginary parts intersect for various values of sigma around a known zero of the zeta function in figure 2 below, we can see that the intersection points are at different values along an apparent single valued curve. This already gives an indication that it is very unlikely that at any point of the zeta function where b$\neq$0, then the points described by (Re(ζ(1/2+b+ai))=Im(ζ(1/2+b+ai))) will be equal to the points described by

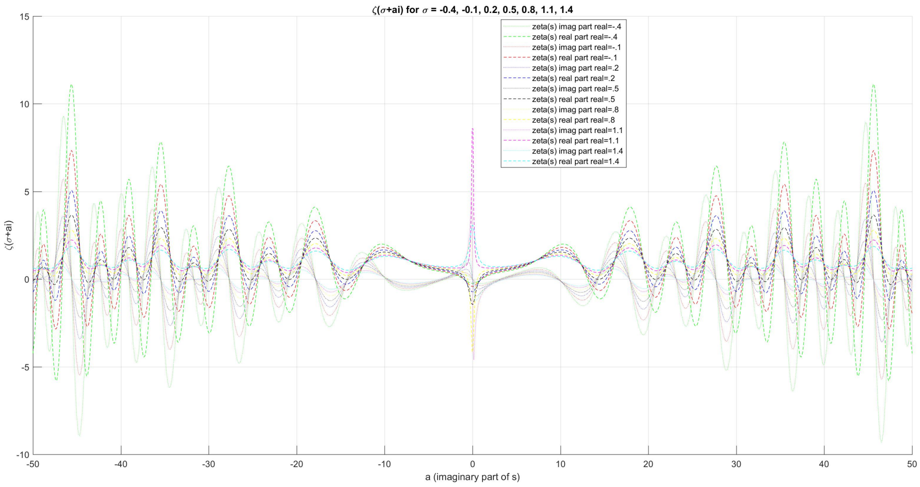

Figure 1. Riemann Zeta Function.

(Re(ζ(1/2-b+ai))=Im(ζ(1/2-b+ai))).

The next step is to follow a more formal approach to showing that the points described by (Re(ζ(1/2+b+ai))=Im(ζ(1/2+b+ai))) will not be equal to the points described by (Re(ζ(1/2-b+ai))=Im(ζ(1/2-b+ai))), where b≠0 - especially the zeros of the real component.

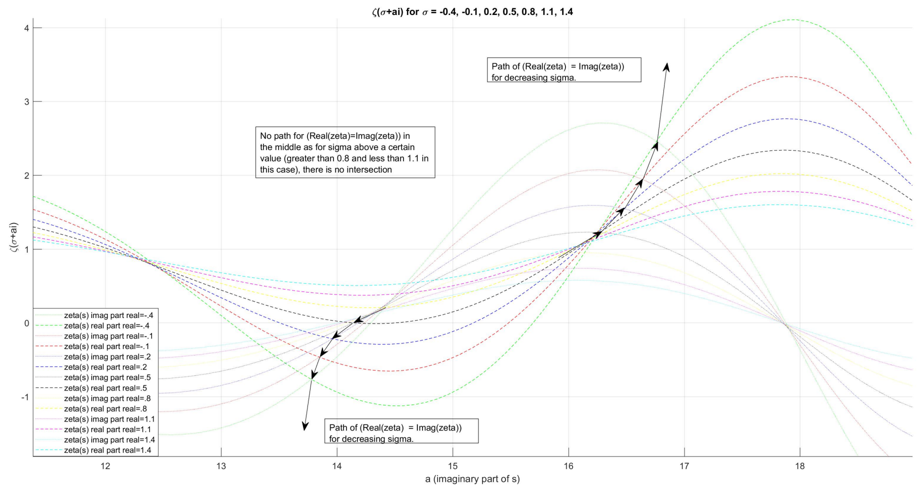

Figure 2. Riemann Zeta Function Detail Around a Known Zero.

2.2. Formal approach to describing the paths of the zeros of the real component of the zeta function in the critical strip.

For all that follows, we shall restrict the value of b between +1/2 to -1/2 (which means restricting σ between 0 and 1). Riemann proved in his original paper that all zeros of the Riemann xi function have t with imaginary parts inside the region of $+\frac{1}{2}$i to $-\frac{1}{2}$i, which is equivalent to restricting b and σ.

2.2.1. Zeta function zeros for 0<σ<1.

In the well known Dirichlet η function [3, p. 25.2.3] (also known as the alternating zeta function), which is related to the zeta function by $\eta(s)$=(1-$2^{(1-s)}$)$\zeta(s)$ and is convergent (uniformly not absolutely) for σ>0, we have an expression that can be used to explore the characteristics of the real component, imaginary component and/or function zeros of the zeta function in the critical strip. It is important to note that (1-$2^{(1-s)}$) does not have any zeros for 0$\leq$ σ<1. It has an infinite number of zeros for σ=1.

It is important to emphasize that the relation between ζ(s) and η(s) shows that the two functions have the same zeros for 0<σ<1.

A zero of η(s) requires coincident zeros of both real and imaginary components of the function.

2.2.2. Eta function real component zeros for σ>0.

Investigating the zeros of the real component of η(s).

Starting with:

$$\eta(s) = \Sigma_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n^s} = (1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}...)$$

Extracting the real part for one term (remembering that s= (σ+ai)):

$$\frac{1}{n^s} = \frac{1}{n^\sigma(cos(alog(n))+isin(alog(n))}$$

$$= \frac{n^\sigma cos(alog(n))-n^\sigma isin(alog(n))}{(n^\sigma cos(alog(n)))^2+(n^\sigma sin(alog(n)))^2}$$

$$= \frac{cos(alog(n))-isin(alog(n))}{n^\sigma}$$

This leads to the series representation of the real part as:

$$1-\frac{cos(alog(2))}{2^\sigma}+\frac{cos(alog(3))}{3^\sigma}-\frac{cos(alog(4))}{4^\sigma}... \text{ Exp 1}$$

2.2.3. Eta function real component with fixed a term.

Further considering expression 1 with a fixed 'a' term (written as A) (ie considering it as an expression with σ as the only variable) and differentiating with respect to σ:

$$\text{For fixed a (A): } \frac{\partial Exp1}{\partial\sigma} = \frac{log(2)cos(Alog(2))}{2^\sigma}-\frac{log(3)cos(Alog(3))}{3^\sigma}+\frac{log(4)cos(Alog(4))}{4^\sigma}-... \text{ Exp 2}$$

This expression is convergent for σ>0 (from the uniform convergence of the eta function series, but can also be seen from the fact that $\frac{log(n)}{n^\sigma}$ eventually becomes a monotonically reducing series tending to zero from a (large) value of n for any value of σ>0, which together with the Dirichlet test shows convergence).

Looking at Expression 1 = 0 (for fixed a) and N=4 (looking at greater values of N later):

$$1-\frac{cos(Alog(2))}{2^\sigma}+\frac{cos(Alog(3))}{3^\sigma}-\frac{cos(Alog(4))}{4^\sigma} = 0$$

$\Longrightarrow \frac{cos(Alog(2))}{2^\sigma}$-$\frac{cos(Alog(3))}{3^\sigma}$+$\frac{cos(Alog(4))}{4^\sigma}$ = 1

This result also shows that:

$\frac{log(2)cos(Alog(2))}{2^\sigma}$ = log(2) + $\frac{log(2)cos(Alog(3))}{3^\sigma}$ - $\frac{log(2)cos(Alog(4))}{4^\sigma}$

-$\frac{log(3)cos(Alog(3))}{3^\sigma}$ = log(3) - $\frac{log(3)cos(Alog(2))}{2^\sigma}$ - $\frac{log(3)cos(Alog(4))}{4^\sigma}$

$\frac{log(4)cos(Alog(4))}{4^\sigma}$ = log(4) + $\frac{log(4)cos(Alog(3))}{3^\sigma}$ - $\frac{log(4)cos(Alog(2))}{2^\sigma}$

Remembering Expression 2, this leads to:

$\frac{\partial Exp1}{\partial\sigma}$ = log(2)+log(3)+log(4)+$\frac{cos(Alog(2))}{2^\sigma}$(-log(3)-log(4))-$\frac{cos(Alog(3))}{3^\sigma}$(-log(2)-log(4)) +$\frac{cos(Alog(4))}{4^\sigma}$(-log(2)-log(3)) Exp 3.

In addition, noting that (again using Exp1 = 0 and substituting):

$\frac{\partial Exp1}{\partial\sigma}$ = log(2) - $\frac{cos(Alog(3))}{3^\sigma}$(log(3)-log(2)) + $\frac{cos(Alog(4))}{4^\sigma}$(log(4)-log(2))

$\frac{\partial Exp1}{\partial\sigma}$ = log(3) + $\frac{cos(Alog(2))}{2^\sigma}$(log(2)-log(3)) + $\frac{cos(Alog(4))}{4^\sigma}$(log(4)-log(3))

$\frac{\partial Exp1}{\partial\sigma}$ = log(4) + $\frac{cos(Alog(2))}{2^\sigma}$(log(2)-log(4)) - $\frac{cos(Alog(3))}{3^\sigma}$(log(3)-log(4))

Summing the above expressions and Expression 3:

4$\frac{\partial Exp1}{\partial\sigma}$ = 2(log(2)+log(3)+log(4)) + 2(log(2)-log(3)-log(4))$\frac{cos(Alog(2))}{2^\sigma}$ - 2(log(3)-log(2)-log(4))$\frac{cos(Alog(3))}{3^\sigma}$ + 2(log(4)-log(2)-log(3))$\frac{cos(Alog(4))}{4^\sigma}$

Extending to the general case (varying N):

N$\frac{\partial Exp1}{\partial\sigma}$ = 2($\sum_{n=2}^{N} log(n)$) + $\sum_{n=2}^{N} \frac{(-1)^n cos(Alog(n))}{n^\sigma} 2((\sum_{m=2}^{N} -log(m)) + 2log(n))$ Exp 4

In the limit, the Right Hand Side (RHS) of expression 4 does not converge. For the next steps of the process, we shall consider partial sums of the Dirichlet eta function (ie n ranges from 2 to N (however large) and not necessarily to ∞).

Looking at expression 4 in more detail:

The magnitude of (2($\sum_{n=2}^{N} log(n)$)) (which is positive) = the magnitude of 2($\sum_{n=2}^{N}((\sum_{m=2}^{N} -log(m)) + 2log(n)))$ (signs of the components unknown).

This can be seen since 2($\sum_{n=2}^{N}((\sum_{m=2}^{N} -log(m)) + 2log(n)))$ = -(2($\sum_{n=2}^{N} log(n)$)).

However, the magnitude of each of the $\frac{(-1)^n cos(Alog(n))}{n^\sigma}$ terms is less than 1 for σ>0 due to the $n^{-\sigma}$ components, which means that $\frac{\partial Exp1}{\partial\sigma}$ is positive when σ>0 for any fixed value of a.

The result of this is that the derivative with respect to σ of the partial sum of the series representing the real component of $\eta(s)$ is always positive when the value of the real component of $\eta(s)$ is zero for a fixed value of a (the imaginary component of s). This means that for any fixed value of a, the real component of $\eta(s)$ can only have one zero (in order to have more zeros, then the derivative would need to be negative or zero at some point when the function is zero). This, combined with the facts that 1) the whole eta function can only be zero when both the real and imaginary parts are zero 2) the eta function zeros are the same as the zeta function zeros and 3) The Riemann xi function shows that a zeta function zero at s means there is a corresponding zero at (1-s), means that s and (1-s) must have the same real component (1/2).

These results hold for any value of a and for any value of N. This means that even though expression 4 does not at first sight appear to converge, we could argue that the derivative will still be positive when N tends to the limit. More rigorously, we can argue that (based on the fact that partial sums of series approach the value of the series with a known estimate of the error as the number of terms in the partial sum increases) for any value of a we can show that the real component of the eta function has a single zero to any required degree of accuracy (by increasing N).

The implication is that the Riemann Hypothesis is true.

3. Conclusions

Known previously - The Riemann zeta function does not have zeros outside the critical strip.

In Section 2.1 the apparent behaviour of the paths of the points where Re($zeta(s)$)=Im($zeta(s)$) were observed, showing that it was unlikely that there would be 2 zeros of $zeta(s)$ for the same value of a (the imaginary component of s). In addition, the property of the Riemann xi function that ξ(s) = ξ(1-s) was noted.

In section 2.2.1 the Dirichlet eta function was introduced as an appropriate mechanism for investigating the zeros of the zeta function for $zeta(s)$ where σ>0.

In Section 2.2.2 the convergent series representation of the real part of the eta function was established. The zeros of the complete eta function exist where both real and imaginary components are zero - therefore it is necessary for the real component to be zero in order for the complete function to be zero.

In section 2.2.2 the convergent series representation of the derivative of the real part of the eta function for a fixed a value was established. Combined with the series representation of the real part of the function (and working with partial sums due to the non-convergence of resulting expressions) it was shown that the derivative will always be positive when the function itself is negative or zero (for a fixed value of a).

This leads to the conclusion that the real component of the eta function has only a single zero for a fixed value of a (the imaginary part of s), which can be shown to any required degree of accuracy by increasing N (the number of terms in the partial sum). It is also implied (by the non-converging expression) that the derivative is positive for any value of N, removing the need for relying on partial sums.

Combining these conclusions, all of the roots of η(s)and therefore ζ(s) are such that for each value of the imaginary component (a) there is at most one root, which means that since $\xi(\sigma + ai)$ = $\xi(1 - (\sigma + ai))$ those roots will be at σ=1/2 - which means that the Riemann Hypothesis is true.

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References

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All figures created in MATLAB.