3-Isoincircles Problem. Trigonometric Analysis of a Hard Sangaku Chalenge.

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Sacred Mathematics : Japanese Temple Geometry

Fukagawa Hidetoshi - Tony Rothman

Still Harder Temple Geometry Problems

Chapter 6 - Problem 3

Remarkable properties in this construction.

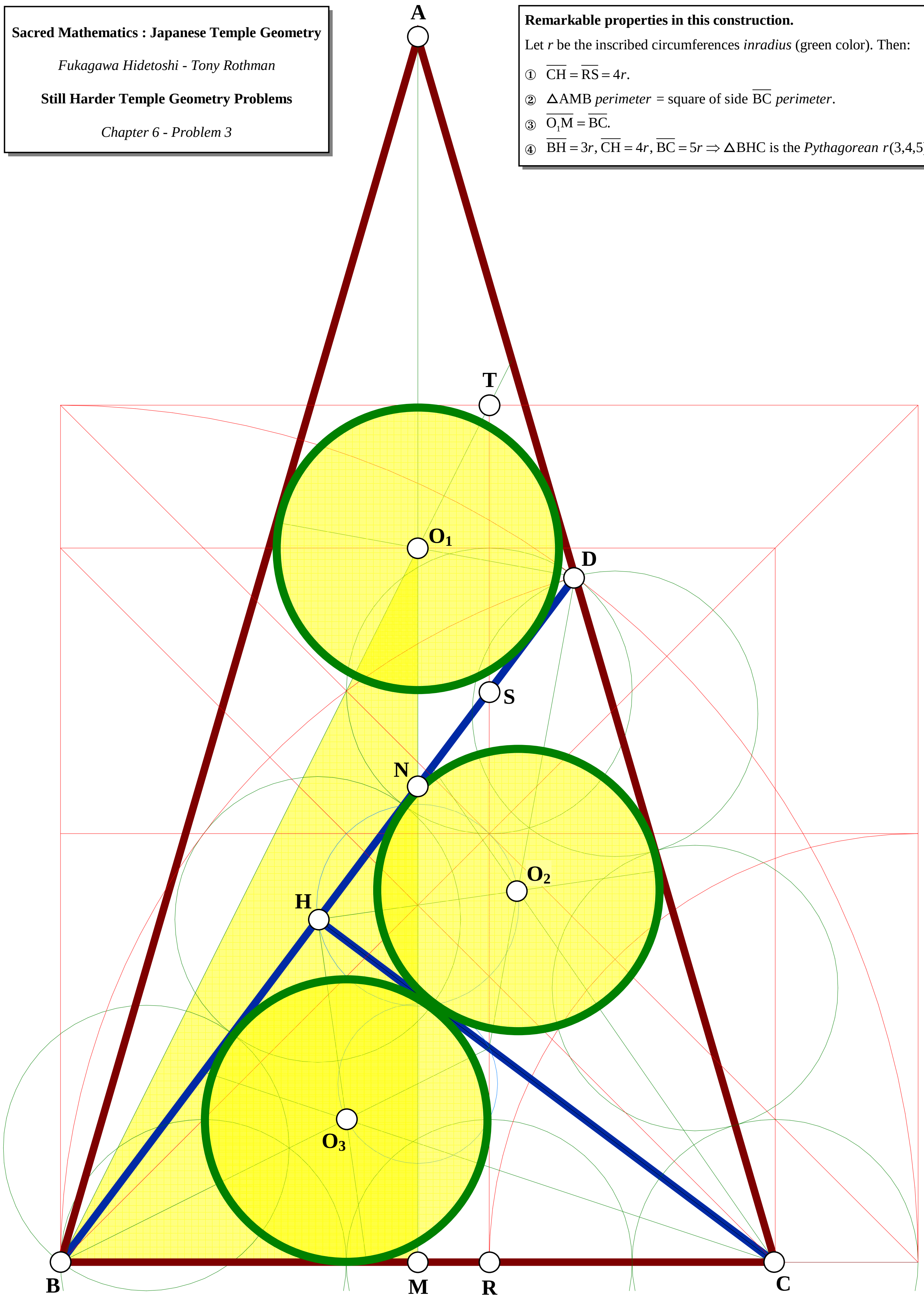
Let *r* be the inscribed circumferences *inradius* (green color). Then:

① $\overline{CH} = \overline{RS} = 4r$.

② ΔAMB *perimeter* = square of side $\overline{BC}$ *perimeter*.

③ $\overline{O_1M} = \overline{BC}$.

④ $\overline{BH} = 3r$, $\overline{CH} = 4r$, $\overline{BC} = 5r \Rightarrow \Delta$BHC is the *Pythagorean* $r(3,4,5)$.



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$$\left.\begin{array}{r}\beta+2\delta=\pi\\ \alpha=\delta+\beta-\alpha\end{array}\right\}\Rightarrow \boxed{\delta=\pi-2\alpha}$$

$$r_1=\overline{O_1P}\cos\delta=\left(\overline{O_1M}-\overline{O_1N}\right)\cos\delta=$$

$$=-a(\tan\alpha+\tan 2\alpha)\cos 2\alpha=$$

$$=-a\tan\alpha\cos 2\alpha-a\sin 2\alpha=$$

$$=-a\tan\alpha(2\cos^2\alpha-1)-a\sin 2\alpha=$$

$$=-2a\sin\alpha\cos\alpha+a\tan\alpha-a\sin 2\alpha.$$

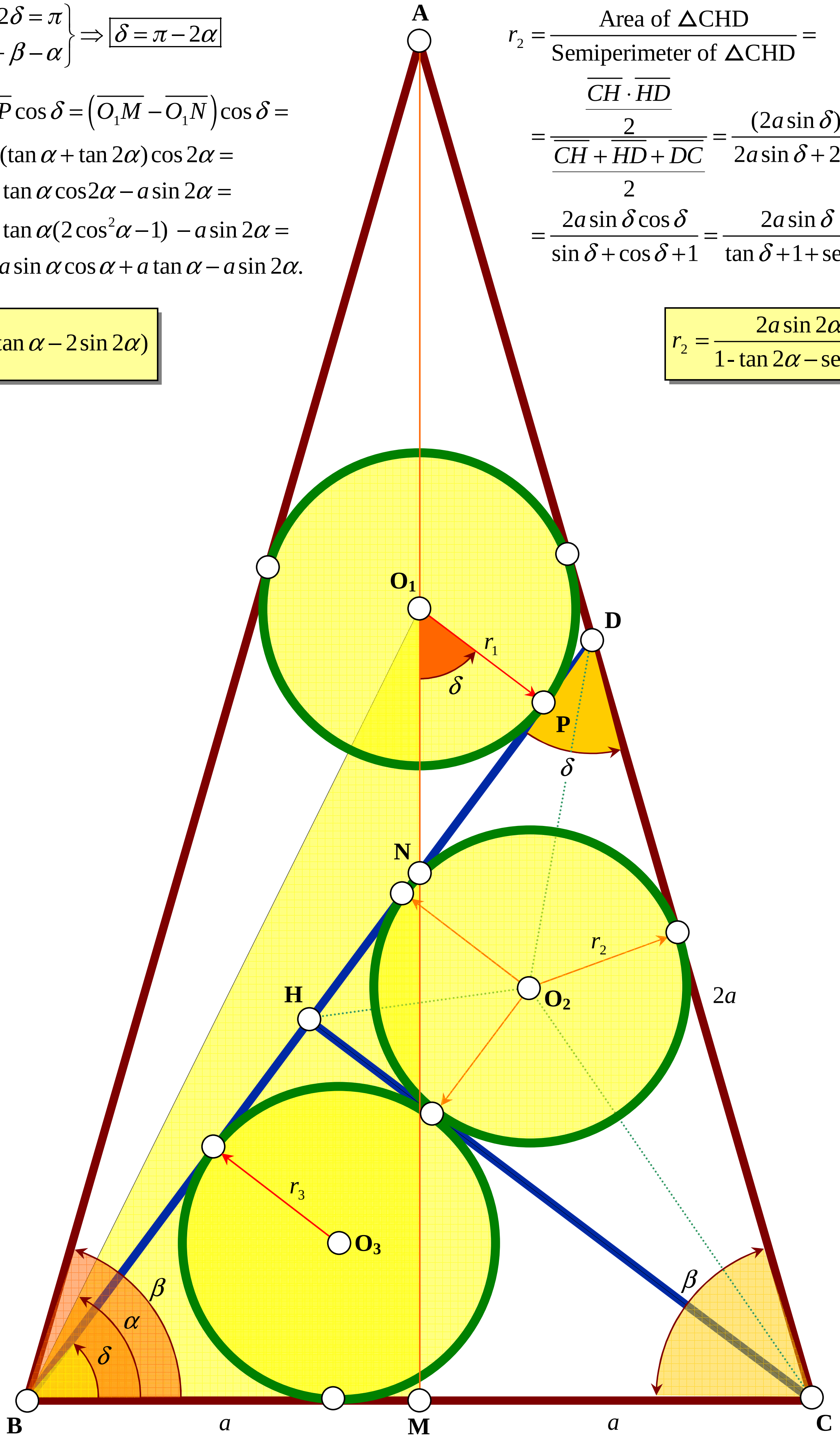
$$\boxed{r_1=a(\tan\alpha-2\sin 2\alpha)}$$

$$r_2=\frac{\text{Area of }\Delta\text{CHD}}{\text{Semiperimeter of }\Delta\text{CHD}}=$$

$$=\frac{\dfrac{\overline{CH}\cdot\overline{HD}}{2}}{\dfrac{\overline{CH}+\overline{HD}+\overline{DC}}{2}}=\frac{(2a\sin\delta)(2a\cos\delta)}{2a\sin\delta+2a\cos\delta+2a}=$$

$$=\frac{2a\sin\delta\cos\delta}{\sin\delta+\cos\delta+1}=\frac{2a\sin\delta}{\tan\delta+1+\sec\delta}.$$

$$\boxed{r_2=\frac{2a\sin 2\alpha}{1-\tan 2\alpha-\sec 2\alpha}}$$



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By equaling the calculated inradius we obtain the following trigonometric equation in the α angle:

$$r_1 \equiv r_2 \equiv r \Rightarrow \tan\alpha - 2\sin 2\alpha = \frac{2\sin 2\alpha}{1-\tan 2\alpha - \sec 2\alpha},$$

or,

$$\frac{\tan\alpha}{2\sin 2\alpha} - 1 = \frac{1}{1-\tan 2\alpha - \sec 2\alpha} \Rightarrow \frac{2\sin 2\alpha}{\tan\alpha - 2\sin 2\alpha} = 1 - \tan 2\alpha - \sec 2\alpha,$$

that can be written as $\dfrac{1}{\dfrac{\sin\alpha/\cos\alpha}{4\sin\alpha\cos\alpha} - 1} = 1 - \tan 2\alpha - \sec 2\alpha,$ i.e., $\dfrac{1}{\dfrac{1}{4\cos^2\alpha} - 1} = 1 - \tan 2\alpha - \sec 2\alpha,$

and this is obtained,

$$1 = \left(\frac{1}{2(1+\cos 2\alpha)} - 1\right)(1-\tan 2\alpha - \sec 2\alpha) = \left(-\frac{1+2\cos 2\alpha}{2(1+\cos 2\alpha)}\right)(1-\tan 2\alpha - \sec 2\alpha) \Rightarrow$$

$$\Rightarrow 2(1+\cos 2\alpha) = -(1+2\cos 2\alpha)(1-\tan 2\alpha - \sec 2\alpha) \Rightarrow 4\cos 2\alpha - 2\sin 2\alpha - \tan 2\alpha - \sec 2\alpha + 1 = 0.$$

By the variable changing $\mathrm{tg}\,\alpha \equiv t,$ the above trigonometric equation becomes the algebraic equation

$$4\frac{1-t^2}{1+t^2} - 2\frac{2t}{1+t^2} - \frac{2t}{1-t^2} - \frac{1+t^2}{1-t^2} + 1 = 0,$$

and simplifying,

$$4\frac{1-t^2-t}{1+t^2} - \frac{(1+t)^{\not 2}}{\cancel{(1+t)}(1-t)} + 1 = 0 \Rightarrow \boxed{t^3 - 5t + 2 = 0}$$

reduced cubic equation, whose exact solutions could be obtained by the Cardano's formula, but it is not necessary since, by simple inspection, we see that $t = 2$ is a root of it. Moreover, this is the only permissible in the context of this problem. Indeed, eliminating this root, the other two, which are obtained from the resulting 2nd grade equation, which are $t = \sqrt{2} - 1, t = -\sqrt{2} - 1,$ whose respective arcs tangent are $\alpha = \pi/8$ y $\alpha = -3\pi/8,$ and it is straightforward that $\triangle$ABC can not be isosceles for these values of α.

Thus, $\tan\alpha = 2, \tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\cdot 2}{1-2^2} = -\dfrac{4}{3}$ and $\tan 4\alpha = \dfrac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \dfrac{2\cdot\left(-\dfrac{4}{3}\right)}{1-\left(-\dfrac{4}{3}\right)^2} = \dfrac{24}{7}.$

Now it is easy to calculate the other parameters according to this results:

① $\boxed{\overline{\mathrm{CH}} = \overline{\mathrm{RS}} = 4r}$

$$\left.\begin{array}{l} \overline{CH} = 2a\sin\delta = 2a\sin 2\alpha = 2a\cdot\dfrac{2t}{1+t^2} = 2a\cdot\dfrac{2\cdot 2}{1+2^2} = \dfrac{8}{5}a \\ r = a\tan\alpha - 2a\sin 2\alpha = 2a - \dfrac{8}{5}a = \dfrac{2}{5}a \\ \overline{RS} = \overline{BR}\tan\delta = 3r\tan(\pi - 2\alpha) = -3r\tan 2\alpha = -3r\cdot\left(-\dfrac{4}{3}\right) = \cancel{-3}\cdot\dfrac{2}{5}a\cdot\left(-\dfrac{4}{3}\right) = \dfrac{8}{5}a \end{array}\right\} \Rightarrow \boxed{\frac{\overline{CH}}{r} = 4}$$

Without further difficulty can be proved the remaining properties:

② $\boxed{\text{perimeter of } \Delta\overline{\text{AMB}} = 4\overline{\text{BC}} = 20r}$

$\boxed{\overline{BM} \equiv a}$

$$\overline{AM} = a\tan\beta = a\tan(\pi - 2\delta) = a\tan\underbrace{\left[\pi - 2(\pi - 2\alpha\right]}_{4\alpha-\pi} = a\tan 4\alpha \Rightarrow \boxed{\overline{AM} = \frac{24}{7}a}$$

$$\overline{AB} = a\sec\beta = a\sec(\pi - 2\delta) = a\sec\underbrace{\left[\pi - 2(\pi - 2\alpha\right]}_{4\alpha-\pi} = -\frac{a}{1-2\sin^2 2\alpha} = -\frac{a}{1-2\left(\frac{2t}{1+t^2}\right)^2} \overset{t=2}{\Rightarrow} \boxed{\overline{AB} = \frac{25}{7}a}$$

Therefore,

$$\overline{\text{BM}} + \overline{\text{AM}} + \overline{\text{AB}} = a + \frac{24}{7}a + \frac{25}{7}a = 8a = 4(2a) = 4\overline{\text{BC}} = 20r.$$

③ $\boxed{\overline{\text{O}_1\text{M}} = \overline{\text{BC}}}$

$$\overline{O_1M} = a\tan\alpha = 2a = \overline{BC}.$$

④ $\boxed{\Delta\overline{\text{BHC}} \equiv \Delta\overline{\text{DHC}} \text{ are } \textit{Pythagorean triangles} \sim (3, 4, 5)}$

$$\left.\begin{array}{l} \overline{BH} = 2a\cos\delta = -2a\cos 2\alpha = -2a\dfrac{1-t^2}{1+t^2} = \dfrac{6}{5}a \Rightarrow \boxed{\overline{BH} = 3r} \\ \overline{HC} = 2a\sin\delta = 2a\sin 2\alpha = 2a\dfrac{2t}{1+t^2} = \dfrac{8}{5}a \Rightarrow \boxed{\overline{HC} = 4r} \\ \overline{BC} = 2a \Rightarrow \boxed{\overline{BC} = 5r} \end{array}\right\} \Rightarrow \boxed{\left(\overline{BH}, \overline{HC}, \overline{BC}\right) = r(3,4,5)}$$

In the first picture you can see many more relationships; eg, $\boxed{\overline{\text{AM}} \text{ bisects } \overline{\text{O}_2\text{H}}}$.

Moreover, from the angular relationships which are deducted directly from the 2nd figure,

$$\left.\begin{array}{r} \beta + 2\delta = \pi \\ \alpha = \delta + \beta - \alpha \end{array}\right\} \Rightarrow \boxed{\beta = 4\alpha - \pi}.$$

Consequently,

$$\tan\beta = \tan(4\alpha - \pi) = \tan 4\alpha = \frac{24}{7},$$

value that had already been obtained to calculate the height of the triangle.

Thus, the construction is only possible in isosceles triangles whose pair of equal angles has a value such that

$$\tan\beta = \frac{24}{7},$$

i.e.,

$$\boxed{\beta \approx 73.74^\circ}$$

To construct an isosceles triangle with these characteristics, we can take 7 units to the side and 12 on the uneven height, or 7 units for the inradio, 35 for the side and 60 on the uneven height.

Alternative demonstration, changing the reference system and the way we define the inradius

$$B(x_B, y_B) = \left(-2a\cos\frac{\beta}{2}, 0\right)$$

$$C(x_C, y_C) = \left(0, -2a\sin\frac{\beta}{2}\right)$$

$$M(x_M, y_M) = \left(\frac{x_B + x_C}{2}, \frac{y_B + y_C}{2}\right)$$

$$M(x_M, y_M) = \left(-a\cos\frac{\beta}{2}, -a\sin\frac{\beta}{2}\right)$$

$$D(x_D, y_D) = \left(0, 2a\sin\frac{\beta}{2}\right)$$

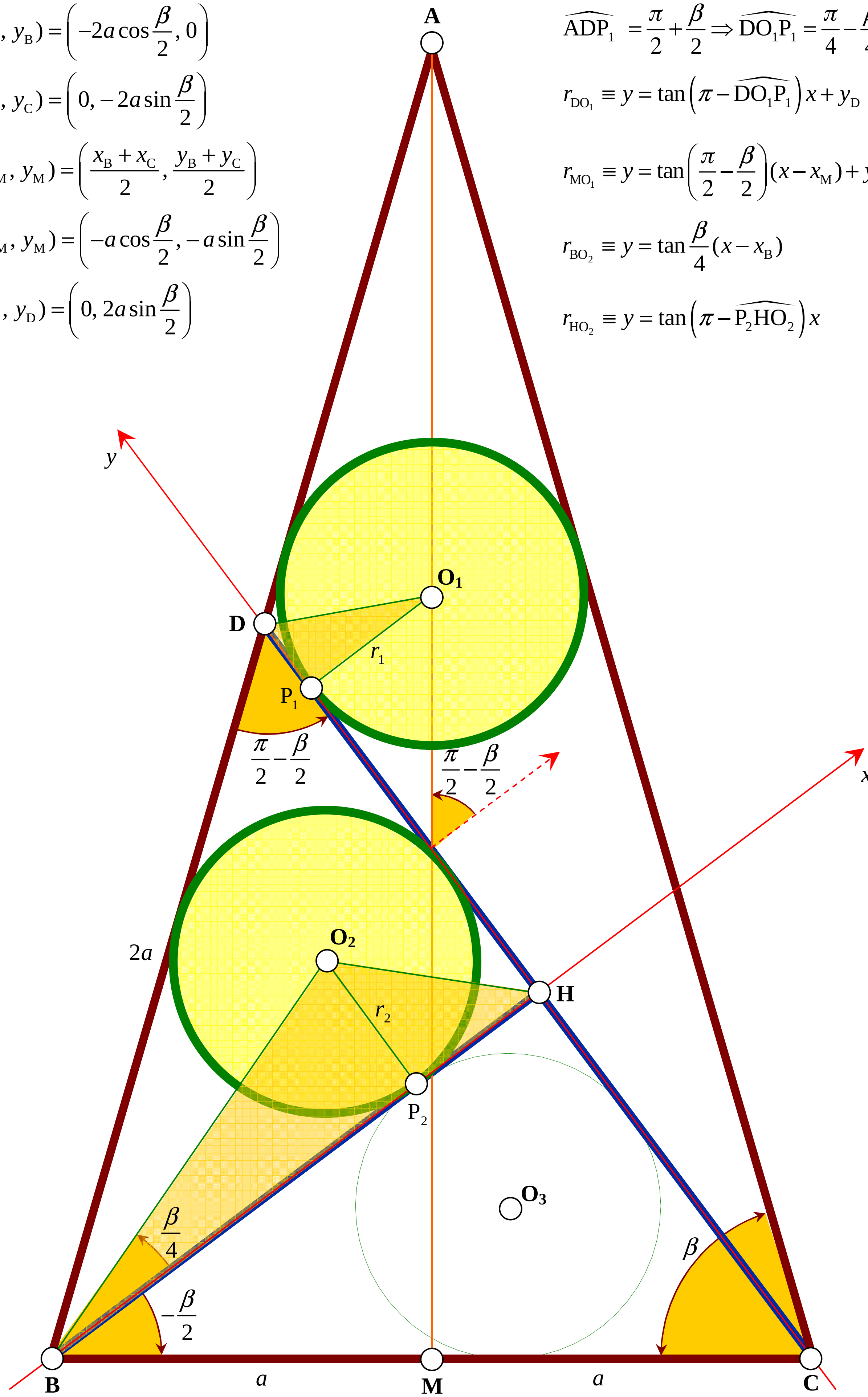
$$\widehat{ADP_1} = \frac{\pi}{2} + \frac{\beta}{2} \Rightarrow \widehat{DO_1P_1} = \frac{\pi}{4} - \frac{\beta}{4}$$

$$r_{DO_1} \equiv y = \tan\left(\pi - \widehat{DO_1P_1}\right)x + y_D$$

$$r_{MO_1} \equiv y = \tan\left(\frac{\pi}{2} - \frac{\beta}{2}\right)(x - x_M) + y_M.$$

$$r_{BO_2} \equiv y = \tan\frac{\beta}{4}(x - x_B)$$

$$r_{HO_2} \equiv y = \tan\left(\pi - \widehat{P_2HO_2}\right)x$$



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Determination of the *incenter* coordinates by intersections of pairs of bisectors:

$$r_{DO_1} \equiv y = -\tan\left(\frac{\pi}{4}-\frac{\beta}{4}\right)x + 2a\sin\frac{\beta}{2} \qquad r_{MO_1} \equiv y = \cot\frac{\beta}{2}\left(x + a\cos\frac{\beta}{2}\right) - a\sin\frac{\beta}{2}$$

$$r_{BO_2} \equiv y = \tan\frac{\beta}{4}\left(x + 2a\cos\frac{\beta}{2}\right) \qquad r_{HO_2} \equiv y = -x$$

$$O_1(x_1, y_1) \equiv r_{DO_1} \cap r_{MO_1} \left\{ \begin{array}{l} y = -\tan\left(\frac{\pi}{4}-\frac{\beta}{4}\right)x + 2a\,\mathrm{sen}\frac{\beta}{2} \\ y = \cot\frac{\beta}{2}\left(x + a\cos\frac{\beta}{2}\right) - a\sin\frac{\beta}{2} \end{array} \right\} \Rightarrow \boxed{r_1 = x_1 = a\cdot\frac{3\sin\frac{\beta}{2} - \cos\frac{\beta}{2}\cot\frac{\beta}{2}}{\tan\left(\frac{\pi}{4}-\frac{\beta}{4}\right) + \cot\frac{\beta}{2}}}$$

$$O_2(x_2, y_2) \equiv r_{BO_2} \cap r_{HO_2} \left\{ \begin{array}{l} y = \tan\frac{\beta}{4}\left(x + 2a\cos\frac{\beta}{2}\right) \\ x = -y \end{array} \right\} \Rightarrow \boxed{r_2 = y_2 = a\cdot\frac{2\cos\frac{\beta}{2}\tan\frac{\beta}{4}}{1+\tan\frac{\beta}{4}}}$$

$$\left. \begin{array}{l} r_1 = a\cdot\dfrac{3\sin\frac{\beta}{2}\tan\frac{\beta}{2} - \cos\frac{\beta}{2}}{\dfrac{1-\tan\frac{\beta}{4}}{1+\tan\frac{\beta}{4}}\cdot\tan\frac{\beta}{2} + 1} \\ r_2 = a\cdot\dfrac{2\cos\frac{\beta}{2}}{\cot\frac{\beta}{4} + 1} \end{array} \right\} r_1 \equiv r_2 \Leftrightarrow \left(3\tan^2\frac{\beta}{2} - 1\right)\left(\cot\frac{\beta}{4} + 1\right) = 2\left(\frac{1-\tan\frac{\beta}{4}}{1+\tan\frac{\beta}{4}}\cdot\tan\frac{\beta}{2} + 1\right).$$

Performing now the change of variable $\boxed{\tan\frac{\beta}{4} \equiv t}$, we have:

$$\left[3\left(\frac{2t}{1-t^2}\right)^2 - 1\right]\left(\frac{1}{t} + 1\right)^2 = 2\left(\frac{1-t}{1+t}\frac{2t}{1-t^2} + 1\right),$$

and simplifying,

$$3t^3 - 7t^2 - t + 1 = 0,$$

equation with no integer roots. By the variable changing $z \equiv 3t$, the above equation becomes

$$z^3 - 7z^2 - 3z + 9 = 0,$$

one of whose roots is $z = 1$. Undoing the change of variable is obtained $t = \frac{1}{3} = \tan\frac{\beta}{4}$. The other two roots are $t = 1 \pm \sqrt{2}$, both inadmissible because they are related to the argument $\beta = \frac{3\pi}{4}$.

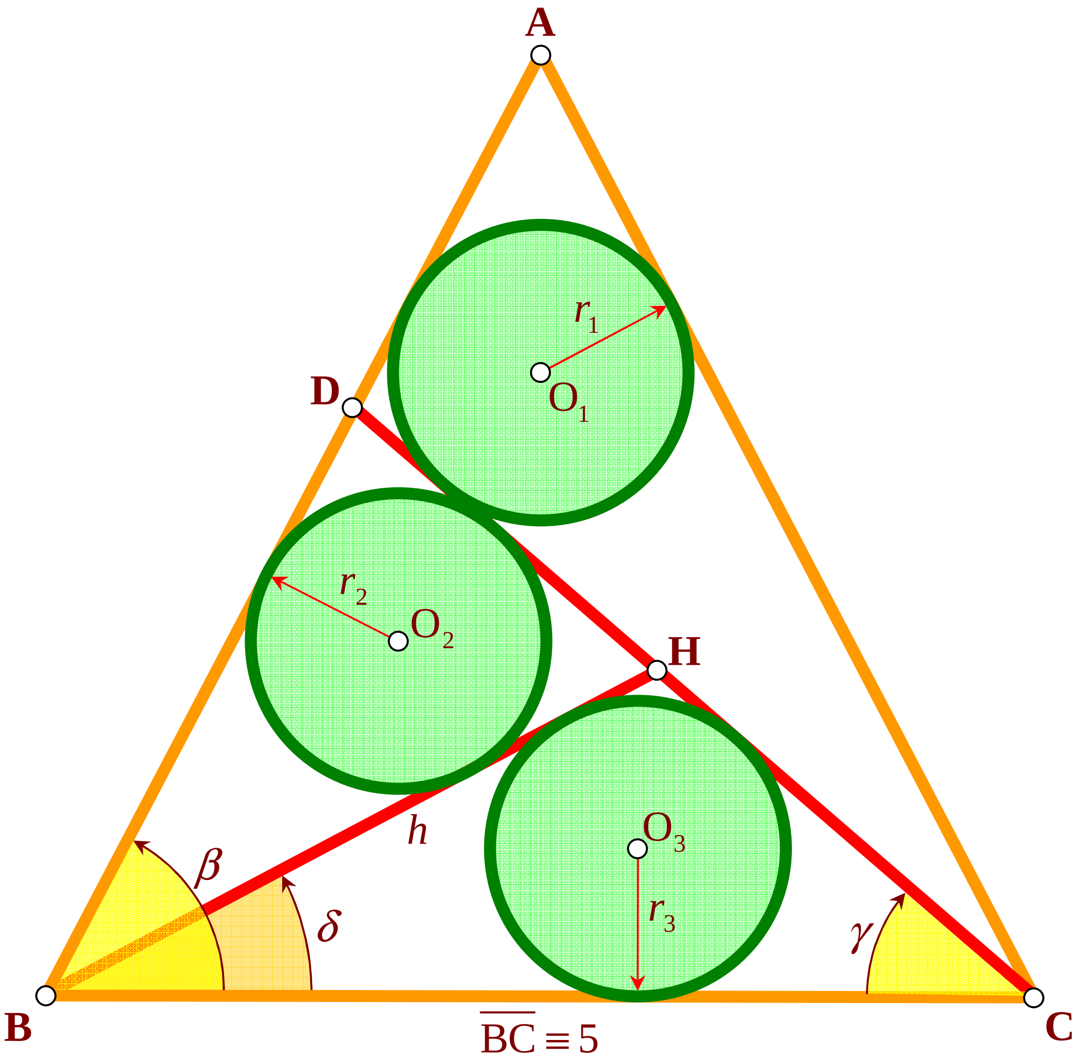
Therefore,

$$\tan\frac{\beta}{2} = \frac{2\tan\frac{\beta}{4}}{1-\tan^2\frac{\beta}{4}} = \frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4} \Rightarrow \tan\beta = \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}} = \frac{2\cdot\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{24}{7}.$$

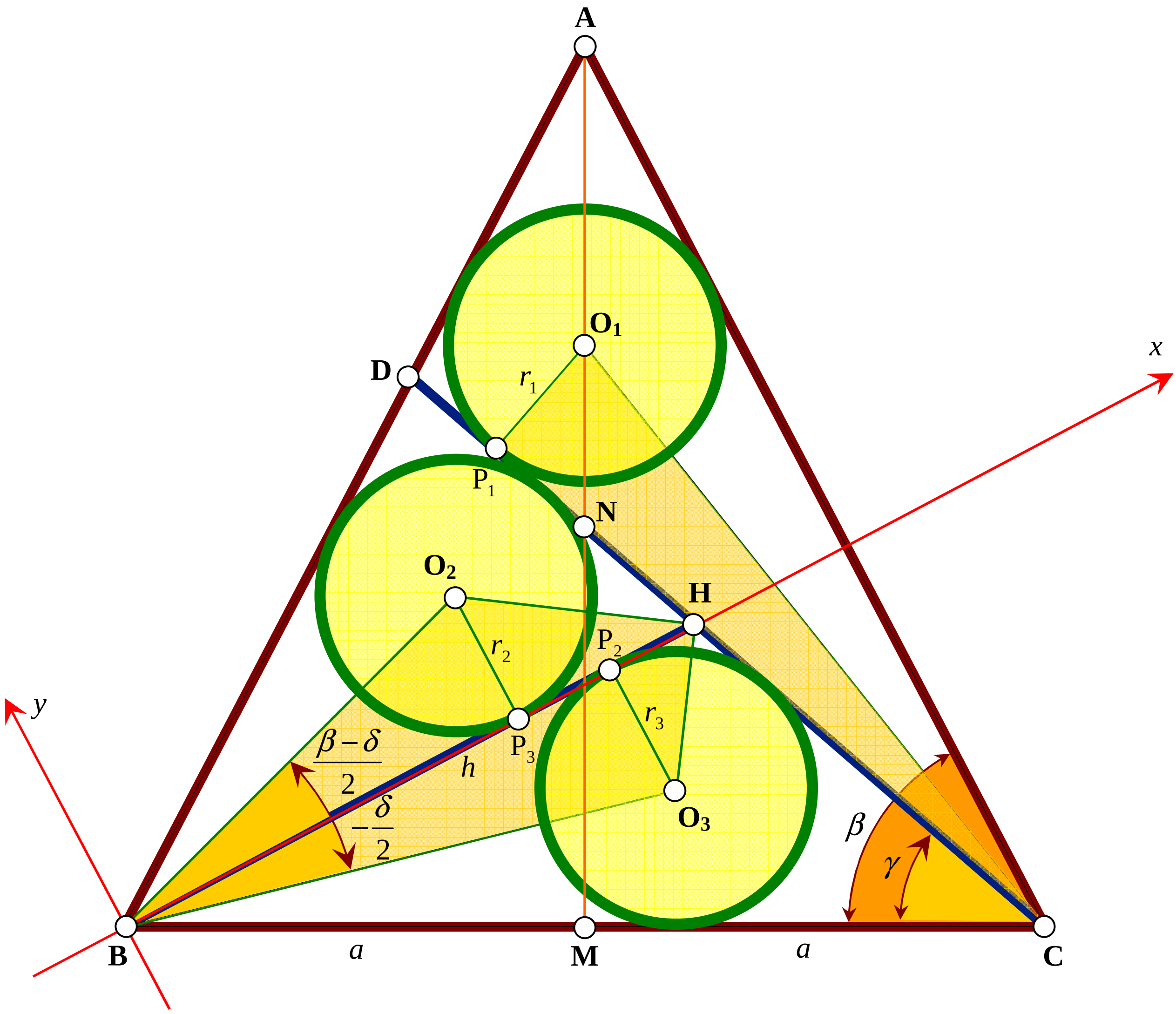
It is proposed a generalization of the problem studied above, eliminating the important restriction that implies the orthogonality condition $\overline{BH} \perp \overline{CD}$.

It is therefore expected to be a significantly harder problem.

After been prefixed the isosceles triangle $\triangle ABC$, there are 2 *degrees of freedom*, that can be *parameterized* by the angles γ and δ, elevation angles of $\overline{CD}$ and $\overline{BH}$, respectively.



For an analytical study, the choice of the *reference system* is critical to the complexity of the resulting equations. After several attempts, it is easy to be convinced that the allocation shown in the figure below is the most convenient since it allows to define the equations of two pairs of bisectors in the simplest possible way, and the *inradius* r_2 and r_3 coincide in absolute value with the ordinate of their *incenters*.



$$\overline{\mathrm{BH}} \equiv h \Rightarrow \mathrm{H} \equiv \mathrm{H}(h, 0)$$

$$\triangle BHD \begin{cases} r_{\mathrm{BO}_2} \equiv y = \left(\tan \dfrac{\beta-\delta}{2} \right) x \\ r_{\mathrm{HO}_2} \equiv y = -\left(\tan \dfrac{\gamma+\delta}{2} \right) (x-h) \end{cases}$$

$$\triangle BHC \begin{cases} r_{\mathrm{BO}_3} \equiv y = -\left(\tan \dfrac{\delta}{2} \right) x \\ r_{\mathrm{HO}_3} \equiv y = \left(\cot \dfrac{\gamma+\delta}{2} \right) (x-h) \end{cases}$$

$$\left. \begin{cases} \triangle \mathrm{NO_1P_1} : r_1 = \overline{\mathrm{O_1N}} \cos \widehat{\mathrm{NO_1P_1}} = (\overline{\mathrm{O_1M}} - \overline{\mathrm{NM}}) \cos \widehat{\mathrm{NO_1P_1}} \\ \overline{\mathrm{O_1M}} = a \tan \widehat{\mathrm{MCO_1}},\ \overline{\mathrm{NM}} = a \tan \widehat{\mathrm{NO_1P_1}} \\ \widehat{\mathrm{NO_1P_1}} = \gamma,\ \widehat{\mathrm{MCO_1}} = \dfrac{\beta+\gamma}{2} \end{cases} \right\} \Rightarrow r_1 = a \left(\tan \frac{\beta+\gamma}{2} \cos \gamma - \sin \gamma \right)$$

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Coordinates of the *incenters* O_2 y O_3 by intersection of bisectors and the *inradius* r_2 and r_3:

$$O_2(x_2, y_2) \equiv r_{BO_2} \cap r_{HO_2} \left\{ \begin{aligned} x_2 &= h \frac{\tan\frac{\gamma+\delta}{2}}{\tan\frac{\beta-\delta}{2} + \tan\frac{\gamma+\delta}{2}} \\ y_2 &= h \frac{\tan\frac{\beta-\delta}{2}\tan\frac{\gamma+\delta}{2}}{\tan\frac{\beta-\delta}{2} + \tan\frac{\gamma+\delta}{2}} \end{aligned} \right\} r_2 = y_2 \Rightarrow \boxed{r_2 = \frac{h}{\cot\frac{\gamma+\delta}{2} + \cot\frac{\beta-\delta}{2}}}$$

$$O_3(x_3, y_3) \equiv r_{BO_3} \cap r_{HO_3} \left\{ \begin{aligned} x_3 &= h \frac{\cot\frac{\gamma+\delta}{2}}{\tan\frac{\delta}{2} + \cot\frac{\gamma+\delta}{2}} \\ y_3 &= h \frac{-\tan\frac{\delta}{2}\cot\frac{\gamma+\delta}{2}}{\tan\frac{\delta}{2} + \cot\frac{\gamma+\delta}{2}} \end{aligned} \right\} r_3 = -y_3 \Rightarrow \boxed{r_3 = \frac{h}{\cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}}}$$

Identifying both inradius, we obtain the equations

$$r_2 \equiv r_3 \Leftrightarrow \cot\frac{\gamma+\delta}{2} + \cot\frac{\beta-\delta}{2} = \cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}.$$

$$r_1 \equiv r_3 \Leftrightarrow \left(\tan\frac{\beta+\gamma}{2}\cos\gamma - \sin\gamma\right)\left(\cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}\right) = \frac{h}{a}.$$

Eliminating the auxiliary parameters a and h,

$$\triangle BHC: \ \frac{h}{\sin\gamma} = \frac{2a}{\sin(\gamma+\delta)} \Rightarrow \frac{h}{a} = \frac{2\sin\gamma}{\sin(\gamma+\delta)},$$

$$r_1 \equiv r_3 \Leftrightarrow \left(\tan\frac{\beta+\gamma}{2}\cos\gamma - \sin\gamma\right)\left(\cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}\right) = \frac{2\sin\gamma}{\sin(\gamma+\delta)},$$

yields a system of two trigonometric equations in the parameters γ and δ, considering β as known:

$$\begin{cases} \cot\frac{\gamma+\delta}{2} + \cot\frac{\beta-\delta}{2} = \cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2} \\ \left(\tan\frac{\beta+\gamma}{2}\cot\gamma - 1\right)\left(\cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}\right)\sin(\gamma+\delta) = 2 \end{cases}$$

or,

$$\begin{cases} \cot\frac{\gamma+\delta}{2} + \cot\frac{\beta-\delta}{2} = \cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2} \\ \left(\tan\frac{\beta+\gamma}{2}\cot\gamma - 1\right)\left(\cot\frac{\delta}{2} + \tan\frac{\gamma+\delta}{2}\right)\tan\frac{\gamma+\delta}{2}\cos^2\frac{\gamma+\delta}{2} = 1 \end{cases}$$

Developing,

$$\begin{cases} ① \dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}+\dfrac{1+\tan\frac{\beta}{2}tg\frac{\delta}{2}}{\tan\frac{\beta}{2}-\tan\frac{\delta}{2}}=\cot\frac{\delta}{2}+\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}} \\ ② \left(\dfrac{\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}}{1-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}\cdot\dfrac{1-\tan^2\frac{\gamma}{2}}{2\tan\frac{\gamma}{2}}-1\right)\left(\cot\frac{\delta}{2}+\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right)\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\cdot\dfrac{1}{1+\left(\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right)^2}=1 \end{cases}$$

$$\begin{cases} ① \left(\dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}\right)^2+\dfrac{1+\tan\frac{\beta}{2}tg\frac{\delta}{2}}{\tan\frac{\beta}{2}-\tan\frac{\delta}{2}}\cdot\dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}=\cot\frac{\delta}{2}\cdot\dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}+1 \\ ② \left(\dfrac{\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}}{1-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}\cdot\dfrac{1-\tan^2\frac{\gamma}{2}}{2\tan\frac{\gamma}{2}}-1\right)\left(\cot\frac{\delta}{2}+\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right)\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}=1+\left(\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right)^2 \end{cases}$$

$$\begin{cases} ① \left(\dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}\right)^2+\left(\dfrac{1+\tan\frac{\beta}{2}tg\frac{\delta}{2}}{\tan\frac{\beta}{2}-\tan\frac{\delta}{2}}-\cot\frac{\delta}{2}\right)\dfrac{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}-1=0 \\ ② \left[\left(\dfrac{\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}}{1-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}\cdot\dfrac{1-\tan^2\frac{\gamma}{2}}{2\tan\frac{\gamma}{2}}-1\right)\left(\cot\frac{\delta}{2}+\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right)-\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}\right]\dfrac{\tan\frac{\gamma}{2}+\tan\frac{\delta}{2}}{1-\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}-1=0 \end{cases}$$

With the changes of variables, $k\equiv\tan\frac{\beta}{2}, u\equiv\tan\frac{\gamma}{2}, v\equiv\tan\frac{\delta}{2}$, this **trigonometric system** becomes an **algebraic system**,

$$\begin{cases} ③ \left(\dfrac{1-uv}{u+v}\right)^2+\left(\dfrac{1+kv}{k-v}-\dfrac{1}{v}\right)\dfrac{1-uv}{u+v}-1=0 \\ ④ \left[\left(\dfrac{k+u}{1-ku}\dfrac{1-u^2}{2u}-1\right)\left(\dfrac{1}{v}+\dfrac{u+v}{1-uv}\right)-\dfrac{u+v}{1-uv}\right]\dfrac{u+v}{1-uv}-1=0 \end{cases}$$

And by changing $\dfrac{1-uv}{u+v}\equiv z$ in the equation ③ we obtain the quadratic equation in z,

$$z^2+\left(\frac{1+kv}{k-v}-\frac{1}{v}\right)z-1=0.$$

Solving,

$$z=\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]\pm\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}.$$

In the context of this problem the solution $z=\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]-\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}$ is not acceptable because $z<0\Rightarrow uv>1$, but this is not possible as γ and δ are acute angles and therefore $0<\tan\frac{\gamma}{2}<1,\ 0<\tan\frac{\delta}{2}<1.$

So, we have:

$$z=\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]+\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}.$$

Undoing the last change of variable, $z\equiv\frac{1-uv}{u+\mathrm{v}}\Rightarrow u=\frac{1-vz}{v+z}$, and replacing in this equation z by its value:

$$⑤\ u=\frac{1-v\left\{\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]+\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}\right\}}{v+\left\{\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]+\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}\right\}}$$

We obtained an equation for u in terms of k (as known) and v, but if we would eliminate u in ④ replacing in this equation the roots of ⑤, it would obviously lead to an analytically intractable equation for v:

$$\left.\begin{array}{l} u=\dfrac{1-v\left\{\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]+\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}\right\}}{v+\left\{\frac{1}{2}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]+\sqrt{1+\frac{1}{4}\left[\frac{1}{v}-\frac{1+kv}{\mathrm{k}-v}\right]^2}\right\}} \\ \left[\left(\frac{k+u}{1-ku}\cdot\frac{1-u^2}{2u}-1\right)\left(\frac{1}{v}+\frac{u+v}{1-uv}\right)-\frac{u+v}{1-uv}\right]\frac{u+v}{1-uv}-1=0 \end{array}\right\}\Rightarrow v=?$$

Furthermore, it is trivial that for any isosceles triangle △ABC there is a *unique solution* because the three inradius are monotonous and continuous functions of the angles γ and δ (anyone can be persuaded of this through the empirical demonstration that this software provides).

Therefore, we conclude that the problem is solvable only by *numerical methods*, and it is not possible to determine the equation that provides $\frac{h}{r}$ as a explicit function of β.

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