Question 422 : A Definite Integral

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Abstract: this note presents certain clases of integrals containing trigonometric and hyperbolic trigonometric functions in their integrands.

Our motivation is the integral

$$\int_0^\infty \frac{x}{\cos x + \cosh x} dx = \frac{\pi}{16}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 \tag{1}$$

For details see Bruce C. Berndt paper 2016, ref. 1.

Related formulas:

$$\int_0^y \frac{x}{\cos x + \cosh x} dx = \frac{1}{4}\sum_{n=0}^{\infty} \frac{c_n}{(2n+1)(4n)!} y^{4n+2} \quad , 0 < y < \frac{\pi}{\sqrt{2}} \tag{2}$$

where

$$c_n = -\sum_{k=1}^{n} \binom{4n}{4k} c_{n-k} \quad , c_0 = 1 \tag{3}$$

For $y > 0.70329..., e^{-2y} + 2e^{-y}\cos y = 1, i = \sqrt{-1}$ we have

$$\int_y^\infty \frac{x}{\cos x + \cosh x} dx = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k} (-1)^n \binom{n}{k}\binom{k}{m} \frac{\left(1 + y\left(2n-k+1+i(2m-k)\right)\right) e^{-y\left(2n-k+1+i(2m-k)\right)}}{\left(2n-k+1+i(2m-k)\right)^2} \tag{4}$$

Remark: $e^{-yi(2m-k)} = (\cos 1 - i\sin 1)^{y(2m-k)} = (\cos y - i\sin y)^{2m-k}$.

For $y \in \mathbb{N} = \{1,2,3,4,...\}$ we have

$$\int_{y\pi}^\infty \frac{x}{\cos x + \cosh x} dx = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k} (-1)^{n+yk} \binom{n}{k}\binom{k}{m} \times$$
$$\times \frac{(2n-k+1)^2 - (2m-k)^2 + y\pi(2n-k+1)\left((2n-k+1)^2 + (2m-k)^2\right)}{\left((2n-k+1)^2 + (2m-k)^2\right)^2} e^{-y\pi(2n-k+1)} \tag{5}$$

For $y > 0.70329..., e^{-2y} + 2e^{-y}\cos y = 1, i = \sqrt{-1}$ we have

$$\int_0^y \frac{x}{\cos x + \cosh x} dx = \frac{\pi}{16}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 -$$
$$2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k} (-1)^n \binom{n}{k}\binom{k}{m} \frac{\left(1 + y\left(2n-k+1+i(2m-k)\right)\right) e^{-y\left(2n-k+1+i(2m-k)\right)}}{\left(2n-k+1+i(2m-k)\right)^2} \tag{6}$$

For $0 < y < \pi/\sqrt{2}$ we have

$$\int_y^{\infty} \frac{x}{\cos x + \cosh x}\, dx = \frac{\pi}{16}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 - \frac{1}{4}\sum_{n=0}^{\infty}\frac{c_n}{(2n+1)(4n)!}\, y^{4n+2} \quad ,0 < y < \frac{\pi}{\sqrt{2}} \tag{7}$$

where c_n is defined by (3).

$$\int_0^{1} \frac{x}{\cos x + \cosh x}\, dx = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{4}\right)^k \sum_{m=0}^{k}\sum_{r=0}^{m}\sum_{s=0}^{r}\binom{k}{m}\binom{m}{r}\binom{r}{s} f(k,m,r,s) \tag{8}$$

where

$$f(k,m,r,s) = \begin{cases} 1/2 & ,k+r = 2m \wedge r = 2s \\ g(k,m,r,s) & ,otherwise \end{cases} \tag{9}$$

$$g(k,m,r,s) = \frac{1+\left(-1+k-2m+r+i(r-2s)\right)e^{k-2m+r}\left(\cos 1 + i\sin 1\right)^{r-2s}}{\left(k-2m+r+i(r-2s)\right)^2} \tag{10}$$

$$\int_1^{\infty} \frac{x}{\cos x + \cosh x}\, dx = 2e^{-1}\sum_{n=0}^{\infty}(-1)^n e^{-n}\alpha^n \sum_{k=0}^{n}\binom{n}{k} e^{-k}\beta^k \sum_{m=0}^{n-k}\binom{n-k}{m}\beta^{2m} h(n,k,m) \tag{11}$$

where

$$\alpha = \cos 1 + i\sin 1 = e^{i} \quad ,\beta = \cos 1 - i\sin 1 = e^{-i} \quad ,\alpha\beta = 1 \tag{12}$$

$$h(n,k,m) = \left(n+k+1+i(2m+k-n)\right)^{-1} + \left(n+k+1+i(2m+k-n)\right)^{-2} \tag{13}$$

$$\frac{\pi}{16}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 = \frac{4}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n \sum_{k=0}^{n}\left(-\frac{2}{3}\right)^k \binom{n}{k}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\left(2k-m+1+i(2r-m)\right)^{-2} \tag{14}$$

References

1. B. C. Berndt: Integrals associated with Ramanujan and elliptic functions, The Ramanujan Journal, vol. 41,2016. http://link.springer.com/content/pdf/10.1007/s11139-016-9793-1.pdf