Euler's Formula for $\zeta(2n)$

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Abstract

In this article we derive a formula for zeta(2) and zeta(2n).

Introduction

In this paper we derive from scratch

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^{2p}} = (-1)^{p-1} \frac{2^{2p-1}}{(2p!)} B_{2p} \pi^{2p} \tag{2}$$

where B_{2p} are the Bernoulli numbers. Both are attributed to Euler [5]. Our treatment is close to that of Eymard [5] and Knopp [6].

The justification for this article is that these texts seem rather unfocused: both authors develop other material sporadically as they prove this result. We wish here to isolate the result and just develop the mathematics necessary for an understanding of this formula. There are many treatments of these result [1]. Here we wish to motivate known, easiest proofs.

Taylor series for sin

At some point someone determined that there is a relationship between nth order derivatives and coefficients of polynomials. This can be anticipated by the easiest observation; if $f(x) = ax^2 + bx + c$, the coefficient of x^0

is given by the zero order derivative evaluated at $x = 0$: $f(0) = c$. As we take ever increasing derivatives the constant of the derivative becomes a new coefficient. So, $f'(x) = 2ax + b$ and $f'(0) = b$. When we repeat this pattern, we notice that a factorial is building by way of the formula $(cx^n)' = cnx^{n-1}$. Factorials need to be divided out. Here it is for the quadratic: $f''(x) = 2a$ gives

$$\frac{f^{(2)}(0)}{2!} = a.$$

In general, for a function $f(x)$ with derivatives

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is termed the Taylor (actually Maclaurin) series expansion of $f(x)$ about the point 0. It is a Maclaurin expansion when the point used, the center is 0.

The power of these power series (an infinite series with x^n) is that they allow for approximations to an arbitrary precision. The transcendental functions in particular are in need of such. What after all can we say about sin(1.2387) and the like? We only have exact evaluations possible for this trigonometric function when the argument is a fraction with π: $\pi/2$, $\pi/3$, etc.. If we have a power series for sin we can evaluate any x value.

We know the derivative of sin is cos and taking nth derivatives is not difficult; the functions just cycle around:

$$\sin' = \cos; \cos' = -\sin; (-\sin)' = -\cos; \text{and } (-\cos)' = \sin.$$

As $\pm\sin(0) = 0$, $\cos(0) = 1$, and $-\cos(0) = -1$, we can easily generate a Maclaurin series for sin:

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \qquad (3)$$

The odd $2n + 1$ follows from the even terms, thanks to $\pm\sin(0) = 0$, vanishing.

Properties of polynomials

Power series are like an infinite polynomial and polynomials have coefficients that are related to their roots – what x values make them 0. So, for example, expanding $(x-a)(x-b)(x-c)$ gives

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \tag{4}$$

We can sense that in general the constant will be the sum of the roots taken all at a time, hence one term, and the coefficient of x will be the sum of the roots taken (or multiplied) $n-1$ at a time. We are obtaining sums that remind us of the goal of determining the sum in (1). In comparing this sum with the ones in (4) and the powers of x in (3), we have a puzzle.

Puzzle of (1)

We'd like to get the polynomial of $\sin x$ to have a x term and a 1 constant. If this were true then, using (4) as a model,

$$\frac{x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc}{abc}$$

gives a coefficient of x equal to $1/c + 1/b + 1/a$, a sum of the reciprocals of the roots. The roots of sin are $\pm n\pi$.

First

$$\sum_{k=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x\sum_{k=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}$$

gives

$$\sin x = x(1 - x^2/3! + x^4/5! - \dots)$$

which gives

$$\frac{\sin x}{x} = (1 - x^2/3! + x^4/5! - \dots).$$

Letting $y = x^2$, we get a infinite polynomial which we set to 0:

$$0 = 1 - y/3! + \dots.$$

This has a constant of 1, so the sum of the roots is $1/3! = 1/6$ and the roots are given by the squares of $\sin x$'s roots (just using $y = x^2$). Thus

$$\frac{1}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2}$$

and this implies (1).

Puzzle of (2)

We start from the observation that there are three ways (at least) to generate a power series: using a division, like $1/(x-1)$; using Taylor series expansions, as above; and using a Fourier expansion. As (2) involves all

$$\zeta(2p) = \sum_{k=1}^{\infty} \frac{1}{k^{2p}}$$

and as all power series expansions have both a sequence of coefficients and powers of z,

$$\sum_{k=1}^{\infty} a_k x^k,$$

if we can use two different means of obtaining a power series expansion for a given same function, we can hope to equate the coefficients and arrive at (2). We would need one set of coefficients to be $\zeta(2p)$. We can use a division to generate that situation.

Theorem 1.

$$\sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2} = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{x^{2p}}{n^{2p}} = \sum_{p=1}^{\infty} \zeta(2p) x^{2p} \tag{5}$$

Proof. First,

$$\frac{x^2}{x^2 - n^2} \frac{1/n^2}{1/n^2} = \frac{x^2}{n^2} \left(\frac{1}{x^2/n^2 - 1} \right).$$

Letting $m = x^2/n^2$, we have

$$\frac{m}{m-1} \frac{1/m}{1/m} = \frac{1}{1 - 1/m}.$$

This last is the formula for a geometric series:

$$\frac{x^2}{x^2 - n^2} = \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}}.$$

Substituting,

$$\sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{2k}}{n^{2k}}$$

and transposing summation gives (5). $\square$

Next is the puzzle of finding a function that is amenable to two methods of power series generation one of which gives (5). This is not as far fetched as it seems. Consider that

$$\frac{x}{x+n} + \frac{x}{x-n} = \frac{2x^2}{x^2-n^2}$$

and

$$x\int \cos[(x+n)t]\ \mathrm{dt} = \frac{x}{x+n}\sin[(x+n)t]$$
$$x\int \cos[(x-n)t]\ \mathrm{dt} = \frac{x}{x-n}\sin[(x-n)t].$$

Using a product to sum trigonometric identity and given that the integration limits when computing Fourier coefficients have an upper limit of π, there is some hope that say the $\cos(nt)$ Fourier expansion of $\cos xt$ might yield the desired function with power series (5).

Theorem 2. *The* $\cos(nt)$ *Fourier expansion of* $\cos(xt)$ *yields*

$$\pi x \cot \pi x = 1 - 2\sum_{k=1}^{\infty} \frac{x^2}{x^2-n^2}.$$

Proof. By definition of a Fourier series (and half series), the Fourier half series expansion of $\cos(xt)$ using $\cos(nt)$ is

$$\cos(xt) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(xt)\cos(nt)\ \mathrm{dt}. \tag{6}$$

Using a trigonometric identity (product to sum) and cos is an even function, this is

$$\begin{aligned}
&\frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos[(x+n)t] + \cos[(x-n)t]\right)\\
&= \frac{1}{\pi}\left(\frac{\sin[(x+n)\pi]}{x+n} + \frac{\sin[(x-n)\pi]}{x-n}\right)\\
&= \frac{\sin(\pi x)\cos(nx)}{\pi}\left(\frac{1}{x+n} + \frac{1}{x-n}\right)\\
&= (-1)^n \frac{2x\sin(\pi x)}{\pi}\frac{1}{x^2-n^2}.
\end{aligned}$$

So we have

$$\cos(xt) = \frac{\sin(\pi x)}{\pi x} + \frac{2x \sin \pi x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^n \cos(nt)}{x^2 - n^2} \tag{7}$$

Setting $t = \pi$, (7) becomes

$$\pi x \cot \pi x = 1 + 2 \sum_{n=1}^{\infty} \frac{x^2}{x^2 - n^2}.$$

□

Corollary 1.

$$\pi x \cot \pi x = 1 - 2 \sum_{k=1}^{\infty} \frac{x^2}{x^2 - n^2} = 1 - 2 \sum_{p=1}^{\infty} \zeta(2p) x^{2p}.$$

Proof. This follows from Theorem 1. □

Now if we just have another way to obtain a power series expansion for $z \cot z$, we could equate the coefficients for each. Why not use Taylor? We have already an expansion for sin and the one for cos can be derived easily. We could divide the two series. This is what Larson in his calculus text [7] does to arrive at the beginnings of a power series expansion for tan. He doesn't follow through and divide 1 by this tan result: $1/\tan = \cot$.

The catch with this approach is that we are not getting a closed form of the coefficients whereby we could compute them at will. Of course the even more natural approach is to take derivatives of $z \cot z$, per Taylor's theorem, and develop the series this wise. The catch is repeated differentiation of cot (and tan) is not nearly so neat and nice as taking such for sin and cos. It is fast a mess.

Can we look it up in a reference book? The following line occurs in Spiegel [8]:

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!}.$$

Multiplying this by x gives

$$x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \cdots - \frac{2^{2n} B_n x^{2n}}{(2n)!}$$

and substituting πx for x gives

$$\pi x \cot \pi x = 1 - \frac{(\pi x)^2}{3} - \frac{(\pi x)^4}{45} - \frac{2(\pi x)^6}{945} - \cdots - \frac{2^{2n} B_n (\pi x)^{2n}}{(2n)!} - \ldots$$

Equating

$$\pi x \cot \pi x = 1 - \sum_{k=1}^{\infty} \frac{2^{2n} B_n (\pi x)^{2n}}{(2n)!} = 1 - 2 \sum_{p=1}^{\infty} \zeta(2p) x^{2p}$$

The same reference book gives $B_1 = 1/6$, so what is $\zeta(2)$? Well

$$\frac{4(1/6)(\pi)^2}{2!} = 2\zeta(2)$$

implies $\zeta(2) = \pi^2/6$.

Bernoulli numbers

Typically calculus textbooks do not include power series expansions for $\tan x$ and $\cot x$. This seems to disallow a tabular and systematic understanding of the subject matter. Instead it favors uninformed understandings of mathematics that just drops natural questions leaving the student thinking that mathematics consists of a sequence of problems without any particular rhyme or reason. Of course the writers of calculus textbooks have a good reason not to include such series; they are difficult. But why can't this be stated? In this section we will derive the series for cot.

Theorem 3.

$$z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}$$

Proof. Let

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

be the Taylor series expansion for $z/(e^z - 1)$. □

References

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