Deriving Euler's Two Great Gems

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Introduction

In this paper we derive from scratch

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^{2p}} = (-1)^{p-1} \frac{2^{2p-1}}{(2p!)} B_{2p} \pi^{2p}$$

where B_{2p} are the Bernoulli numbers. Both are attributed to Euler [3].

Taylor series for sin

At some point someone determined that there is a relationship between nth order derivatives and coefficients of polynomials. This can be anticipated by the easiest observation; if $f(x) = ax^2 + bx + c$, the coefficient of x^0 is given by the zero order derivative evaluated at $x = 0$: $f(0) = c$. As we take ever increasing derivatives the constant of the derivative becomes a new coefficient. So, $f'(x) = 2ax + b$ and $f'(0) = b$. When we repeat this pattern, we notice that a factorial is building by way of the formula $(cx^n)' = cnx^{n-1}$. Factorials need to be divided out. Here it is for the quadratic: $f''(x) = 2a$ gives

$$\frac{f^{(2)}(0)}{2!} = a.$$

In general, for a function $f(x)$ with derivatives

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is termed the Taylor (actually Maclaurin) series expansion of $f(x)$ about the point 0. It is a Maclaurin expansion when the point used, the center is 0.

The power of these power series (an infinite series with x^n) is that they allow for approximations to an arbitrary precision. The transcendental functions in particular are in need of such. What after all can we say about sin(1.2387) and the like? We only have exact evaluations possible for this trigonometric function when the argument is a fraction with π: $\pi/2$, $\pi/3$, etc.. If we have a power series for sin we can evaluate any x value.

We know the derivative of sin is cos and taking nth derivatives is not difficult; the functions just cycle around:

$$\sin' = \cos; \cos' = -\sin; (-\sin)' = -\cos; \text{and } (-\cos)' = \sin.$$

As $\pm\sin(0) = 0$, $\cos(0) = 1$, and $-\cos(0) = -1$, we can easily generate a Maclaurin series for sin:

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{2}$$

The odd $2n + 1$ follows from the even terms, thanks to $\pm\sin(0) = 0$, vanishing.

Properties of polynomials

Power series are like an infinite polynomial and polynomials have coefficients that are related to their roots – what x values make them 0. So, for example, expanding $(x - a)(x - b)(x - c)$ gives

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc. \tag{3}$$

We can sense that in general the constant will be the sum of the roots taken all at a time, hence one term, and the coefficient of x will be the sum of the roots taken (or multiplied) $n - 1$ at a time. We are obtaining sums that remind us of the goal of determining the sum in (1). In comparing this sum with the ones in (3) and the powers of x in (2), we have a puzzle.

Puzzle of (1)

We'd like to get the polynomial of $\sin x$ to have a x term and a 1 constant. If this were true then, using (3) as a model,

$$\frac{x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc}{abc}$$

gives a coefficient of x equal to $1/c + 1/b + 1/a$, a sum of the reciprocals of the roots. The roots of sin are $\pm n\pi$.

First

$$\sum_{k=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x \sum_{k=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}$$

gives

$$\sin x = x(1 - x^2/3! + x^4/5! - \dots)$$

which gives

$$\frac{\sin x}{x} = (1 - x^2/3! + x^4/5! - \dots).$$

Letting $y = x^2$, we get a infinite polynomial which we set to 0:

$$0 = 1 - y/3! + \dots.$$

This has a constant of 1, so the sum of the roots is $1/3! = 1/6$ and the roots are given by the squares of $\sin x$'s roots (just using $y = x^2$). Thus

$$\frac{1}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2}$$

and this implies (1).

References

[1] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[2] G. Chrystal, *Algebra: An Elementary Textbook*, 7th ed., vol. 1, American Mathematical Society, Providence, RI, 1964.

[3] P. Eymard and J.-P. Lafon, *The Number π*, American Mathematical Society, Providence, RI, 2004.