SMARANDACHE HYPER ALGEBRAS

YOUNG BAE JUN AND EUN HWAN ROH*

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ABSTRACT. The notion of Smarandache hyper I-algebra and Smarandache hyper K-algebra are introduced, and related properties are investigated.

1. INTRODUCTION

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure **W** on A such that there exists a proper subset B of A which is embedded with a strong structure **S**. In [5], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [4]. In this paper, we introduce the notion of Smarandache hyper K- and I-algebras, and investigate its properties.

2. PRELIMINARIES

We include some elementary aspects of hyper K-algebras that are necessary for this paper, and for more details we refer to [1] and [6]. Let H be a non-empty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$.

By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

By a *hyper I-algebra* we mean a non-empty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(H1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x < x$,
(H4) $x < y$ and $y < x$ imply $x = y$

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*Corresponding author: Tel. +82 55 740 1232.

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I-algebra $(H, \circ, 0)$ satisfies an additional condition:

(H5) $0 < x$ for all $x \in H$,

then $(H, \circ, 0)$ is called a *hyper K-algebra* (see [1]).

Note that every hyper BCK-algebra is a hyper K-algebra.

In a hyper I-algebra H, the following hold (see [1, Proposition 3.4]):

(a1) $(A \circ B) \circ C = (A \circ C) \circ B$,
(a2) $A \circ B < C \Leftrightarrow A \circ C < B$,
(a3) $A \subseteq B$ implies $A < B$

for all nonempty subsets A, B and C of H.

In a hyper K-algebra H, the following hold (see [1, Proposition 3.6]):

(a4) $x \in x \circ 0$ for all $x \in H$.

3. Smarandache Hyper Algebras and Smarandache Hyper Ideals

Definition 3.1. A *Smarandache hyper I-algebra* is defined to be a hyper I-algebra $(H, \circ, 0)$ in which there exists a proper subset Ω of H such that $(\Omega, \circ, 0)$ is a non-trivial hyper K-algebra.

Example 3.2. Let $H = \{0, 1, 2\}$ and define an hyper operations "$\circ_1$" and "$\circ_2$" on H as follows:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Table a1

$\circ_2$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

Table a2

Then $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are Smarandache hyper I-algebras.

Example 3.3. Let $H = \{0, a, b\}$ and define an hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b
0	$\{0,a,b\}$	$\{a,b\}$	$\{a,b\}$
a	$\{0,a,b\}$	$\{0,a,b\}$	$\{a,b\}$
b	$\{0,a,b\}$	$\{a,b\}$	$\{0,a,b\}$

Table a4

Then $(H, \circ, 0)$ is not a Smarandache hyper I-algebra since $(\Omega_1 = \{0, a\}, \circ, 0)$ and $(\Omega_2 = \{0, b\}, \circ, 0)$ are not hyper K-algebras.

Definition 3.4. A *Smarandache hyper K-algebra* is defined to be a hyper K-algebra $(H, \circ, 0)$ in which there exists a proper subset Ω of H such that $(\Omega, \circ, 0)$ is a non-trivial hyper BCK-algebra.

Example 3.5. Let $H = \{0, a, b, c\}$ and define an hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$	$\{0,a\}$
c	$\{c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{0,b,c\}$

Table a3

Then $(H,\circ,0)$ is a Smarandache hyper K-algebra because $(\Omega = \{0,a,b\},\circ,0)$ is a hyper BCK-algebra.

Example 3.6. Let $H = \{0,a,b\}$ and define an hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Table a4

Then $(H,\circ,0)$ is not a Smarandache hyper K-algebra since $(\Omega_1 = \{0,a\},\circ,0)$ and $(\Omega_2 = \{0,b\},\circ,0)$ are not hyper BCK-algebras.

In what follows, let H and Ω denote a Smarandache hyper K-algebra and a non-trivial hyper BCK-algebra which is properly contained in H, respectively.

Definition 3.7. A nonempty subset I of H is called a *Smarandache hyper* $(\ll,\in)$*-ideal* of H related to Ω (or briefly, Ω*-Smarandache hyper* $(\ll,\in)$*-ideal* of H) if it satisfies:

(c1) $0 \in I$,

(c2) $(\forall x \in \Omega)\ (\forall y \in I)\ (x \circ y \ll I \Rightarrow x \in I)$.

If I is a Smarandache hyper $(\ll,\in)$-ideal of H related to every hyper BCK-algebra contained in H, we simply say that I is a *Smarandache hyper* $(\ll,\in)$*-ideal* of H.

Example 3.8. Let $H = \{0,a,b,c\}$ and define a hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{b\}$
c	$\{c\}$	$\{a,b\}$	$\{0,a\}$	$\{0,b\}$

Table 3

Then $(H,\circ,0)$ is a Smarandache hyper K-algebra because $(\Omega = \{0,b\},\circ,0)$ is a hyper BCK-algebra. Moreover, a subset $\{0,c\}$ of H is an Ω-Smarandache hyper $(\ll,\in)$-ideal of H.

Definition 3.9. A nonempty subset I of H is called a *Smarandache hyper* $(<,\in)$*-ideal* of H related to Ω (or briefly, Ω*-Smarandache hyper* $(<,\in)$*-ideal* of H) if it satisfies:

(c1) $0 \in I$,

(c2*) $(\forall x \in \Omega)\ (\forall y \in I)\ (x \circ y < I \Rightarrow x \in I)$.

If I is a Smarandache hyper $(<,\in)$-ideal of H related to every hyper BCK-algebra contained in H, we simply say that I is a *Smarandache hyper* $(<,\in)$*-ideal* of H.

Example 3.10. Consider a Smarandache hyper K-algebra $H = \{0, a, b, c\}$ described in Example 3.8. Then a subset $\{0, a\}$ of H is an Ω-Smarandache hyper $(<, \in)$-ideal of H, where $\Omega = \{0, a, b\}$. Moreover, a subset $\{0, c\}$ of H is an Ω-Smarandache hyper $(<, \in)$-ideal of H, where $\Omega = \{0, b\}$.

Theorem 3.11. *Every Ω-Smarandache hyper $(<, \in)$-ideal is a Ω-Smarandache hyper $(\ll, \in)$-ideal.*

Proof. Straightforward. □

Proposition 3.12. *Let A be a subset of $H$and let I be an Ω-Smarandache hyper $(\ll, \in)$-ideal of H which is contained in Ω. If $A \subseteq \Omega$ and $A \ll I$, then A is contained in I.*

Proof. Assume that $A \subseteq \Omega$ and $A \ll I$. Let $x \in A$. Then $x \circ 0 = \{x\} \ll I$, and so $x \in I$ by (c2). Therefore A is contained in I. □

Proposition 3.13. *Let H be a Smarandache hyper K-algebra with Ω as a hyper BCK-algebra. Then*

$$(\forall x, y, z, u \in \Omega)\,(((x \circ z) \circ (y \circ z)) \circ u \ll (x \circ y) \circ u).$$

Proof. Let $x, y, z, u \in \Omega$. Since Ω is a hyper BCK-algebra, we have

$$\begin{aligned} ((x \circ z) \circ (y \circ z)) \circ u &= ((x \circ u) \circ z) \circ (y \circ z) \\ &= \cup\{(w \circ z) \circ (y \circ z) \mid w \in x \circ u\} \\ &\ll \cup\{w \circ y \mid w \in x \circ u\} = (x \circ u) \circ y = (x \circ y) \circ u, \end{aligned}$$

which is the desired result. □

Definition 3.14. A nonempty subset I of H is called a *Smarandache hyper* $(\subseteq, \in)$*-ideal* of H related to Ω (or briefly, Ω*-Smarandache hyper* $(\subseteq, \in)$*-ideal* of H) if it satisfies:

(c1) $0 \in I$,
(cw) $(\forall x \in \Omega)\ (\forall y \in I)\ (x \circ y \subseteq I \Rightarrow x \in I)$.

If I is a Smarandache hyper $(\subseteq, \in)$-ideal of H related to every hyper BCK-algebra contained in H, we simply say that I is a *Smarandache hyper* $(\subseteq, \in)$*-ideal* of H.

Example 3.15. Let $H = \{0, a, b, c\}$ and define an hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0, a\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Table a5

Then $(H, \circ, 0)$ is a Smarandache hyper K-algebra because $(\Omega = \{0, b, c\}, \circ, 0)$ is a hyper BCK-algebra. Moreover, a subset $\{0, b\}$ of H is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal of H.

Theorem 3.16. *Every Smarandache hyper $(\ll, \in)$-ideal is a Smarandache hyper $(\subseteq, \in)$-ideal.*

Proof. Note that $A \subseteq B \Rightarrow A \ll B$ for every subsets A and B of H. Thus the result is obvious. □

The converse of Theorem 3.16 is not true in general as seen in the following example.

Example 3.17. Consider Example 3.15. Then $\{0, c\}$ is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal of H which is not an Ω-Smarandache hyper $(\ll, \in)$-ideal of H, where $\Omega = \{0, b, c\}$.

Theorem 3.18. *Let Ω_1 and Ω_2 be hyper BCK-algebras which are properly contained in H such that $\Omega_1 \subset \Omega_2$. Then every Ω_2-Smarandache hyper $(\ll, \in)$-ideal is an Ω_1-Smarandache hyper $(\ll, \in)$-ideal.*

Proof. Straightforward. □

Theorem 3.19. *Let Ω_1 and Ω_2 be hyper BCK-algebras which are properly contained in H such that $\Omega_1 \subset \Omega_2$. Then every Ω_2-Smarandache hyper $(<, \in)$-ideal is an Ω_1-Smarandache hyper $(<, \in)$-ideal.*

Proof. Straightforward. □

Theorem 3.20. *Let Ω_1 and Ω_2 be hyper BCK-algebras which are properly contained in H such that $\Omega_1 \subset \Omega_2$. Then every Ω_2-Smarandache hyper $(\subseteq, \in)$-ideal is an Ω_1-Smarandache hyper $(\subseteq, \in)$-ideal.*

Proof. Straightforward. □

Definition 3.21. A nonempty subset I of H is called a *Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal* of H related to Ω (or briefly, *Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal* of H) if it satisfies:

(c1) $0 \in I$,

(cs) $(\forall x, y, z \in \Omega)$ $((x \circ y) \circ z \ll I,\ y \circ z \subseteq I \Rightarrow x \circ z \subseteq I)$.

Theorem 3.22. *Every Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal is an Ω-Sma- randache hyper $(\ll, \in)$-ideal.*

Proof. Let I be an Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal of H and assume that $x \circ y \ll I$ for all $x \in \Omega$ and $y \in I$. Taking $z = 0$ in (cs), we have $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \subseteq I$. Using (cs), we conclude that $\{x\} = x \circ 0 \subseteq I$. Hence I is an Ω-Smarandache hyper $(\ll, \in)$-ideal of H. □

The following example shows that the converse of Theorem 3.22 is not true in general.

Example 3.23. Let $H = \{0, a, b, c\}$ and define an hyper operation "$\circ$" on H by the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0, a\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0, b\}$

Table a6

Then $(H, \circ, 0)$ is a Smarandache hyper K-algebra because $(\Omega = \{0, b, c\}, \circ, 0)$ is a hyper BCK-algebra. Moreover, a subset $\{0\}$ of H is an Ω-Smarandache hyper $(\ll, \in)$-ideal of H which is not an an Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal of H.

Theorem 3.24. *If I is an Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal of H, then the set*

$$\Omega(I_a) := \{x \in \Omega \mid x \circ a \subseteq I\}$$

is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal of H for all $a \in H$.

Proof. Obviously $0 \in \Omega(I_a)$. Assume that $x \circ y \subseteq \Omega(I_a)$ for all $x \in \Omega$ and $y \in \Omega(I_a)$. Then $(x \circ y) \circ a \subseteq I$, and hence $(x \circ y) \circ a \ll I$, and $y \circ a \subseteq I$. It follows from (cs) that $x \circ a \subseteq I$, that is, $x \in \Omega(I_a)$. Therefore $\Omega(I_a)$ is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal of H. □

Theorem 3.25. *Let I be an Ω-Smarandache hyper $(\ll, \in)$-ideal of H which is contained in Ω such that the set*

$$\Omega(I_a) := \{x \in \Omega \mid x \circ a \subseteq I\},\ a \in H$$

is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal of H. Then I is an Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal of H.

Proof. Assume that $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ for all $x, y, z \in \Omega$. From $y \circ z \subseteq I$, we get $y \in \Omega(I_z)$. Since $(x \circ y) \circ z \subseteq \Omega$, it follows from Proposition 3.12 that $(x \circ y) \circ z \subseteq I$ so that $w \circ z \subseteq I$, that is, $w \in \Omega(I_z)$ for each $w \in x \circ y$. Hence $x \circ y \subseteq \Omega(I_z)$. Since $\Omega(I_z)$ is an Ω-Smarandache hyper $(\subseteq, \in)$-ideal, we have $x \in \Omega(I_z)$, and so $x \circ z \subseteq I$. Therefore I is an Ω-Smarandache hyper $(\ll, \subseteq, \subseteq)$-ideal of H. □

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Y. B. Jun, Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea
E-mail address: ybjun@gsnu.ac.kr

E. H. Roh, Department of Mathematics Education, Chinju National University of Education, Jinju 660-756, Korea
E-mail address: ehroh@cue.ac.kr