A Quasi-Exactly Solvable Non-Polynomial, Non-Confining Potential Well

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Abstract

Using a momentum scale, we construct an n-independent, non-polynomial, symmetrized finite well, which, with the addition of a delta potential with n-dependent coupling, becomes quasi-exactly solvable. Making a polynomial ansatz for the closed-form eigenfunctions, we obtain a three-term recursion relation, from which the known energies are derived and the polynomial coefficients are factorized. The coupling is then written in terms of a continued fraction, which, as n tends to infinity, reveals a triangular symmetry and converges. Finally, the location of the closed-form eigenfunctions is determined and the first ones are examined.

Keywords: non-polynomial potentials, symmetrized potentials, piecewise analytic potentials, potential wells, delta potential, quasi-exactly solvable potentials, closed-form eigenfunctions, convergent coupling

Contents

1. Introduction

The search for exact solutions to the Schrödinger equation has resulted in the discovery of a new class of potentials, for which only a finite part of the energy spectrum, along with the respective eigenfunctions, can be found in closed-form [1-8]. These potentials are called quasi-exactly solvable and occupy an intermediate place, between the few which are exactly solvable (such as the harmonic oscillator and the Coulomb potential), for which the entire energy spectrum and all eigenfunctions are known, and the many which are non-solvable, for which none eigenvalue and none eigenfunction can be exactly determined.

Apart from analytic quasi-exactly solvable potentials [1-4], symmetrized (piecewise analytic) ones have also been proposed and studied [5-8].

In this framework, we introduce the potential

$$V(x) = -\frac{\varepsilon_0}{\frac{|x|}{x_0}+1} - \lambda\delta(x),$$

where the energy and length scales ε_0 and x_0 are related to a positive momentum scale p_0, such that

$$\varepsilon_0 = \frac{p_0^{\ 2}}{m} \text{ and } x_0 = \frac{\hbar}{p_0}$$

For convenience, we write the coupling constant λ as $\frac{\lambda\hbar^2}{mx_0}$, where λ is now dimensionless*, and the potential is then written as

$$V(x) = -\frac{\varepsilon_0}{\frac{|x|}{x_0}+1} - \frac{\lambda\hbar^2\delta(x)}{mx_0} \quad (1)$$

* Since $\int_{-\infty}^{\infty} dx\delta(x) = 1$, the delta function has dimensions of inverse length, then the coupling of the delta potential has dimensions of energy times length.

Then, since $\left[\frac{\hbar^2}{mx_0}\right] = \frac{(PL)^2}{ML} = \frac{P^2L}{M} = EL$, the new λ is dimensionless.

In (1), the first term $-\frac{\varepsilon_0}{\frac{|x|}{x_0}+1}$ is a symmetric well of finite depth equal to ε_0, and it is continuous everywhere.

The delta potential $-\frac{\lambda\hbar^2\delta(x)}{mx_0}$ creates, at $x=0$, a spike of infinite depth, provided that $\lambda>0$.

The energy eigenfunctions $\psi_n(x)$ are continuous everywhere [9-12], so that the integral $\int_{-\infty}^{\infty} dx|\psi_n(x)|^2$ is well defined, while the presence of the delta function induces a discontinuity in the first derivatives of $\psi_n(x)$ at zero, equal to $-\frac{2\lambda\psi_n(0)}{x_0}$ [9, 11], i.e.

$$\psi_n'(0^+)-\psi_n'(0^-)=-\frac{2\lambda\psi_n(0)}{x_0} \tag{2}$$

Proof of (2)

As an energy eigenfunction of the potential (1), $\psi_n(x)$ satisfies the time-independent Schrödinger equation

$$\psi_n''(x)+\frac{2m}{\hbar^2}(E_n-V(x))\psi_n(x)=0,$$

with E_n being the energy of the eigenstate described by $\psi_n(x)$.

If $\varepsilon>0$, then we have

$$\int_{-\varepsilon}^{\varepsilon} dx\left(\psi_n''(x)+\frac{2m}{\hbar^2}(E_n-V(x))\psi_n(x)\right)=0\Rightarrow$$

$$\Rightarrow\int_{-\varepsilon}^{\varepsilon} dx\left(\psi_n''(x)+\frac{2m}{\hbar^2}\left(E_n+\frac{\varepsilon_0}{\frac{|x|}{x_0}+1}+\frac{\lambda\hbar^2\delta(x)}{mx_0}\right)\psi_n(x)\right)=0\Rightarrow$$

$$\Rightarrow\int_{-\varepsilon}^{\varepsilon} dx\psi_n''(x)+\frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} dx\left(E_n+\frac{\varepsilon_0}{\frac{|x|}{x_0}+1}\right)\psi_n(x)+\frac{2\lambda}{x_0}\underbrace{\int_{-\varepsilon}^{\varepsilon} dx\delta(x)\psi_n(x)}_{\psi_n(0)}=0\Rightarrow$$

$$\Rightarrow \psi_n'(x)\Big|_{-\varepsilon}^{\varepsilon} + \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} dx \left(E_n + \frac{\varepsilon_0}{\frac{|x|}{x_0}+1} \right) \psi_n(x) + \frac{2\lambda \psi_n(0)}{x_0} = 0 \Rightarrow$$

$$\Rightarrow \psi_n'(\varepsilon) - \psi_n'(-\varepsilon) + \frac{2\lambda \psi_n(0)}{x_0} + \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} dx \left(E_n + \frac{\varepsilon_0}{\frac{|x|}{x_0}+1} \right) \psi_n(x) = 0$$

We assume that the eigenfunction $\psi_n(x)$ is also bounded everywhere, so that the probability density is finite everywhere.

Then the function $\left(E_n + \frac{\varepsilon_0}{\frac{|x|}{x_0}+1} \right) \psi_n(x)$ is continuous and bounded in the interval $[-\varepsilon, \varepsilon]$, thus it has a maximum and a minimum value in $[-\varepsilon, \varepsilon]$, which we denote by a and b, respectively.

Thus

$$2\varepsilon b \le \int_{-\varepsilon}^{\varepsilon} dx \left(E_n + \frac{\varepsilon_0}{\frac{|x|}{x_0}+1} \right) \psi_n(x) \le 2\varepsilon a$$

Taking the limit $\varepsilon \to 0^+$, the previous integral vanishes, and then the last equation gives

$$\psi_n'(0^+) - \psi_n'(0^-) + \frac{2\lambda \psi_n(0)}{x_0} = 0$$

Thus

$$\psi_n'(0^+) - \psi_n'(0^-) = -\frac{2\lambda \psi_n(0)}{x_0}$$

To find the closed-form eigenfunctions of the potential (1), we use the ansatz*

$$\psi_n(x) = A_n P_n(x) \left(\frac{|x|}{x_0} + 1 \right) \exp\left(-\frac{a_n |x|}{x_0} \right) \quad (3)$$

where a_n are dimensionless positive constants, A_n are normalization constants with the same dimensions as $\psi_n(x)$, i.e. $L^{-1/2}$, and $P_n(x)$ are dimensionless polynomials of degree $n = 0,1,...$

* Since the potential vanishes at infinity, at large distances $\psi_n(x) \sim \exp(-k|x|)$, where k depends on the energy and has dimensions of inverse length.
We'll see that the constants a_n are energy-dependent.
The factor $\frac{|x|}{x_0}+1$ is put in because the potential (1) has a fraction part with the same denominator.

Since $a_n > 0$, the exponential term $\exp\left(-\frac{a_n|x|}{x_0}\right)$ ensures that the eigenfunctions are square integrable, and thus they describe bound eigenstates.
Since the potential (1) is symmetric, its energy eigenfunctions have definite parity [9, 10], i.e. they are of either even or odd parity.
Then, since the functions $\frac{|x|}{x_0}+1$ and $\exp\left(-\frac{a_n|x|}{x_0}\right)$ are both of even parity, the polynomials $P_n(x)$ have the same parity as the eigenfunctions $\psi_n(x)$.

2. Discontinuity condition

As explained, the eigenfunctions $\psi_n(x)$ must be continuous at zero, while their first derivatives must have a discontinuity, at zero, equal to $-\frac{2\lambda\psi_n(0)}{x_0}$.

Since the functions $\frac{|x|}{x_0}+1$ and $\exp\left(-\frac{a_n|x|}{x_0}\right)$ are continuous at zero, from (3) we see that the polynomials $P_n(x)$ must be continuous at zero, i.e.

$$P_n(0^-) = P_n(0^+) \equiv P_n(0) \qquad (4)$$

For $x < 0$, from (3) we obtain

$$\psi_n(x) = A_n P_n(x)\left(-\frac{x}{x_0}+1\right)\exp\left(\frac{a_n x}{x_0}\right)$$

Then, the first derivative in the negative region is

$$\psi_n'(x) = A_n\left(P_n'(x)\left(-\frac{x}{x_0}+1\right)-\frac{1}{x_0}P_n(x)+\frac{a_n}{x_0}P_n(x)\left(-\frac{x}{x_0}+1\right)\right)\exp\left(\frac{a_n x}{x_0}\right)=$$
$$= A_n\left(P_n'(x)\left(-\frac{x}{x_0}+1\right)+\left(-\frac{a_n x}{x_0^2}+\frac{a_n-1}{x_0}\right)P_n(x)\right)\exp\left(\frac{a_n x}{x_0}\right)$$

That is

$$\psi_n'(x) = A_n\left(\left(-\frac{a_n x}{x_0^2}+\frac{a_n-1}{x_0}\right)P_n(x)+\left(-\frac{x}{x_0}+1\right)P_n'(x)\right)\exp\left(\frac{a_n x}{x_0}\right)$$

and

$$\psi_n'(0^-) = A_n\left(\frac{a_n-1}{x_0}P_n(0)+P_n'(0^-)\right) \qquad (5)$$

where we also used (4).

For $x>0$, from (3) we obtain

$$\psi_n(x) = A_n P_n(x)\left(\frac{x}{x_0}+1\right)\exp\left(-\frac{a_n x}{x_0}\right)$$

Then, the first derivative in the positive region is

$$\psi_n'(x) = A_n\left(P_n'(x)\left(\frac{x}{x_0}+1\right)+\frac{1}{x_0}P_n(x)-\frac{a_n}{x_0}P_n(x)\left(\frac{x}{x_0}+1\right)\right)\exp\left(-\frac{a_n x}{x_0}\right)=$$
$$= A_n\left(\left(\frac{x}{x_0}+1\right)P_n'(x)+\left(-\frac{a_n x}{x_0^2}+\frac{1-a_n}{x_0}\right)P_n(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)$$

That is

$$\psi_n'(x) = A_n\left(\left(-\frac{a_n x}{x_0^2}+\frac{1-a_n}{x_0}\right)P_n(x)+\left(\frac{x}{x_0}+1\right)P_n'(x)\right)\exp\left(-\frac{a_n x}{x_0}\right) \qquad (6)$$

and

$$\psi_n'(0^+) = A_n\left(\frac{1-a_n}{x_0}P_n(0)+P_n'(0^+)\right) \qquad (7)$$

where we also used (4).

Also, from (3), we obtain

$$\psi_n(0) = A_n P_n(0) \tag{8}$$

By means of (5), (7), and (8), the discontinuity condition (2) is written as

$$A_n\left(\frac{1-a_n}{x_0}P_n(0)+P_n'(0^+)\right)-A_n\left(\frac{a_n-1}{x_0}P_n(0)+P_n'(0^-)\right)=-\frac{2\lambda A_n P_n(0)}{x_0}$$

The normalization constant A_n cannot be zero, because then the eigenfunction is identically zero, and thus it is linearly dependent, and then it cannot be eigenfunction. Since $A_n \neq 0$, dividing both members of the last equation by A_n, we obtain

$$\frac{1-a_n}{x_0}P_n(0)+P_n'(0^+)-\frac{a_n-1}{x_0}P_n(0)-P_n'(0^-)=-\frac{2\lambda P_n(0)}{x_0}\Rightarrow$$
$$\Rightarrow P_n'(0^+)-P_n'(0^-)+\frac{2(1-a_n)}{x_0}P_n(0)=-\frac{2\lambda P_n(0)}{x_0}\Rightarrow$$
$$\Rightarrow P_n'(0^+)-P_n'(0^-)=-2\left(\frac{\lambda}{x_0}+\frac{1-a_n}{x_0}\right)P_n(0)$$

Thus

$$P_n'(0^+)-P_n'(0^-)=-\frac{2(\lambda+1-a_n)}{x_0}P_n(0) \tag{9}$$

The equation (9) is the discontinuity condition of the derivative of $P_n(x)$.

3. The differential equation for the polynomials P_n in the region x>0

Using (6), the second derivative of $\psi_n(x)$ in the region $x>0$ is

$$\psi_n''(x)=A_n\left(-\frac{a_n}{x_0^2}P_n(x)+\left(-\frac{a_n x}{x_0^2}+\frac{1-a_n}{x_0}\right)P_n'(x)+\frac{1}{x_0}P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)+$$
$$+A_n\left(-\frac{a_n}{x_0}\right)\left(\left(-\frac{a_n x}{x_0^2}+\frac{1-a_n}{x_0}\right)P_n(x)+\left(\frac{x}{x_0}+1\right)P_n'(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)=$$
$$=A_n\left(-\frac{a_n}{x_0^2}P_n(x)+\left(-\frac{a_n x}{x_0^2}+\frac{2-a_n}{x_0}\right)P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)+$$
$$+A_n\left(\left(\frac{a_n^2 x}{x_0^3}+\frac{a_n^2-a_n}{x_0^2}\right)P_n(x)+\left(-\frac{a_n x}{x_0^2}-\frac{a_n}{x_0}\right)P_n'(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)=$$

$$= A_n\left(\left(\frac{a_n^{\ 2}x}{x_0^{\ 3}}+\frac{a_n^{\ 2}-2a_n}{x_0^{\ 2}}\right)P_n(x)+2\left(-\frac{a_n x}{x_0^{\ 2}}+\frac{1-a_n}{x_0}\right)P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)=$$
$$= A_n\left(\frac{a_n}{x_0^{\ 2}}\left(\frac{a_n x}{x_0}+a_n-2\right)P_n(x)+\frac{2}{x_0}\left(-\frac{a_n x}{x_0}+1-a_n\right)P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)$$

That is, in the region $x>0$,

$$\psi_n''(x)= A_n\left(\frac{a_n}{x_0^{\ 2}}\left(\frac{a_n x}{x_0}+a_n-2\right)P_n(x)+\frac{2}{x_0}\left(-\frac{a_n x}{x_0}+1-a_n\right)P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)\right)\exp\left(-\frac{a_n x}{x_0}\right)$$

As an energy eigenfunction, $\psi_n(x)$ satisfies the time-independent Schrödinger equation

$$\psi_n''(x)+\frac{2m}{\hbar^2}\left(E_n-V(x)\right)\psi_n(x)=0,$$

with E_n being the energy of the eigenstate described by $\psi_n(x)$.

Plugging into the previous equation, the expression of $\psi_n''(x)$ along with the expressions of $\psi_n(x)$ – from (3) – and the potential – from (1) – and dividing by $A_n\exp\left(-\frac{a_n x}{x_0}\right)$, we obtain, in the region $x>0$*,

$$\frac{a_n}{x_0^{\ 2}}\left(\frac{a_n x}{x_0}+a_n-2\right)P_n(x)+\frac{2}{x_0}\left(-\frac{a_n x}{x_0}+1-a_n\right)P_n'(x)+\left(\frac{x}{x_0}+1\right)P_n''(x)+$$
$$+\frac{2m}{\hbar^2}\left(E_n+\frac{\varepsilon_0}{\frac{x}{x_0}+1}\right)\left(\frac{x}{x_0}+1\right)P_n(x)=0$$

* The delta function vanishes in the region $x>0$ (and in the region $x<0$ too).

The last equation is written as

$$\left(\frac{x}{x_0}+1\right)P_n''(x)+\frac{2}{x_0}\left(-\frac{a_n x}{x_0}+1-a_n\right)P_n'(x)+\left(\frac{a_n}{x_0^{\ 2}}\left(\frac{a_n x}{x_0}+a_n-2\right)+\frac{2mE_n}{\hbar^2}\left(\frac{x}{x_0}+1\right)+\frac{2m\varepsilon_0}{\hbar^2}\right)P_n(x)=0$$

To make things easier, we introduce the dimensionless variable*

$$\tilde{x} = \frac{x}{x_0} \qquad (10)$$

and we set

$$k_n = \sqrt{-\frac{2mE_n}{\hbar^2}} \text{ and } \tilde{k}_0 = \sqrt{\frac{2m\varepsilon_0}{\hbar^2}}$$

* 1. Since $x_0 > 0$, $\tilde{x}$ and x have the same sign, thus $\tilde{x} > 0$ in the region $x > 0$.

2. Since $V(x) < 0$, the bound energies are negative, i.e. $E_n < 0$.

3. Observe that $[k_n] = [\tilde{k}_0] = L^{-1}$.

From (10), we obtain

$$\frac{d}{dx} = \frac{d}{d\tilde{x}}\frac{d\tilde{x}}{dx} = \frac{1}{x_0}\frac{d}{d\tilde{x}}$$

and thus

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{x_0{}^2}\frac{d^2}{d\tilde{x}^2}$$

Then, the last differential equation becomes

$$(\tilde{x}+1)\frac{1}{x_0{}^2}P_n''(\tilde{x}) + \frac{2}{x_0}(-a_n\tilde{x}+1-a_n)\frac{1}{x_0}P_n'(\tilde{x}) + \left(\frac{a_n}{x_0{}^2}(a_n\tilde{x}+a_n-2) - k_n{}^2(\tilde{x}+1) + \tilde{k}_0{}^2\right)P_n(\tilde{x}) = 0$$

where now the prime denotes differentiation with respect to $\tilde{x}$.

The last equation is written as

$$(\tilde{x}+1)P_n''(\tilde{x}) + 2(-a_n\tilde{x}+1-a_n)P_n'(\tilde{x}) + \left(a_n(a_n\tilde{x}+a_n-2) - (k_nx_0)^2(\tilde{x}+1) + (\tilde{k}_0x_0)^2\right)P_n(\tilde{x}) = 0$$

But

$$\left(\tilde{k}_0 x_0\right)^2 = \left(\sqrt{\frac{2m\varepsilon_0}{\hbar^2}}\frac{\hbar}{p_0}\right)^2 = \left(\sqrt{\frac{2m\frac{p_0{}^2}{m}}{\hbar^2}}\frac{\hbar}{p_0}\right)^2 = \left(\sqrt{\frac{2p_0{}^2}{\hbar^2}}\frac{\hbar}{p_0}\right)^2 = 2$$

Then, we have

$$a_n\left(a_n\tilde{x}+a_n-2\right)-\left(k_n x_0\right)^2\left(\tilde{x}+1\right)+\left(\tilde{k}_0 x_0\right)^2 = a_n{}^2\tilde{x}+a_n\left(a_n-2\right)-\left(k_n x_0\right)^2\tilde{x}-\left(k_n x_0\right)^2+2=$$
$$=\left(a_n{}^2-\left(k_n x_0\right)^2\right)\tilde{x}+a_n\left(a_n-2\right)+2-\left(k_n x_0\right)^2$$

and the last differential equation is written as

$$\left(\tilde{x}+1\right)P_n''\left(\tilde{x}\right)+2\left(-a_n\tilde{x}+1-a_n\right)P_n'\left(\tilde{x}\right)+\left(\left(a_n{}^2-\left(k_n x_0\right)^2\right)\tilde{x}+a_n\left(a_n-2\right)+2-\left(k_n x_0\right)^2\right)P_n\left(\tilde{x}\right)=0 \quad (11)$$

The polynomials $P_n\left(\tilde{x}\right)$ are dimensionless and the variable $\tilde{x}$ is also dimensionless.

The differential equation (11) is linear and homogeneous, and thus each of the polynomials $P_n\left(\tilde{x}\right)$ is calculated up to a multiplicative constant.

Then, without loss of generality, we can assume that the polynomials $P_n\left(\tilde{x}\right)$ are monic in the region $\tilde{x}>0$, i.e.

$P_n\left(\tilde{x}\right)=\sum_{k=0}^{n} p_k\tilde{x}^k$, with $p_n=1$.

Then, in the left-hand side of (11), we have a polynomial of degree $n+1$, with the coefficient of its highest term $\tilde{x}^{n+1}$ being $a_n{}^2-\left(k_n x_0\right)^2$.

Since the left-hand-side polynomial in (11) equals zero for every $\tilde{x}>0$, all its coefficients must be zero, and thus

$$a_n{}^2-\left(k_n x_0\right)^2=0$$

Since a_n, k_n, x_0 are positive, we obtain

$$a_n = k_n x_0 \quad (12)$$

By means of (12), (11) becomes

$$\left(\tilde{x}+1\right)P_n''\left(\tilde{x}\right)+2\left(-k_n x_0\tilde{x}+1-k_n x_0\right)P_n'\left(\tilde{x}\right)+\left(k_n x_0\left(k_n x_0-2\right)+2-\left(k_n x_0\right)^2\right)P_n\left(\tilde{x}\right)=0$$

and thus

$$\left(\tilde{x}+1\right)P_n''\left(\tilde{x}\right)+2\left(-k_nx_0\tilde{x}+1-k_nx_0\right)P_n'\left(\tilde{x}\right)+2\left(1-k_nx_0\right)P_n\left(\tilde{x}\right)=0 \qquad (13)$$

Now, the left-hand side of (13) is a polynomial of degree n, with the coefficients of its highest term $\tilde{x}^n$ being

$$2\left(-k_nx_0\right)n+2\left(1-k_nx_0\right)=2\left(1-\left(n+1\right)k_nx_0\right),$$

since the coefficient of the highest term $\tilde{x}^{n-1}$ of $P_n'\left(\tilde{x}\right)$ is n, because $P_n\left(\tilde{x}\right)$ is monic in the region $\tilde{x}>0$.

The left-hand-side polynomial in (13) equals zero for every $\tilde{x}>0$, and thus all its coefficients must also be zero, and then

$$1-\left(n+1\right)k_nx_0=0$$

and thus

$$k_n=\frac{1}{\left(n+1\right)x_0} \qquad (14)$$

for $n=0,1,...$

Substituting (14) into $k_n=\sqrt{-\frac{2mE_n}{\hbar^2}}$ and solving for E_n, we obtain

$$E_n=-\frac{\hbar^2}{2\left(n+1\right)^2 m{x_0}^2} \qquad (15)$$

for $n=0,1,...$

These are the energies of those eigenstates of the potential (1) which are described by the closed-form eigenfunctions (3).

Besides, substituting (14) into (12) yields

$$a_n=\frac{1}{n+1} \qquad (16)$$

Using (16), the closed-form eigenfunctions (3) and the discontinuity condition (9) are respectively written as

$$\psi_n(x) = A_n P_n(x)\left(\frac{|x|}{x_0}+1\right)\exp\left(-\frac{|x|}{(n+1)x_0}\right) \quad (17)$$

$$P_n'(0^+) - P_n'(0^-) = -\frac{2}{x_0}\left(\lambda + \frac{n}{n+1}\right)P_n(0) \quad (18)$$

for $n = 0,1,...$

Finally, by means of (14), the differential equation (13) becomes, after a little algebra,

$$(n+1)(\tilde{x}+1)P_n''(\tilde{x}) - 2(\tilde{x}-n)P_n'(\tilde{x}) + 2nP_n(\tilde{x}) = 0 \quad (19)$$

for $n = 0,1,...$, and $\tilde{x} > 0$.

4. The recursion relation and the polynomial coefficients

As $P_n(\tilde{x})$ is monic, the coefficients of the terms $\tilde{x}^k$, with $k = 0,1,...,n$, in the left-hand side of (19) are, respectively,

$$2nP_n(\tilde{x}) \to 2np_k$$

$$-2(\tilde{x}-n)P_n'(\tilde{x}) \to -2\left(kp_k - n(k+1)p_{k+1}\right)$$

$$(n+1)(\tilde{x}+1)P_n''(\tilde{x}) \to (n+1)\left((k+1)kp_{k+1} + (k+2)(k+1)p_{k+2}\right)$$

Since the polynomial in the left-hand side of (19) equals zero for every $\tilde{x} > 0$, all its coefficients must be zero, and thus

$$(n+1)\left(k(k+1)p_{k+1} + (k+1)(k+2)p_{k+2}\right) - 2\left(kp_k - n(k+1)p_{k+1}\right) + 2np_k = 0 \Rightarrow$$
$$\Rightarrow (n+1)k(k+1)p_{k+1} + (n+1)(k+1)(k+2)p_{k+2} - 2kp_k + 2n(k+1)p_{k+1} + 2np_k = 0 \Rightarrow$$
$$\Rightarrow (n+1)(k+1)(k+2)p_{k+2} + (k+1)\left(k(n+1)+2n\right)p_{k+1} - 2(k-n)p_k = 0$$

Therefore, we end up to the recursion relation

$$(n+1)(k+1)(k+2)p_{k+2} = -(k+1)\left(k(n+1)+2n\right)p_{k+1} + 2(k-n)p_k \quad (20)$$

which holds in the region $\tilde{x} > 0$, for $k = 0,1,...,n$.

The recursion relation (20) is a three-term recursion relation, which is indicative of a quasi-exactly solvable potential [4].

For $k=n$, dropping the coefficients p_{n+1} and p_{n+2}, as their indices exceed the degree of $P_n(\tilde{x})$, and using that $p_n=1$, we see that (20) holds identically, as expected, since we used this equation, in the previous section, to fix the constants a_n.

Thus, the recursion relation (20) gives n non-trivial – i.e. linearly independent – equations, for $k=0,1,...n-1$, which are as many as the unknown coefficients $p_0, p_1, ..., p_{n-1}$ of the monic polynomial $P_n(\tilde{x})$.

For $k=n-1$, (20) gives

$$0=-n\left((n-1)(n+1)+2n\right)+2(n-1-n)\,p_{n-1},$$

since $p_n=1$ and $p_{n+1}=0$.

Solving the last equation for p_{n-1}, we obtain

$$p_{n-1}=\frac{-n\left(n^2+2n-1\right)}{2} \tag{21}$$

For $n=0$, p_{n-1} vanishes, as it should, since then the index $n-1$ is negative.

For $n\geq 1$, the index $n-1$ is non-negative and then the coefficient p_{n-1} appears in the polynomial $P_n(\tilde{x})$, and then, as we see from (21), p_{n-1} is negative, i.e. it has opposite sign from that of p_n.

For $k=n-2$, (20) gives, by means of (21),

$$(n+1)(n-1)n=-(n-1)\left((n-2)(n+1)+2n\right)\left(-\frac{n\left(n^2+2n-1\right)}{2}\right)-4p_{n-2}$$

Solving for p_{n-2} we obtain, after some algebra,

$$p_{n-2}=\frac{(n-1)n\left(\left(n^2+2n-1\right)\left((n-2)(n+1)+2n\right)-2(n+1)\right)}{8} \tag{22}$$

For $n=0,1$, p_{n-2} vanishes, as it should, since then the index $n-2$ is negative.

For $n\geq 2$, the index $n-2$ is non-negative and then p_{n-2} appears in $P_n(\tilde{x})$, and then, as we can easily derive from (22), p_{n-2} is positive, i.e. it has opposite sign from that of p_{n-1}.

We observe that (21) and (22) are respectively written as

$$p_{n-1} = \frac{(-1)^1 n}{2^1 * 1!}\left(n^2 + 2n - 1\right) = \frac{(-1)^1 n!}{2^1 * 1!(n-1)!}\left(\underbrace{n^2 + 2n - 1}_{\substack{\text{Monic polynomial} \\ \text{of degree } 2*1}}\right)$$

$$p_{n-2} = \frac{(-1)^2 (n-1)n}{2^2 * 2!}\left(\left(n^2 + 2n - 1\right)\left((n-2)(n+1) + 2n\right) - 2(n+1)\right) =$$
$$= \frac{(-1)^2 n!}{2^2 * 2!(n-2)!}\left(\underbrace{\left(n^2 + 2n - 1\right)\left((n-2)(n+1) + 2n\right) - 2(n+1)}_{\text{Monic polynomial of degree } 2*2}\right)$$

We'll show, by induction, that the coefficient p_{n-k}, where $k = 1, 2, ..., n$, is written as

$$p_{n-k} = \frac{(-1)^k n!}{2^k k!(n-k)!} f_{2k}(n) \qquad (23)$$

where $f_{2k}(n)$ monic polynomial of degree $2k$ in n.

For $k = 1, 2$, (23) holds, as shown, for

$f_2(n) = n^2 + 2n - 1$ and $f_4(n) = \left(n^2 + 2n - 1\right)\left((n-2)(n+1) + 2n\right) - 2(n+1)$

Assuming that (23) holds for k and $k+1$, we have

$$p_{n-k} = \frac{(-1)^k n!}{2^k k!(n-k)!} f_{2k}(n)$$

$$p_{n-(k+1)} = \frac{(-1)^{k+1} n!}{2^{k+1}(k+1)!(n-(k+1))!} f_{2(k+1)}(n),$$

where $f_{2k}(n), f_{2(k+1)}(n)$ monic polynomials of degrees $2k$ and $2(k+1)$, respectively.

We'll show that (23) holds for $k+2$.

For $k \to n-(k+2)$, the recursion relation (20) is written as

$$(n+1)(n-(k+2)+1)(n-(k+2)+2)\, p_{n-(k+2)+2} = -(n-(k+2)+1)\left((n-(k+2))(n+1) + 2n\right) p_{n-(k+2)+1} +$$
$$+2(n-(k+2)-n)\, p_{n-(k+2)} \Rightarrow$$
$$\Rightarrow (n+1)(n-(k+1))(n-k)\, p_{n-k} = -(n-(k+1))\left((n-(k+2))(n+1) + 2n\right) p_{n-(k+1)} - 2(k+2)\, p_{n-(k+2)}$$

Substituting p_{n-k} and $p_{n-(k+1)}$, the last recursion relation becomes

$$(n+1)(n-(k+1))(n-k)\frac{(-1)^k n!}{2^k k!(n-k)!} f_{2k}(n) =$$
$$= -(n-(k+1))((n-(k+2))(n+1)+2n)\frac{(-1)^{k+1} n!}{2^{k+1}(k+1)!(n-(k+1))!} f_{2(k+1)}(n) - 2(k+2)\, p_{n-(k+2)}$$

But

$$\frac{(n-(k+1))(n-k)}{(n-k)!} = \frac{1}{(n-(k+2))!}, \quad \frac{(n-(k+1))}{(n-(k+1))!} = \frac{1}{(n-(k+2))!},$$

and then

$$(n+1)\frac{(-1)^k n!}{2^k k!(n-(k+2))!} f_{2k}(n) =$$
$$= -((n-(k+2))(n+1)+2n)\frac{(-1)^{k+1} n!}{2^{k+1}(k+1)!(n-(k+2))!} f_{2(k+1)}(n) - 2(k+2)\, p_{n-(k+2)} \Rightarrow$$
$$\Rightarrow 2(k+2)\, p_{n-(k+2)} = -\frac{(-1)^{k+1} n!}{2^{k+1}(k+1)!(n-(k+2))!}((n-(k+2))(n+1)+2n) f_{2(k+1)}(n) -$$
$$-\frac{(-1)^k n!}{2^k k!(n-(k+2))!}(n+1) f_{2k}(n) \overset{\substack{-(-1)^{k+1}=(-1)^{k+2} \\ (-1)^k=(-1)^{k+2}}}{=}$$
$$= \frac{(-1)^{k+2} n!}{2^{k+1}(k+1)!(n-(k+2))!}((n-(k+2))(n+1)+2n) f_{2(k+1)}(n) - \frac{(-1)^{k+2} n!}{2^k k!(n-(k+2))!}(n+1) f_{2k}(n) =$$
$$= \frac{(-1)^{k+2} n!}{2^{k+1}(k+1)!(n-(k+2))!}(((n-(k+2))(n+1)+2n) f_{2(k+1)}(n) - 2(k+1)(n+1) f_{2k}(n))$$

Thus

$$p_{n-(k+2)} = \frac{(-1)^{k+2} n!}{2^{k+2}(k+2)!(n-(k+2))!}(((n-(k+2))(n+1)+2n) f_{2(k+1)}(n) - 2(k+1)(n+1) f_{2k}(n))$$

Since $f_{2k}(n), f_{2(k+1)}(n)$ are monic polynomials of degrees $2k$ and $2(k+1)$, respectively, the expression

$$\left(\left(n-\left(k+2\right)\right)\left(n+1\right)+2n\right)f_{2(k+1)}\left(n\right)-2\left(k+1\right)\left(n+1\right)f_{2k}\left(n\right)$$

is a monic polynomial of degree $2\left(k+1\right)+2=2\left(k+2\right)$.

Thus, the coefficient $p_{n-(k+2)}$ is written as

$$p_{n-(k+2)}=\frac{\left(-1\right)^{k+2}n!}{2^{k+2}\left(k+2\right)!\left(n-\left(k+2\right)\right)!}f_{2(k+2)}\left(n\right),$$

where

$$f_{2(k+2)}\left(n\right)=\left(\left(n-\left(k+2\right)\right)\left(n+1\right)+2n\right)f_{2(k+1)}\left(n\right)-2\left(k+1\right)\left(n+1\right)f_{2k}\left(n\right) \qquad (24)$$

a monic polynomial of degree $2\left(k+2\right)$.

Therefore, (23) holds for $k=1,2,...,n$.

For $k=0$, (23) gives

$$p_n=\frac{\left(-1\right)^0 n!}{2^0 0!n!}f_0\left(n\right)\overset{\overbrace{0!=1}}{=}f_0\left(n\right)=1,$$

if $f_0\left(n\right)$ is a monic polynomial of zero degree.

Thus, for $f_0\left(n\right)=1$, (23) holds for $k=0$ too.

The polynomials $f_{2k}\left(n\right)$ satisfy the recursion relation (24), with

$f_0\left(n\right)=1$ and $f_2\left(n\right)=n^2+2n-1$

Next, we'll show, also by induction, that

$$f_{2(k+1)}\left(n\right)>\left(n+1\right)f_{2k}\left(n\right)>0 \qquad (25)$$

for $n\geq k+2$.

For $k=0$ and $n\geq 2$, we have

$$f_2\left(n\right)>\left(n+1\right)f_0\left(n\right)\Leftrightarrow n^2+2n-1>n+1\Leftrightarrow n^2+n-2>0$$

The last inequality holds for $n\geq 2$, since the sequence n^2+n-2 is strictly increasing, and thus, for $n\geq 2$,

$$n^2+n-2>4+2-2=4>0$$

Thus $f_2(n) > (n+1) f_0(n)$.

Also, $(n+1) f_0(n) = n+1 > 0$.

Thus, for $k=0$ and $n \geq 2$, (25) holds.
Assuming that

$f_{2(k+1)}(n) > (n+1) f_{2k}(n) > 0$, for $n \geq k+2$,

we'll show that

$f_{2(k+2)}(n) > (n+1) f_{2(k+1)}(n) > 0$, for $n \geq k+3$.

Using the recursion relation (24), the inequality $f_{2(k+2)}(n) > (n+1) f_{2(k+1)}(n)$ is equivalently written as

$$\left(\left(n-(k+2)\right)(n+1)+2n\right) f_{2(k+1)}(n) - 2(k+1)(n+1) f_{2k}(n) > (n+1) f_{2(k+1)}(n) \Leftrightarrow$$
$$\Leftrightarrow \left(\left(n-(k+2)\right)(n+1)+2n-(n+1)\right) f_{2(k+1)}(n) - 2(k+1)(n+1) f_{2k}(n) > 0 \Leftrightarrow$$
$$\Leftrightarrow \left(\left(n-(k+2)\right)(n+1)+n-1\right) f_{2(k+1)}(n) > 2(k+1)(n+1) f_{2k}(n) \qquad (26)$$

Besides, for $n \geq k+3 \Rightarrow n-(k+2) \geq 1$, and thus, since $n+1>0$,

$$\left(n-(k+2)\right)(n+1) \geq n+1 \Leftrightarrow \left(n-(k+2)\right)(n+1)+n-1 \geq n+1+n-1 = 2n \geq 2(k+3) > 2(k+1)$$

Thus

$$\left(n-(k+2)\right)(n+1)+n-1 > 2(k+1) > 0$$

Then, since $f_{2(k+1)}(n) > 0$ (by assumption),

$$\left(\left(n-(k+2)\right)(n+1)+n-1\right) f_{2(k+1)}(n) > 2(k+1) f_{2(k+1)}(n) > 2(k+1)(n+1) f_{2k}(n)$$

Thus

$$\left(\left(n-(k+2)\right)(n+1)+n-1\right) f_{2(k+1)}(n) > 2(k+1)(n+1) f_{2k}(n)$$

Then (26) holds, and thus $f_{2(k+2)}(n) > (n+1) f_{2(k+1)}(n)$.

Also, since $f_{2(k+1)}(n) > 0$ (by assumption), we have $(n+1) f_{2(k+1)}(n) > 0$, and thus

$f_{2(k+2)}(n) > (n+1) f_{2(k+1)}(n) > 0$, for $n \geq k+3$.

Therefore, (25) holds for $n \geq k+2$.

Besides, from (23), we have

$$p_{n-(k+1)} = \frac{(-1)^{k+1} n!}{2^{k+1}(k+1)!(n-(k+1))!} f_{2(k+1)}(n),$$

and, from (25), $f_{2(k+1)}(n) > 0$, for $n \geq k+2$.

Thus, for $k+1 \leq n-1$, the coefficient $p_{n-(k+1)}$ is non-zero and it has sign $(-1)^{k+1}$.

Then, we obtain that p_{n-1} is negative ($k+1=1$), p_{n-2} is positive ($k+1=2$), and so on, until we reach p_1 ($k+1=n-1$).

Including the coefficient p_n, which is positive, we see that the coefficients from p_n up to p_1 are all non-zero and any two successive coefficients have opposite signs, starting from p_n which is positive.

Next, we'll show that p_0 is also included in the previous sequence.

For $k=0$, the recursion relation (20) is written as

$$2(n+1) p_2 = -2np_1 - 2np_0$$

Solving the last equation for p_0, we obtain

$$p_0 = -\left(p_1 + \frac{n+1}{n} p_2 \right) \tag{27}$$

Besides, for $k=n-1$ and $k=n-2$, (23) is respectively written as

$$p_1 = \frac{(-1)^{n-1} n!}{2^{n-1}(n-1)!(n-(n-1))!} f_{2(n-1)}(n) = \frac{(-1)^{n-1} n}{2^{n-1}} f_{2(n-1)}(n)$$

$$p_2 = \frac{(-1)^{n-2} n!}{2^{n-2}(n-2)!(n-(n-2))!} f_{2(n-2)}(n) = \frac{(-1)^{n-2}(n-1)n}{2^{n-1}} f_{2(n-2)}(n)$$

Dividing the second equation by the first, we obtain

$$\frac{p_2}{p_1} = \frac{(-1)^{n-2}(n-1)}{(-1)^{n-1}} \frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} = -(n-1)\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}$$

Thus

$$p_2 = -(n-1)\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}p_1$$

Substituting into (27) yields

$$p_0 = -\left(1 - \frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)p_1 \qquad (28)$$

For $n=0$, the polynomial $P_0(\tilde{x})$ is of zero degree and monic, and thus it has only a constant term, which is 1, i.e. $p_0 = 1$.

For $n=1$, the polynomial $P_1(\tilde{x})$ is of first degree and monic, and thus $p_1 = 1$.

Also, for $n=1$, (28) gives $p_0 = -p_1$, and thus $p_0 = -1$.

We see that p_0 and p_1 have opposite signs.

For $n \geq 2$, from (25) we obtain, for $k = n-2$,

$$0 < \frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} < \frac{1}{n+1}$$

Then, multiplying the previous inequalities by $\frac{(n-1)(n+1)}{n}$, which is positive for $n \geq 2$, we obtain

$$0 < \frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} < \frac{n-1}{n} = 1 - \frac{1}{n}$$

and thus

$$1 - \frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} > \frac{1}{n} > 0$$

Then, from (28), we see that p_0 is non-zero, as p_1 is non-zero, and it has opposite sign from that of p_1.

Therefore, for every $n \in \mathbb{N}$, all coefficients of the polynomial $P_n(\tilde{x})$ are non-zero and any two successive coefficients have opposite signs, starting from p_n, which is positive.

Also, since the constant term p_0 of $P_n\left(\tilde{x}\right)$ is non-zero, $P_n\left(\tilde{x}\right)$ cannot be of odd parity, and thus, since it has definite parity, it is of even parity, for every $n \in \mathbb{N}$.

5. Calculation of the delta potential coupling λ

As explained, the polynomials $P_n(\tilde{x})$ are of even parity, and thus they are written as

$$P_n(\tilde{x}) = p_0 + p_1|\tilde{x}| + \sum_{k=2}^{n} p_k |\tilde{x}|^k$$

with $p_n = 1$.

Then

$$P_n(x) = p_0 + p_1\frac{|x|}{x_0} + \sum_{k=2}^{n} p_k\left(\frac{|x|}{x_0}\right)^k \qquad (29)$$

with $p_n = 1$.

For $n = 0$, both the series and p_1 vanish, while for $n = 1$, only the series vanishes.

Since the polynomials $P_n(x)$ must be continuous at $x = 0$, (29) holds at $x = 0$ too, and thus it holds for every $x \in \mathbb{R}$.

In the region $x > 0$, the constant term of $P_n'(x)$ (the prime now denotes differentiation with respect to x) is $\frac{p_1}{x_0}$, while in the region $x < 0$, its constant term is $-\frac{p_1}{x_0}$, and then $P_n'(0^+) = \frac{p_1}{x_0}$ and $P_n'(0^-) = -\frac{p_1}{x_0}$.

Also, $P_n(0) = p_0$, and then the discontinuity condition (18) is written as

$$\frac{2p_1}{x_0} = -\frac{2}{x_0}\left(\lambda + \frac{n}{n+1}\right)p_0,$$

where p_0 – and p_1 – is non-zero.

Solving the previous equation for λ yields

$$\lambda = -\frac{p_1}{p_0} - \frac{n}{n+1} \qquad (30)$$

By means of (28), (30) is written as

$$\lambda = \frac{1}{1 - \frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}} - \frac{n}{n+1} \qquad (31)$$

Using the recursion relation (24), which holds for $k+2 \le n \Rightarrow k \le n-2$, we can write the expression $\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}$ as a continued fraction.

For $k = n-3$, $n \ge 3$, (24) gives

$$f_{2(n-1)}(n) = \left(\left(n-(n-3+2)\right)(n+1)+2n\right)f_{2(n-2)}(n) - 2(n-3+1)(n+1)f_{2(n-3)}(n) = (3n+1)f_{2(n-2)}(n) - 2(n-2)(n+1)f_{2(n-3)}(n)$$

Thus

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{(3n+1)f_{2(n-2)}(n)-2(n-2)(n+1)f_{2(n-3)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{2(n-2)(n+1)}{3n+1}\frac{f_{2(n-3)}(n)}{f_{2(n-2)}(n)}}$$

That is

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{2(n-2)(n+1)}{3n+1}\frac{f_{2(n-3)}(n)}{f_{2(n-2)}(n)}} \qquad (32)$$

For $k=n-4$, $n\geq 4$, (24) gives

$$f_{2(n-2)}(n)=\left(\left(n-(n-4+2)\right)(n+1)+2n\right)f_{2(n-3)}(n)-2(n-4+1)(n+1)f_{2(n-4)}(n)=(4n+2)f_{2(n-3)}(n)-2(n-3)(n+1)f_{2(n-4)}(n)=$$
$$=2\left((2n+1)f_{2(n-3)}(n)-(n-3)(n+1)f_{2(n-4)}(n)\right)$$

Thus

$$\frac{2(n-2)(n+1)}{3n+1}\frac{f_{2(n-3)}(n)}{f_{2(n-2)}(n)}=\frac{2(n-2)(n+1)}{3n+1}\frac{f_{2(n-3)}(n)}{2\left((2n+1)f_{2(n-3)}(n)-(n-3)(n+1)f_{2(n-4)}(n)\right)}=\frac{(n-2)(n+1)}{(3n+1)(2n+1)}\frac{1}{1-\frac{(n-3)(n+1)}{2n+1}\frac{f_{2(n-4)}(n)}{f_{2(n-3)}(n)}}$$

Substituting into (32), we obtain

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{(n-2)(n+1)}{(3n+1)(2n+1)}\frac{1}{1-\frac{(n-3)(n+1)}{2n+1}\frac{f_{2(n-4)}(n)}{f_{2(n-3)}(n)}}} \quad (33)$$

For $k=n-5$, $n\geq 5$, (24) gives

$$f_{2(n-3)}(n)=\left((n-(n-5+2))(n+1)+2n\right)f_{2(n-4)}(n)-2(n-5+1)(n+1)f_{2(n-5)}(n)=(5n+3)f_{2(n-4)}(n)-2(n-4)(n+1)f_{2(n-5)}(n)$$

Thus

$$\frac{(n-3)(n+1)}{2n+1}\frac{f_{2(n-4)}(n)}{f_{2(n-3)}(n)}=\frac{(n-3)(n+1)}{2n+1}\frac{f_{2(n-4)}(n)}{(5n+3)f_{2(n-4)}(n)-2(n-4)(n+1)f_{2(n-5)}(n)}=\frac{(n-3)(n+1)}{(2n+1)(5n+3)}\frac{1}{1-\frac{2(n-4)(n+1)}{5n+3}\frac{f_{2(n-5)}(n)}{f_{2(n-4)}(n)}}$$

Substituting into (33), we obtain

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{(n-2)(n+1)}{(3n+1)(2n+1)}\frac{1}{1-\frac{(n-3)(n+1)}{(2n+1)(5n+3)}\frac{1}{1-\frac{2(n-4)(n+1)}{5n+3}\frac{f_{2(n-5)}(n)}{f_{2(n-4)}(n)}}}} \quad (34)$$

For $k=n-6$, $n\geq 6$, (24) gives

$$f_{2(n-4)}(n)=\left(\left(n-(n-6+2)\right)(n+1)+2n\right)f_{2(n-5)}(n)-2(n-6+1)(n+1)f_{2(n-6)}(n)=(6n+4)f_{2(n-5)}(n)-2(n-5)(n+1)f_{2(n-6)}(n)=$$
$$=2\left((3n+2)f_{2(n-5)}(n)-(n-5)(n+1)f_{2(n-6)}(n)\right)$$

Thus

$$\frac{2(n-4)(n+1)}{5n+3}\frac{f_{2(n-5)}(n)}{f_{2(n-4)}(n)}=\frac{2(n-4)(n+1)}{5n+3}\frac{f_{2(n-5)}(n)}{2\left((3n+2)f_{2(n-5)}(n)-(n-5)(n+1)f_{2(n-6)}(n)\right)}=\frac{(n-4)(n+1)}{(5n+3)(3n+2)}\frac{1}{1-\frac{(n-5)(n+1)}{3n+2}\frac{f_{2(n-6)}(n)}{f_{2(n-5)}(n)}}$$

Substituting into (34), we obtain

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\cfrac{1}{1-\cfrac{(n-2)(n+1)}{(3n+1)(2n+1)}\cfrac{1}{1-\cfrac{(n-3)(n+1)}{(2n+1)(5n+3)}\cfrac{1}{1-\cfrac{(n-4)(n+1)}{(5n+3)(3n+2)}\cfrac{1}{1-\cfrac{(n-5)(n+1)}{3n+2}\cfrac{f_{2(n-6)}(n)}{f_{2(n-5)}(n)}}}}} \quad (35)$$

Using (35), we'll calculate the coupling λ for $n=2,3,4,5$.

For $n=0$, (35) is not valid, since the denominator of $\frac{(n-1)(n+1)}{n}$ vanishes.

For $n=0$, we can calculate the coupling λ – let us denote it λ_0 – using (30).

Indeed, for $n=0$, $P_0(\tilde{x})=1$, and thus $p_0=1$ and $p_1=0$, and then (30) gives

$\lambda_0 = 0$

For $n = 1$, $\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} = 0$ and then (31) gives

$$\lambda_1 = \frac{1}{1-0} - \frac{1}{2} = 0.5$$

For $n = 2$, (35) reduces to

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} = \frac{(n-1)(n+1)}{n(3n+1)} = \frac{3}{2*7} = \frac{3}{14}$$

Then (31) gives

$$\lambda_2 = \frac{1}{1-\frac{3}{14}} - \frac{2}{3} = \frac{14}{11} - \frac{2}{3} = \frac{42-22}{33} = \frac{20}{33} \simeq 0.6061$$

For $n = 3$, (35) reduces to

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} = \frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{(n-2)(n+1)}{(3n+1)(2n+1)}} = \frac{8}{3*10}\frac{1}{1-\frac{4}{10*7}} = \frac{4}{15}\frac{1}{1-\frac{2}{35}} = \frac{4}{15}\frac{35}{33} = \frac{4}{3}\frac{7}{33} = \frac{28}{99}$$

Then (31) gives

$$\lambda_3 = \frac{1}{1-\frac{28}{99}} - \frac{3}{4} = \frac{99}{71} - \frac{3}{4} \simeq 1.3944 - 0.7500 = 0.6444$$

For $n = 4$, (35) reduces to

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)} = \frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{(n-2)(n+1)}{(3n+1)(2n+1)}\frac{1}{1-\frac{(n-3)(n+1)}{(2n+1)(5n+3)}}} = \frac{15}{4*13}\frac{1}{1-\frac{10}{13*9}\frac{1}{1-\frac{5}{9*23}}} = \frac{15}{4*13}\frac{1}{1-\frac{10}{13*9}\frac{9*23}{202}} =$$

$$= \frac{15}{4*13}\frac{1}{1-\frac{10}{13}\frac{23}{202}} = \frac{15}{4*13}\frac{1}{1-\frac{5}{13}\frac{23}{101}} = \frac{15}{4*13}\frac{13*101}{1313-115} = \frac{15}{4}\frac{101}{1313-115} = \frac{15}{4}\frac{101}{1198} = \frac{1515}{4792}$$

Then (31) gives

$$\lambda_4 = \frac{1}{1-\frac{1515}{4792}} - \frac{4}{5} = \frac{4792}{3277} - \frac{4}{5} \simeq 1.4623 - 0.8000 = 0.6623$$

For $n = 5$, (35) reduces to

$$\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}=\frac{(n-1)(n+1)}{n(3n+1)}\frac{1}{1-\frac{(n-2)(n+1)}{(3n+1)(2n+1)}\frac{1}{1-\frac{(n-3)(n+1)}{(2n+1)(5n+3)}\frac{1}{1-\frac{(n-4)(n+1)}{(5n+3)(3n+2)}}}}=\frac{24}{5*16}\frac{1}{1-\frac{3*6}{16*11}\frac{1}{1-\frac{2*6}{11*28}\frac{1}{1-\frac{6}{28*17}}}}=$$

$$=\frac{3}{5*2}\frac{1}{1-\frac{3*3}{8*11}\frac{1}{1-\frac{3}{11*7}\frac{28*17}{470}}}=\frac{3}{5*2}\frac{1}{1-\frac{3*3}{8*11}\frac{1}{1-\frac{3}{11}\frac{4*17}{470}}}=\frac{3}{5*2}\frac{1}{1-\frac{3*3}{8*11}\frac{1}{1-\frac{3}{11}\frac{2*17}{235}}}=\frac{3}{5*2}\frac{1}{1-\frac{3*3}{8*11}\frac{11*235}{2585-102}}=$$

$$=\frac{3}{5*2}\frac{1}{1-\frac{3*3}{8}\frac{235}{2483}}=\frac{3}{5*2}\frac{8*2483}{19864-2115}=\frac{3*4*2483}{5*17749}=\frac{29796}{88745}$$

Then (31) gives

$$\lambda_5=\frac{1}{1-\frac{29796}{88745}}-\frac{5}{6}=\frac{88745}{58949}-\frac{5}{6}\simeq 1.5055-0.8333\simeq 0.6722$$

Summarizing the previous results, we have

Table I

n	λ
0	0

1	0.5
2	0.6061
3	0.6444
4	0.6623
5	0.6722

The coupling λ starts from zero and increases, as n increases, but the difference of the successive couplings decreases indicating that the sequence of λ's converges, i.e. $\lambda_\infty < \infty$.

Next, using (35), we'll derive the expression of λ_∞.

Taking the limit $n \to \infty$, we have

$$\frac{(n-1)(n+1)}{n(3n+1)} \to \frac{1}{3}$$

$$\frac{(n-2)(n+1)}{(3n+1)(2n+1)} \to \frac{1}{6}$$

$$\frac{(n-3)(n+1)}{(2n+1)(5n+3)} \to \frac{1}{10}$$

$$\frac{(n-4)(n+1)}{(5n+3)(3n+2)} \to \frac{1}{15}$$

and (35) is written as

$$\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)=\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\cfrac{1}{1-\lim\limits_{n\to\infty}\left(\frac{(n-5)(n+1)}{3n+2}\frac{f_{2(n-6)}(n)}{f_{2(n-5)}(n)}\right)}}}}$$

or

$$-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)=-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\cfrac{1}{1-\lim\limits_{n\to\infty}\left(\frac{(n-5)(n+1)}{3n+2}\frac{f_{2(n-6)}(n)}{f_{2(n-5)}(n)}\right)}}}}$$

We see that, in the limit $n\to\infty$, the expression $-\lim\limits_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)$ is written as an infinite generalized continued fraction with integer part equal to zero, all partial denominators equal to one, and partial numerators being given by the sequence

$-\frac{1}{3}, -\frac{1}{6}, -\frac{1}{10}, -\frac{1}{15}, \ldots, -\frac{2}{(s+1)(s+2)}$, with $s = 1, 2, \ldots$,

and we see that the absolute values of the reciprocals of the partial numerators of $-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)$ are all triangular numbers except 1*.

* Indeed, the absolute values of the reciprocals of the partial numerators of $-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)$ are

$a_2 = 3 + 2 + 1$

$a_3 = a_2 + 3 + 1$

$a_4 = a_3 + 4 + 1$

…

$a_s = a_{s-1} + s + 1$

Adding the previous equations, the terms from a_2 up to a_{s-1} are canceled out, and we obtain

$$a_s = 3 + (\underbrace{2+3+\ldots+s}_{s-1\text{ terms}}) + (\underbrace{1+1+\ldots+1}_{s-1\text{ terms}}) = 3 + \frac{(2+s)(s-1)}{2} + s - 1 = 3 + (s-1)\left(\frac{s+2}{2}+1\right) = 3 + (s-1)\frac{s+4}{2} = 3 + \frac{(s-1)(s+4)}{2} =$$
$$= \frac{s^2+3s-4+6}{2} = \frac{s^2+3s+2}{2} = \frac{(s+1)(s+2)}{2}$$

That is

$$a_s = \frac{(s+1)(s+2)}{2}, \text{ with } s = 1, 2, \ldots$$

Setting $s' = s+1$, the previous sequence is written as

$$a_{s'} = \frac{s'(s'+1)}{2}, \text{ with } s' = 2, 3, \ldots$$

These are all triangular numbers except 1.

The first terms of the sequence a_s are then

3,6,10,15,21,28,36,45,55,66,78,91,105,…

Besides, in the limit $n \to \infty$, (31) gives

$$\lambda_\infty = \frac{1}{1 - \lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)} - 1 \qquad (36)$$

since $\lim_{n\to\infty}\frac{n}{n+1} = 1$.

Substituting into (36) the expression of $-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)$, we obtain

$$\lambda_{\infty} = \cfrac{1}{1-\cfrac{1}{3}\cfrac{1}{1-\cfrac{1}{6}\cfrac{1}{1-\cfrac{1}{10}\cfrac{1}{1-\cfrac{1}{15}\cfrac{1}{1-\cfrac{1}{21}\cfrac{1}{1-\cfrac{1}{28}\cfrac{1}{1-\cfrac{1}{36}\dots}}}}}}} - 1 \tag{37}$$

Using Gauss' notation, we write $-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right)$ as

$$-\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{n}\frac{f_{2(n-2)}(n)}{f_{2(n-1)}(n)}\right) = \mathop{K}_{s=1}^{\infty}\frac{-\frac{2}{(s+1)(s+2)}}{1},$$

and substituting into (36), we write λ_{∞} as

$$\lambda_{\infty} = \frac{1}{1+\mathop{K}_{s=1}^{\infty}\frac{-\frac{2}{(s+1)(s+2)}}{1}} - 1 \tag{38}$$

We observe that for $s=2,3,\ldots$, $\left|-\frac{2}{(s+1)(s+2)}\right|<\frac{1}{4}$, and thus, from Worpitzky's theorem [13], the continued fraction $\underset{s=2}{\overset{\infty}{K}}\frac{-\frac{2}{(s+1)(s+2)}}{1}$ converges, and then $\underset{s=1}{\overset{\infty}{K}}\frac{-\frac{2}{(s+1)(s+2)}}{1}$ also converges, and thus $\lambda_\infty<\infty$, i.e. the delta potential coupling converges too.

Using the expression (37) or (38), we can make successive approximations to λ_∞.

Thus, we have

$s=0$,

$$\lambda_\infty \simeq \frac{1}{1}-1=0$$

$s=1$,

$$\lambda_\infty \simeq \frac{1}{1-\frac{1}{3}}-1=\frac{3}{2}-1=0.5$$

Observe that the two previous approximations are the exact values of the coupling for $n=0,1$.

$s=2$,

$$\lambda_\infty + 1 \simeq \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}}} = \frac{1}{1-\frac{1}{3}\frac{6}{5}} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$$

Thus

$$\lambda_\infty \simeq \frac{5}{3} - 1 = \frac{2}{3} \simeq 0.66667$$

$s = 3$,

$$\lambda_\infty + 1 \simeq \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{10}{9}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{5}{27}}} = \frac{1}{1-\frac{1}{3}\frac{27}{22}} = \frac{1}{1-\frac{9}{22}} = \frac{22}{13}$$

Thus

$$\lambda_\infty \simeq \frac{22}{13} - 1 = \frac{9}{13} \simeq 0.69231$$

$s = 4$,

$$\lambda_\infty+1\simeq\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{15}{14}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{2}\frac{3}{14}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{3}{28}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{28}{25}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{14}{3*25}}}=$$

$$=\frac{1}{1-\frac{1}{3}\frac{3*25}{61}}=\frac{1}{1-\frac{25}{61}}=\frac{61}{36}$$

Thus

$$\lambda_\infty\simeq\frac{61}{36}-1=\frac{25}{36}\simeq 0.69444$$

$s=5$,

$$\lambda_\infty+1\simeq\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{21}}}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{21}{20}}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{5}\frac{7}{20}}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{100}{93}}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{10}{93}}}}=$$

$$=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{93}{83}}}=\frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{2}\frac{31}{83}}}=\frac{1}{1-\frac{1}{3}\frac{166}{135}}=\frac{405}{239}$$

Thus

$$\lambda_\infty \simeq \frac{405}{239} - 1 = \frac{166}{239} \simeq 0.69456$$

$s = 6$,

$$\lambda_\infty + 1 \simeq \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\cfrac{1}{1-\frac{1}{21}\cfrac{1}{1-\frac{1}{28}}}}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\cfrac{1}{1-\frac{1}{21}\frac{28}{27}}}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\cfrac{1}{1-\frac{1}{3}\frac{4}{27}}}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{1}{15}\frac{3*27}{77}}}}} =$$

$$= \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\cfrac{1}{1-\frac{27}{5*77}}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{10}\frac{5*77}{358}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\cfrac{1}{1-\frac{1}{2}\frac{77}{358}}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{1}{6}\frac{2*358}{639}}} = \cfrac{1}{1-\frac{1}{3}\cfrac{1}{1-\frac{358}{3*639}}} = \cfrac{1}{1-\frac{1}{3}\frac{3*639}{1559}} = \cfrac{1}{1-\frac{639}{1559}} = \frac{1559}{920}$$

Thus

$$\lambda_\infty \simeq \frac{1559}{920} - 1 = \frac{639}{920} \simeq 0.69456$$

$s = 7$,

$$\lambda_\infty + 1 \simeq \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{21}\frac{1}{1-\frac{1}{28}\frac{1}{1-\frac{1}{36}}}}}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{21}\frac{1}{1-\frac{1}{28}\frac{36}{35}}}}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{21}\frac{1}{1-\frac{1}{7}\frac{9}{35}}}}}}} =$$

$$= \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{21}\frac{7*35}{236}}}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{1}{1-\frac{1}{3}\frac{35}{236}}}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{15}\frac{3*236}{673}}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{1}{1-\frac{1}{5}\frac{236}{673}}}}} =$$

$$= \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{10}\frac{5*673}{3129}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{1}{1-\frac{1}{2}\frac{673}{3129}}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1}{6}\frac{2*3129}{5585}}} = \frac{1}{1-\frac{1}{3}\frac{1}{1-\frac{1043}{5585}}} = \frac{1}{1-\frac{1}{3}\frac{5585}{4542}} = \frac{13626}{8041}$$

Thus

$$\lambda_\infty \simeq \frac{13626}{8041} - 1 = \frac{5585}{8041} \simeq 0.69456$$

Summarizing the previous approximations, we have

Table II

s	λ_∞
0	0
1	0.5
2	0.66667
3	0.69231
4	0.69444
5	0.69456
6	0.69456
7	0.69456

6. The location of the known eigenstates

From the expression (35), and the Table I, we see that the coupling λ increases as n increases.

Thus, as n increases, the delta potential becomes more attractive, and then, since the other term of the potential (1) is n-independent, the negative potential becomes more attractive too, in a sense the infinite attractive well becomes deeper.

Also, from (15), we see that the energies increase, as n increases.

That is, as n increases, we find eigenstates of deeper wells with higher energies.

These higher energies correspond to higher excited states, otherwise we would have (at least) two attractive wells with the energy of a lower state of the deeper well being greater than the energy of a higher state of the shallower well, which is physically unacceptable.

The higher excited states correspond to closed-form eigenfunctions (17) with more real zeros [10, 14].

Thus, as n increases, the number of real zeros of the closed-form eigenfunction (17) increases.

The zeros of the eigenfunction (17) are the zeros of the polynomial $P_n(x)$, which is of even parity and thus it has an even number of zeros.

If n increases by one, the degree of $P_n(x)$ also increases by one, and since it is of even parity, its real zeros can increase only by two.

Thus, since an increase in n increases the number of real zeros of $P_n(x)$, an increase in n by one increases the number of real zeros of $P_n(x)$ by two, and the number of real zeros of the eigenfunction also by two.

For $n=0$, from (29) we obtain $P_0(x)=1$ and the eigenfunction $\psi_0(x)$ has no real zeros, and thus it is the ground-state wave function.

Then, $\psi_1(x)$ has two real zeros, and thus it is the second-excited-state wave function, $\psi_2(x)$ has four real zeros, and thus it is the fourth-excited-state wave function, and so on.

Therefore, for each value of $n \in \mathbb{N}$, (15) and (17) give, respectively, the energy and the wave function of the $2n$ th-excited state of the respective potential (1) [10, 14].

7. Examples

We'll quasi-exactly solve the potential (1) for $n=0,1,2,3$, and 4.

n=0

For $n=0$, from (29) we obtain $P_0\left(x\right)=1$, and then, from (17),

$$\psi_0\left(x\right)=A_0\left(\frac{|x|}{x_0}+1\right)\exp\left(-\frac{|x|}{x_0}\right),$$

which has no real zeros and it describes the ground state of the potential

$$V\left(x\right)=-\frac{\varepsilon_0}{\frac{|x|}{x_0}+1},$$

since $\lambda_0=0$ (see Table I).

The ground-state energy of the previous potential is given by (15) for $n=0$, i.e.

$$E_0=-\frac{\hbar^2}{2mx_0^{\ 2}}$$

Substituting $x_0=\frac{\hbar}{p_0}$ into E_0 yields $E_0=-\frac{p_0^{\ 2}}{2m}=-\frac{\varepsilon_0}{2}$, i.e. the ground-state energy lies in the middle of the well $V\left(x\right)=-\frac{\varepsilon_0}{\frac{|x|}{x_0}+1}$.

n=1

The polynomial $P_1\left(\tilde{x}\right)$ is of first degree and monic, and from (28), we saw that $p_0=-p_1$. Then, since $p_1=1$, $p_0=-1$, and thus, from (29),

$$P_1\left(x\right)=\frac{|x|}{x_0}-1$$

Then, from (17) we obtain the eigenfunction

$$\psi_1\left(x\right)=A_1\left(\left(\frac{x}{x_0}\right)^2-1\right)\exp\left(-\frac{|x|}{2x_0}\right),$$

which has two real zeros, at $\pm x_0$, and it describes the second-excited state of the potential

$$V(x) = -\frac{\varepsilon_0}{\frac{|x|}{x_0}+1} - \frac{0.5\hbar^2\delta(x)}{mx_0},$$

since $\lambda_1 = 0.5$ (see Table I).

The second-excited-state energy of the previous potential is given by (15) for $n=1$, i.e.

$$E_1 = -\frac{\hbar^2}{8m{x_0}^2}$$

n=2

For $n=2$, (23) is written as

$$p_{2-k} = \frac{(-1)^k\, 2!}{2^k\, k!(2-k)!} f_{2k}(2)$$

with $k = 0,1,2$

Thus, we have

$$p_2 = 1$$

$$p_{2-1} = \frac{(-1)^1\, 2!}{2^1 1!(2-1)!} f_2(2) = -f_2(2)$$

with $f_2(n) = n^2 + 2n - 1$, and thus $f_2(2) = 7$, and then

$$p_{2-1} = -7$$

And

$$p_{2-2} = \frac{(-1)^2\, 2!}{2^2 2!(2-2)!} f_4(2) \overset{0!=1}{=} \frac{1}{4} f_4(2)$$

For $n=2$ and $k=0$, the recursion relation (24) gives

$$f_4(2) = 4f_2(2) - 6f_0(2) = 4*7-6 = 22,$$

since $f_0(n)=1$, and $f_2(2)=7$ from the previous calculation.

Thus

$$p_{2-2}=\frac{22}{4}=\frac{11}{2}$$

Thus, from (29),

$$P_2(x)=\left(\frac{x}{x_0}\right)^2-7\frac{|x|}{x_0}+\frac{11}{2}$$

Then, from (17) we obtain the eigenfunction

$$\psi_2(x)=A_2\left(\left(\frac{x}{x_0}\right)^2-\frac{7|x|}{x_0}+\frac{11}{2}\right)\left(\frac{|x|}{x_0}+1\right)\exp\left(-\frac{|x|}{3x_0}\right)$$

The polynomial $P_2(x)$ has four real zeros.

Indeed, the discriminant of the trinomial $\tilde{x}^2-7\tilde{x}+\frac{11}{2}$ is positive, and thus the trinomial has two real zeros, the sum and product of which are both positive, and thus the two zeros are positive.

Then, $P_2(x)$ has two positive zeros and the opposite ones, i.e. four in total.

Therefore, the eigenfunction $\psi_2(x)$ has four real zeros, and thus it describes the fourth-excited state of the potential

$$V(x)\simeq-\frac{\varepsilon_0}{\frac{|x|}{x_0}+1}-\frac{0.6061\hbar^2\delta(x)}{mx_0},$$

since $\lambda_2\simeq 0.6061$ (see Table I).

The fourth-excited-state energy of the previous potential is given by (15) for $n=2$, i.e.

$$E_2=-\frac{\hbar^2}{18m{x_0}^2}$$

n=3

For $n=3$, (23) is written as

$$p_{3-k} = \frac{(-1)^k 3!}{2^k k!(3-k)!} f_{2k}(3)$$

with $k = 0,1,2,3$

Thus, we have

$$p_3 = 1$$

$$p_{3-1} = \frac{(-1)^1 3!}{2^1 1!(3-1)!} f_2(3) = -\frac{3!}{2*2!} f_2(3) = -\frac{3}{2} f_2(3)$$

with $f_2(n) = n^2 + 2n - 1$, and thus $f_2(3) = 14$, and then

$$p_{3-1} = -21$$

Also

$$p_{3-2} = \frac{(-1)^2 3!}{2^2 2!(3-2)!} f_4(3) = \frac{3!}{2^2 2!} f_4(3) = \frac{3}{4} f_4(3)$$

For $n = 3$ and $k = 0$, the recursion relation (24) gives

$$f_4(3) = ((3+1)+6) f_2(3) - 2*4 f_0(3) = 10 f_2(3) - 8 f_0(3) = 10*14 - 8 = 132,$$

since $f_0(n) = 1$, and $f_2(3) = 14$ from the previous calculation.

Thus

$$p_{3-2} = \frac{3}{4} 132 = 3*33 = 99$$

And

$$p_{3-3} = \frac{(-1)^3 3!}{2^3 3!(3-3)!} f_6(3) \overset{0!=1}{\overbrace{=}} -\frac{1}{8} f_6(3)$$

For $n = 3$ and $k = 1$, the recursion relation (24) gives

$$f_6(3) = ((3-(1+2))(3+1)+6) f_4(3) - 2(1+1)(3+1) f_2(3) = 6 f_4(3) - 16 f_2(3) =$$
$$= 6*132 - 16*14 = 792 - 224 = 568,$$

where we used that $f_2(3) = 14$ and $f_4(3) = 132$, from the previous calculations.

Thus

$$p_{3-3} = -\frac{1}{8}568 = -71$$

Thus, from (29),

$$P_3(x) = \left(\frac{|x|}{x_0}\right)^3 - 21\left(\frac{x}{x_0}\right)^2 + 99\frac{|x|}{x_0} - 71$$

Then, from (17) we obtain the eigenfunction

$$\psi_3(x) = A_3\left(\left(\frac{|x|}{x_0}\right)^3 - 21\left(\frac{x}{x_0}\right)^2 + 99\frac{|x|}{x_0} - 71\right)\left(\frac{|x|}{x_0}+1\right)\exp\left(-\frac{|x|}{4x_0}\right)$$

The polynomial $P_3(x)$ has six real zeros.

Indeed, for the polynomial $\tilde{x}^3 - 21\tilde{x}^2 + 99\tilde{x} - 71$, we have

$$P_3(0) = -71 < 0$$

$$P_3(1) = 1 - 21 + 99 - 71 = 8 > 0$$

$$P_3(6) = 216 - 756 + 594 - 71 = -17 < 0$$

and

$$P_3(\infty) = \infty > 0$$

Thus, the polynomial $P_3(\tilde{x})$ has at least one zero in each of the intervals $(0,1)$, $(1,6)$, and $(6,\infty)$, and since it is of third degree, it has exactly one zero in each of these intervals.

Then, the polynomial $P_3(x)$ has three positive zeros and the opposite ones, i.e. six in total.

Therefore, the eigenfunction $\psi_3(x)$ has six real zeros, and thus it describes the sixth-excited state of the potential

$$V(x) \simeq -\frac{\varepsilon_0}{\frac{|x|}{x_0}+1} - \frac{0.6444\hbar^2\delta(x)}{mx_0},$$

since $\lambda_3 \simeq 0.6444$ (see Table I).

The sixth-excited-state energy of the previous potential is given by (15) for $n=3$, i.e.

$$E_3 = -\frac{\hbar^2}{32mx_0{}^2}$$

n=4

For $n=4$, (23) is written as

$$p_{4-k} = \frac{(-1)^k 4!}{2^k k!(4-k)!} f_{2k}(4) \qquad (39)$$

with $k=0,1,2,3,4$

We'll need $f_2(4), f_4(4), f_6(4)$, and $f_8(4)$.

We remind that $f_0(n)=1$, and $f_2(n)=n^2+2n-1$, and thus

$$f_2(4)=16+8-1=23$$

For $n=4$, the recursion relation (24) is written as

$$f_{2(k+2)}(4) = (5(2-k)+8) f_{2(k+1)}(4) - 10(k+1) f_{2k}(4) \qquad (40)$$

For $k=0$, (40) gives

$$f_4(4) = 18 f_2(4) - 10 f_0(4) = 18*23-10 = 404$$

For $k=1$, (40) gives

$$f_6(4) = 13 f_4(4) - 20 f_2(4) = 13*404-20*23 = 4792$$

For $k=2$, (40) gives

$$f_8(4) = 8 f_6(4) - 30 f_4(4) = 8*4792-30*404 = 38336-12120 = 26216$$

We are now ready to calculate the polynomial coefficients.

For $k=0$, (39) gives

$$p_4 = 1$$

For $k=1$, (39) gives

$$p_{4-1} = \frac{(-1)^1 4!}{2^1 1!(4-1)!} f_2(4) = -2*23 = -46$$

For $k=2$, (39) gives

$$p_{4-2} = \frac{(-1)^2 4!}{2^2 2!(4-2)!} f_4(4) = \frac{4!}{2^2 2!2!} f_4(4) = \frac{12}{8} f_4(4) = \frac{3}{2} 404 = 606$$

For $k=3$, (39) gives

$$p_{4-3} = \frac{(-1)^3 4!}{2^3 3!(4-3)!} f_6(4) = -\frac{4}{2^3} f_6(4) = -\frac{1}{2} 4792 = -2396$$

For $k=4$, (39) gives

$$p_{4-4} = \frac{(-1)^4 4!}{2^4 4!(4-4)!} f_8(4) = \frac{1}{16} 26216 = 1638.5$$

Thus, from (29),

$$P_4(x) = \left(\frac{x}{x_0}\right)^4 - 46\left(\frac{|x|}{x_0}\right)^3 + 606\left(\frac{x}{x_0}\right)^2 - 2396\frac{|x|}{x_0} + 1638.5$$

Then, from (17) we obtain the eigenfunction

$$\psi_4(x) = A_4\left(\left(\frac{x}{x_0}\right)^4 - 46\left(\frac{|x|}{x_0}\right)^3 + 606\left(\frac{x}{x_0}\right)^2 - 2396\frac{|x|}{x_0} + 1638.5\right)\left(\frac{|x|}{x_0}+1\right)\exp\left(-\frac{|x|}{5x_0}\right)$$

The polynomial $\tilde{x}^4 - 46\tilde{x}^3 + 606\tilde{x}^2 - 2396\tilde{x} + 1638.5$ has four positive zeros*, and thus the polynomial $P_4(x)$ has eight real zeros, which are also the real zeros of $\psi_4(x)$.

* The zeros of the polynomial $\tilde{x}^4 - 46\tilde{x}^3 + 606\tilde{x}^2 - 2396\tilde{x} + 1638.5$ are approximately at 0.85823, 5.42909, 13.32795, and at 26.38474, https://tinyurl.com/y7x8xgog.

Thus, the previous eigenfunction describes the eighth-excited state of the potential

$$V(x) \simeq -\frac{\varepsilon_0}{\frac{|x|}{x_0}+1} - \frac{0.6623\hbar^2\delta(x)}{mx_0},$$

since $\lambda_4 \simeq 0.6623$ (see Table I).

The eighth-excited-state energy of the previous potential is given by (15) for $n=4$, i.e.

$$E_4 = -\frac{\hbar^2}{50m{x_0}^2}$$

8. References

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