A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Apery's $\zeta(3)$ proof is the only proof that a specific odd argument for $\zeta(n)$ is irrational. Even arguments are a natural consequence of Bernoulli formula [2] for $\zeta(2n)$.

Apery also showed $\zeta(2)$ is irrational, and Beuker, based on the work of Apery, simplified both proofs [3]. These proofs for $\zeta(2)$ and $\zeta(3)$ require the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. The puzzle is, then, if you can use Apery's idea for an easier, as it turns out case, that of $\zeta(2)$, why can't you generalize Apery's ideas to the general $\zeta(n)$ cases? Both the evens and odds?

Proving the general case using Apery's [1] central idea seems hopelessly elusive. It is not for a lack of trying. Apery's and other ideas can be seen in the very difficult results of Rivoal and Zudilin [7, 10]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational, seem less than encouraging.

In this paper we explore a wholly different direction. Sondow gives a very easy *geometric* proof of the irrationality of e [9]. This proof uses what could be called an *eliminate as you define* idea. You build a number by a geometric process that eliminates other numbers (other rational numbers) from being possible convergence points. In the case of e there is a clear and easy connection between terms. Each term is a proportion of the previous and moves and squeezes partials from the left and right in a neat, orderly fashion. Trying the same trick with $\zeta(2)$, the right boundary doesn't necessarily contract in from a single boundary. Here's the key: it doesn't contract from a *single* boundary, but it does from a set of boundaries. For $\zeta(2)$, for example, $1/4 + 1/9$ is neither at $.x$ base 4 or $.y$ base 9. Continuing $1/4 + 1/9 + 1/16 + 1/5^2 + 1/6^2 + 1/7^2 = 282/551$ which is between 1/4s and 1/9s, but not between, any more, 1/4 and 1/2 and 3/9 and 4/9; it has blown passed these multiples of 1/4 and 1/9, but is still between some such multiples – implying not equal to any such multiples! In fact: in any number base, as the base is used to express a series, eventually decimal digits become fixed. For $\zeta(2)-1$ in base 10, we can say $.6 < \zeta(2)-1 < .7$. In another base, larger than 10, similar boundaries, fixed, will have to exist for sufficiently large upper limits of partials and will have to move in from .6 and .7, base 10. This is the idea we pursue in this paper.

This is an open number theory problem, so, for those that like challenges, I'll give here a sequence of problems to solve. I.e. see if you can do it before you read about how it was done. We will need two definitions:

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

Show that every rational number in $(0, 1)$ can be written as a single decimal using the denominators of the terms of any z_n. Next show the partial sums, s_k^n, can't be expressed as a single decimal in any of the terms of s_k^n. This implies that the precision of s_k^n increases. This is unlike something simple like $1/4 + 1/4 = 1/2$ – the precision or fineness of terms is 1/4 and that of the sum is 1/2, less precision – wider decimal intervals: base 2 versus base 4. Note that if a series converges to a rational number, its partials will get close to a number of less precision, in this sense. For example, $.\overline{1}$ base 4 converges to .1 base 3 – less precision than base 4. So, having shown the denominators of the terms cover all rational possible convergence points and that the partials escape their terms, show that the partials can't converge to a number with finite precision and hence must converge to an irrational number.

2 Terms cover rationals

We start with something relatively easy.

Definition 1. *A decimal set, base j^n, is defined by*

$$D_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \text{ base } j^n.$$

That is D_{j^n} consists of all single decimals greater than 0 and less than 1 in base j^n.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Lemma 1.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Note: Sondow's e is irrational proof gives this same idea. To wit, given a rational $0 < p/q < 1$

$$\frac{p}{q} = \frac{p(q-1)!}{q!}.$$

That is the denominators of the terms of e taken as number bases express all rational numbers in $(0,1)$.

3 Partials escape terms

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

We will use z_2 to motivate the development. The partials of z_2, as they include all even k^2 in their denominators, will have a reduced form that has a greater power of two in the partial's denominator. This result is given in Lemma 2; it is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem. Next, if we can show that there is at least one prime that does not recur in the k^2 denominators, then that prime will occur in the partial sum's reduced fraction. This result is given in Lemma 3. Such a prime does exist: Lemma 4, Bertrand's postulate.

The idea is simple. Consider $1/4 + 1/9 + 1/16 + 1/25$. There will be a power of 2 and of a relatively large prime in the denominator of the reduced sum. Indeed, the sum is $1669/3600$ and the denominator of this reduced form has the prime factorization of $2^4 3^2 5^2$; it has a relatively large power of 2 and the prime 5. The prime is between 3 and 6 as Bertrand's postulate stipulates. As $2^2 5^2$ exceeds the largest denominator in this partial sum, the partial sum can't be expressed as a single decimal in any of the denominators of the terms of the partial. Simply put: 2 times something past the middle exceeds the last.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k,p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate [4]. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. □

One can get a geometric like idea similar to Sondow's. The partial s_k^n resides between decimal points in all the decimal sets in Ξ_k^n. Unlike the case of e, the intervals don't nest neatly. In fact, they migrate and overlap. Consider that z_2 has partials in the interval $[1/4, 2/4]$, but as $z_2 = .6\cdots > 2/4$, partials don't stay in this interval. But they do stay in some interval of the form

$[.(x-1),.x]$ of D_4. Although D_4 and D_{16} overlap, in this sense, s_k^2 will not be at any endpoint of D_{16}.

What happens when decimals become fixed? In every base they will become fixed. This means $z_n \in (.(x-1),.x)$ where single decimal digits are indicated. These intervals narrow as the precision of the decimal base increases. Eventually they all nest and like Sondow's e proof z_n gets trapped between all possible rational convergence points. The details are coming.

4 A Suggestive Table

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article. See [6].

Lemma 1 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms. As all rational numbers between 0 and 1 are in Ξ_k^n for some k sufficiently large this says partials are being, so to speak, chased out of the Ξ_k^n park – possible rational convergence points. Where could they go but to the irrational zoo, sorry!

5 A Simple Proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Theorem 2. *z_n is irrational.*

Proof. Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. So, taking limits,

$$\lim_{k\to\infty} s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n,$$

and, using Lemma 3 and $\lim_{k\to\infty} s_k^n = z_n$, this gives

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1).$$

That is z_n is irrational. □

Some mathematicians, including the author, find this proof unsatisfactory, but it is simple. I suggest that mathematicians have been trained to be unduly suspicious of proofs like the above. Consider that the above is just the same as

$$\lim_{n\to\infty} (-1/n, 1/n) = \emptyset,$$

where $(-1/n, 1/n)$ designates an open interval. I'm with you in wanting to viscerally feel a squeeze at work.

Another suggestive table

We can make a table, Table 2, that orients us to the various issues involved with the proof just given. As with Table 1, we focus on the z_2 case. This table

	D_4	D_9	D_{16}	$\dots$	D_{k^2}	$\dots$
s_3	$\epsilon_{4,3}$	$\epsilon_{9,3}$	$\epsilon_{16,3}$	$\dots$	$\epsilon_{k^2,3}$	0
s_4	$\epsilon_{4,4}$					$\dots 0$
$\vdots$						$\dots\dots 0$
s_k						
$\vdots$						

Table 2: The epsilon values give the minimum distance of partials to elements of decimal sets.

gives the minimum distance between partial sums of z_2, s_ks, and decimal sets. So, for example, $\epsilon_{4,4}$ gives

$$\epsilon_{4,4} = \min |s_4 - x|,$$

where $x \in D_4$. As $s_4 = \sum_{j=2}^{4} 1/j^2 = 61/144$ and the minimum of

$$\{|61/144 - 1/4|, |61/144 - 2/4|, |61/144 - 3/4|\}$$

is $|61/144 - 2/4| = 11/144$, we have $\epsilon_{4,4} = 11/144$.

We can witness certain properties of this situation. As each partial is a fraction and the fractions cover, per Lemma 3, all rationals, there will be a first zero in each row. We agree to exclude repetitions of fractions in decimal sets, so there is one zero. Also this 0, per Corollary 1, occurs to the right of the last decimal set used in forming the partial. This follows as the decimal sets contain terms of the partials and partials escape their terms. Within a single row, before the zero, epsilon values may get large and small, but there will always be an epsilon value below and to the right that will be smaller. This follows as the precision given by decimal sets grows with the base used. This follows from observing that in any base, powers of the base are also bases and have more precision; any number can be written as an infinite decimal. We can then construct a descending sequence of epsilon values proceeding from the upper left to the lower right; in fact s_k with D_{k^2} will form such a descending sequence – with different primes in k showing the convergence point is not getting close to any rational. The convergence point is not in any decimal set on the upper row eventually, so the convergence point must be irrational.

It is instructive to make tables for other series. For example, a table for $.\overline{1}$ base 4 converges to 1/3. Placing D_3 in the top row, the decreasing gradient of epsilon values goes to the upper left – convergence is to a rational. In this example the terms don't cover the rationals. The telescoping series

$$\sum_{j=2}^{\infty} \frac{1}{j} - \frac{1}{j+1}$$

has terms that cover the rationals, but the partials don't escape their terms consistently; 0 values move around and don't progress consistently to the right. Consider

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} = \frac{1}{2} - \frac{1}{n+1}, \text{ if } n \text{ is odd, say } n = 2m+1$$

so

$$\frac{1}{2} - \frac{1}{n+1} = \frac{m}{2(m+1)}$$

and $2(m+1) < n(n+1)$, the denominator of the last term. Also the terms of the telescoping series have always even denominators – decimal sets that contain the convergence point 1/2. The partials are not escaping their terms.

The most telling feature of both of these examples (potentially counter examples) is that greater precision numbers are getting close to numbers of less precision. Take precision of a decimal base to be the distance $1/b$, where b is the base. Then, in order for a number to be rational the ever more precise numbers of the partials has to be getting close to a unique, number of less precision. In the above examples, powers of 4 bases get close to .1 base 3 or 1/3, a number of finite precision. The telescoping series also gets close to 1/2, .1 base 2, although the partials waver around in precision.

One more time. Consider any rational number in all number bases. It will have decimal representations of the point 9 repeating variety, pure repeating, and mixed – when such a number is forced into infinite decimals. But in all cases partials get ever more precise, but approach a number, the rational, of less precision. We can tighten up these observations.

6 Towards greater precision

We drop the n subscript used previously with z_n and slightly modify such use of subscripts and superscripts in this section. The context should make meanings clear. In that regard, we use the bases k as a fill in for base k^n, to further simplify notation.

Definition 3. *Let $D_k^{\epsilon_k}$ be the set of all D_k decimal sets having an element within ϵ_k of s_k.*

Lemma 5. *Let z be the convergence point of the series with partials s_k. Then z is irrational if there exists a monotonically decreasing sequence ϵ_k such that*

$$\lim_{k\to\infty} \epsilon_k = 0,$$

and

$$\bigcap_{k=2}^{\infty} D_k^{\epsilon_k} = \emptyset. \tag{3}$$

Proof. We use proof by contraposition: $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$. Suppose z is rational then $z \in D_k^*$, a specific decimal set. Define

$$\epsilon_m = z - s_m$$

and set

$$\epsilon_k = 2\epsilon_m.$$

Then

$$D_k^* \subset \bigcap_{k=2}^{\infty} D_k^{\epsilon_k},$$

so the intersection is not empty. □

Definition 4. *The precision of a decimal base, b, is $1/b$.*

Lemma 6. *Given any ϵ there exists a decimal base b of greater precision than ϵ; that is*

$$\frac{1}{b} < \epsilon.$$

Proof. This is the Archimedean property of the reals [8]. □

Theorem 3. *z_n is irrational.*

Proof. We need to define a sequence ϵ_k. Let

$$\epsilon_k^* = \min\{|x - s_k| : x \in \Xi_2^k\}.$$

We know by Corollary 1 that $\epsilon_k^* > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon_3^*$. This excludes the decimal sets of Ξ_2^3 at this our first iteration. Assume we can generally do this for the kth iteration. For the $k+1$st iteration, using Lemma 6, there exists a base in Ξ_2^{k+r}, for some r such that $\epsilon_{k+r}^* < \epsilon_k/2$. Set $\epsilon_{k+1} = \epsilon_{k+r}^*$. The procedure gives ϵ values that exclude ever more decimal sets from $D_k^{\epsilon_k}$. Regroup the series. By Lemma 1, the exclusions are exhaustive, so

$$\bigcap_{k=2}^{\infty} D_k^{\epsilon_k} = \emptyset,$$

as needed. □

7 Conclusion

The alternate proof definitely has a squeeze action to it and is more satisfying. It seems like Sondow's e proof with epsilon reasoning added.

The proof distills to the observation that a series is given as decimals in any base with partials of ever greater precision. Those partials can approach a rational number of finite precision or an irrational number of infinite precision. Show that the partials of z_n approach numbers of ever greater precision.

This in turn, I think, is the fundamental reaction to adding a convergent form of all fractions, one over all natural numbers to a power. The complexity of the partials and the convergence point must reflect the complexity of the terms. When I first saw this series I thought that they all – z_n in the plural sense – must converge to irrational numbers. Sure enough half of them do via Bernoulli's formula for even ns. How could odds possible not so converge when they share so much with the evens. There must be a way, I thought, to strip all the extraneous aspects of these series and find a common thread. The common thread must be that partials are escaping terms in the sense of having ever larger denominators. But all series have partials with denominators that get larger and larger. That is true, but ones converging to a rational have partials that get close to a fixed rational number with one relatively small denominator. If the partials are moving away from all such rational numbers then by exhaustion they are moving toward an irrational.

One other behind the scenes confession or reflection: looking at Beuker's (descendent of Apery's) and even Bernoulli's work (easier) on the subject, I thought their analysis was way too difficult for something so simple. Then I attempted to read Rivoal and Zudilin on the subject and became convinced that this problem could not possible be this hard: a totally new angle was needed.

It struck me that there must be a very simple proof of this, something like a law of nature idea – rationals added give an irrational. It struck me that the proof would be something like the evolution of positional notation for real numbers – so easy, yet so profoundly initially mysterious (even awkward and annoying), until it becomes totally natural and obviously the only way to do it. Could it be at some future date, the irrationality of a series becomes trivial to adduce, that of the square root of 2 harder?

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