A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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May 12, 2019

Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [2]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (early chapters of Apostol and Hardy, [1, 3]) and treats all cases at once.

We use the following notation: for positive integers $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. Lemma 1 is similar to Apostol's chapter 1, problem 30. See [4] for a solution to this problem.

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [3]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

In light of this result we give the following definitions and corollary.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \ \textit{base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. $\square$

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article.

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

Lemma 3.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Loosely speaking, Lemma 3 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms.

4 Set theoretical proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Theorem 2. *z_n is irrational.*

Proof. Idea: Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1). \tag{3}$$

That is z_n is irrational.

Details: $\mathbb{R}(0,1) \setminus \Xi_k^n$ consists of a union of open intervals with rational endpoints given by elements of Ξ_k^n. So for example, considering z_2, we have

$$I_1 = \mathbb{R}(0,1) \setminus D_4. \tag{4}$$

This gives gives $I_1 = (0,1/4) \cup (1/4,2/4) \cup (2/4,3/4) \cup (3/4,1)$. This is $(0,1)$ with rational points of the form $x/4$, $x = 1$, 2, and 3 removed. Now let

$$I_2 = \mathbb{R}(0,1) \setminus D_4 \cup D_5.$$

When the fractions are sorted in ascending order they are

$$\frac{1}{9}, \frac{2}{9}, \frac{1}{4}, \frac{3}{9}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{6}{9}, \frac{3}{4}, \frac{7}{9}, \frac{8}{9},$$

so

$$I_2 = (0,1/9) \cup (1/9,2/9) \cup (2/9,1/4) \cup (1/4,3/9) \cup (3/9,4/9) \text{ and so on.}$$

Clearly the diameter of each of these open intervals is going to 0 and all rational numbers in $(0,1)$ occur as endpoints in these intervals.

As $s_k \in \mathbb{R}(0,1) \setminus \Xi_k^n$, this implies that $\frac{p}{q} < s_k < \frac{r}{s}$ where $(\frac{p}{q}, \frac{r}{s})$ is an open interval and $\frac{p}{q}$ and $\frac{r}{s}$ are in Ξ_k^n. For any such unequal rationals $|\frac{p}{q} - \frac{r}{s}| > \epsilon$, for every ϵ. So z_n is forced into an interval of diameter 0 with all rational values in $(0,1)$ either less than it or greater. It must be irrational. □

5 Conclusion

The proof is very much like that of Sondow's geometric proof of the irrationality of e [6]. One can understand his proof by considering the series given by $.\overline{1}$ base 4:

$$\sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots.$$

Divide the unit interval into $\frac{1}{4}$. Skip the first interval as we know the convergence point must be in $(.1, .2)$, base 4. Then divide $[.1, .2]$ into four intervals and skip the first one to give $[.11, .12]$. Repeat this process. The diameter of the intervals goes to 0. Leaving the convergence point between all finite base 4 decimals between 0 and 1: it is $\frac{1}{3}$, a number not expressible as a finite decimal base 4. Sondow does the same thing dividing up the unit interval with $n!$ rather than fourths. Skipping the first two terms in the series for e, the unit interval is divided into $3! = 6$, the first interval is skipped, and the next sixth is divided into 4 giving $\frac{1}{24}$ intervals. The first of these is skipped and the process continues. Once again the intervals shrink to 0 with all possible rational convergence points to the left and right of the point. $e - 2$ must be irrational, so e is. Sondow is using results from analysis: in particular, [5, page 82, problem 21].

The situation for z_n is more complicated than $.\overline{1}$, base 4, and the irrationality of e. For these numbers the intervals are perpetually offset, moving to the right. None the less the diameters of the intervals are shrinking and the excluded finite decimals in all bases are increasing to $\mathbb{Q}(0, 1)$, so all rationals are excluded. These numbers must be irrational. It's a nest within a nest: referring to (4), $I_{n+1} \subset I_n$ and within each of these intervals is the one that contains s_k. As each of the finite union of intervals has a diameter that is going to 0, the union's diameter suddenly vanishes at ∞. That's the mystery of (3). But one can infer in the case of e and z_n their irrationality without using nested intervals: their irrational because they define Dedekind cuts for irrational numbers [5, chapter 1 appendix].

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, **11**, (1979), 268–272.

[3] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[4] G. Hurst, Solutions to Introduction to Analytic Number Theory by Tom M. Apostol, Available at: https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[5] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.

[6] J. Sondow, A geometric proof that e is irrational and a new measure of its irrationality, *Amer. Math. Mon.* 113 (2006), 637-641.