A Simple Proof that $\zeta(2)$ is Irrational

Timothy W. Jones

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Abstract

We prove that a partial sum of $\zeta(2) - 1 = z_2$ is not given by any single decimal in a number base given by a denominator of its terms. This result, applied to all partials, shows that there are an infinite number of partial sums in one interval of the form $[.(x-1), .x]$ where $.x$ is single decimal in a number base of the denominators of the terms of z_2. We show that $\zeta(2) - 1$ is contained in an open interval inside $[.(x-1), .x]$. As all possible rational values of $\zeta(2) - 1$ are in these intervals, z_2 must be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [5]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. Here we give a simpler proof that uses just basic number theory.

2 Bertrand

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

The first lemma is a little more difficult than an exercise in Apostol's Introduction to Analytic Number Theory [2, p. 23, problem 30], its inspiration.

Lemma 1. *The reduced fraction, r/s giving*

$$s_k^m = \sum_{j=2}^{k} \frac{1}{j^m} = \frac{r}{s} \tag{1}$$

is such that 2^m divides s.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^m, 3^m, \ldots, k^m\}$ will have a greatest power of 2, ma. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^m$ will have a greatest power of 2 exponent of mb. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + (k!)^m/3^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{2}$$

The term $(k!)^m/2^{ma}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $mb - ma$ for 2. As all other terms but this term will have more than an exponent of 2^{mb-ma} in their prime factorization, we have the numerator of (2) has the form

$$2^{mb-ma}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^m/2^{ma}$. The denominator, meanwhile, has the factored form

$$2^{mb}C,$$

where $2 \nmid C$. This leaves 2^{ma} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If p is a prime such that $k > p > k/2$, then p^m divides s in (1).*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + \cdots + (k!)^m/p^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{3}$$

As $(k, p) = 1$, only the term $(k!)^m/p^m$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^m/p^m$. As $p < k$, p^m divides $(k!)^m$, the denominator of r/s, as needed. $\square$

Theorem 1. *If*

$$s_k^m = \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{k^m} = \frac{r}{s}, \tag{4}$$

with r/s reduced, then $s > k^m$.

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [10]. If k of (4) is even we are assured that there exists a prime p such that $k > p > k/2$. If k is odd $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^m p^m$ divides the denominator of (4) and as $2^m p^m > k^m$, the proof is completed. $\square$

So, for z_2, we have the following.

Definition 1.

$$D_{k^2} = \{0, 1/k^2, \ldots, (k^2-1)/k^2\} = \{0, .1, \ldots, .(k^2-1)\} \textit{ base } k^2$$

Corollary 1.

$$s_n^2 \notin \bigcup_{k=2}^{n} D_{k^2}$$

Proof. Immediate. $\square$

3 Cantor

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1. The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9,

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 modified to exclude them all via partial sums of z_2.

or D_{16}, etc.. Can we conclude that z_2 is irrational? The table should remind readers of Cantor's diagonal method. The catch with this conclusion is that we are not working with a single decimal system and verification via decimal notation is wanting.[1] We can, however, build a proof using the property that this table indicates.

Lemma 3. *For every natural number k greater than 1, there exists a natural number N_k such that*

$$s_n^2 \in ((x-1)/k^2, x/k^2) \tag{5}$$

for all $n > N_k$.

Proof. We know $0 < z_2 < 1$. For a given k, we can partition the interval $[0, 1]$:

$$\bigcup_{j=1}^{k} \left[\frac{j-1}{k^2}, \frac{j}{k^2}\right] = [0, 1].$$

Also, as no partial equals an endpoint and s_n^2 is a strictly increasing, convergent sequence, there will be an endpoint that separates those intervals with

[1] Earlier drafts stopped here. The list of rationals are unambiguous and exhaustive and partials go through points moving further into the Southeast section of the table. Precision must increase implying an irrational number. Theorem 2 develops this idea conclusively.

a finite number of partials in it from the one with an infinite number, a tail of the series. This interval is the one of (5), establishing the theorem. ☐

4 Knopp

In this section we develop an upper bound for z_2 based on the treatment of this subject given in Knopp [11, p. 260].

Lemma 4.

$$z_2 < s_n^2 + \frac{1}{n} \tag{6}$$

Proof. As $(n+1)^2 > n(n+1)$

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$$

As

$$z_2 - s_n^2 = \sum_{j=n+1}^{\infty} \frac{1}{j},$$

$$z_2 - s_n^2 < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots. \tag{7}$$

Now

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so the series in (7) telescopes to $1/n$ giving (6), as needed. ☐

5 Proof for z_2

In this section,

$$S_n = \sum_{m=2}^{n} \frac{1}{m^2}.$$

Lemma 5. *For every natural number $k > 2$, there exists $x \in \{1, 2, \dots, k\}$ and N such that for all $n > N$*

$$\left[S_n, S_n + \frac{1}{n}\right] \subset \left(\frac{x-1}{k^2}, \frac{x}{k^2}\right) \tag{8}$$

Proof. The left endpoint. As partials S_n never go through dots on C_{k^2} circles, if $k < n$, we know there is a N such that

$$S_n \in \left(\frac{x-1}{k^2}, \frac{x}{k^2} \right]$$

for all $n > N$. A last radii crosses through a dot and the remaining radii are counter-clockwise of that rational boundary.

The right endpoint. We use that $S_n + 1/n$ is a decreasing sequence converging to z_2 [11]. First if $S_n + 1/n = x/k^2$, then

$$S_{n+1} + \frac{1}{n+1} \in \left(\frac{x-1}{k^2}, \frac{x}{k^2} \right)$$

and the Lemma is established. Next, suppose there exists $n^* > N$ such that

$$\frac{x}{k^2} < S_{n^*+j} + \frac{1}{n^*+j} < S_{n^*} + \frac{1}{n^*}$$

for all natural number $j > 0$, then the interval $[x/k^2, (x+1)/k^2]$ would be the unique interval for k^2 that has an infinity of points in it, a contradiction. So there must be j such that

$$S_{n^*+j} + \frac{1}{n^*+j} < \frac{x}{k^2}$$

and thus the right hand limit is established by taking N large enough. The lemma, (8), is established.

□

Theorem 2. *z_2 is irrational.*

Proof. Suppose to obtain a contradiction that $z_2 = a/b$. Then, using Lemma 5, $z_2 \in [S_n, S_n + 1/n]$, and all possible fractions for z_2 are endpoints of intervals of the form $[(x-1)/k^2, x/k^2]$, we have a contradiction. □

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