Machin-type formulas for pi

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Abstract. This note presents some Machin-type formulas for pi.

J. Machin (1680-1751):

$$\frac{\pi}{4}=4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right)$$

Part 1

1. The polynomial equation.

$$f(x)=x^5-4x+1=0 \qquad (1)$$

2. $f(x)=0$ is not solvable by radicals.

3. Theorem (Galois). A polinomial $f\in\mathbb{Q}[x]$ is solvable by radicals if and only if its Galois group $Gal(f)$ is soluble.

4. Maple: $Gal(f)$:

$$"5T5",\{"S(5)"\},"-",120,$$
$$\{"(1\,5)","(2\,5)","(3\,5)","(4\,5)"\}$$

$$Order(G)=120$$

G is not soluble

5. Roots

$$f(x)=0\Rightarrow x=\{x_1,x_2,x_3,x_4,x_5\} \qquad (2)$$

$$x_1,x_2,x_3\in\mathbb{R}$$
$$x_4=a+bi\in\mathbb{C},\, i=\sqrt{-1}$$
$$x_5=a-bi\in\mathbb{C}$$

$$x_1=\frac{1}{4}+\frac{1}{4}\left(\frac{1}{4}+\frac{1}{4}\left(\frac{1}{4}+...\right)^5\right)^5 \qquad (3)$$

$$x_2=\sqrt[4]{4-\frac{1}{\sqrt[4]{4-\frac{1}{\sqrt[4]{4-...}}}}} \qquad (4)$$

$$x_3=-\sqrt[5]{1+4\sqrt[5]{1+4\sqrt[5]{1+...}}} \qquad (5)$$

$$a=-\left(\frac{x_1+x_2+x_3}{2}\right) \qquad (6)$$

$$b=\sqrt{-(x_1x_2x_3)^{-1}-\left(\frac{x_1+x_2+x_3}{2}\right)^2} \qquad (7)$$

6. Roots: approximate values

$$x_1=0.25024534...$$

$$x_2=1.34324608...$$

$$x_3=-1.47081820...$$

$$x_4=-0.06133...+i\times 1.42087...$$

$$x_5=-0.06133...-i\times 1.42087...$$

7. Graphics.

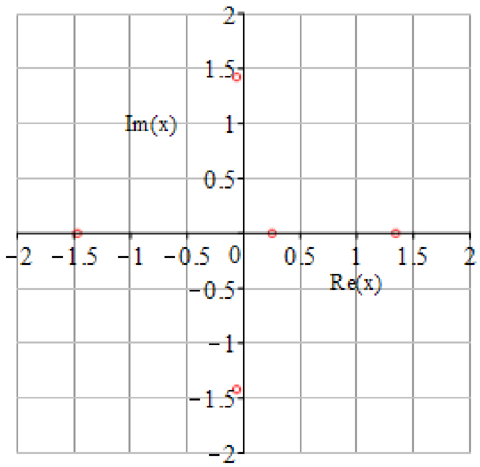

Fig. 1 , • $f(x)=0$.

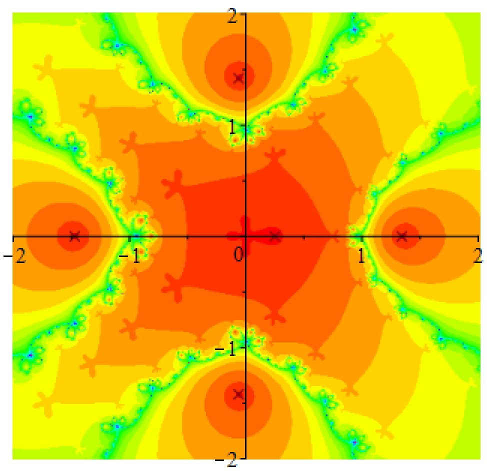

Fig. 2 , Newton-Julia set for $f(z)$.

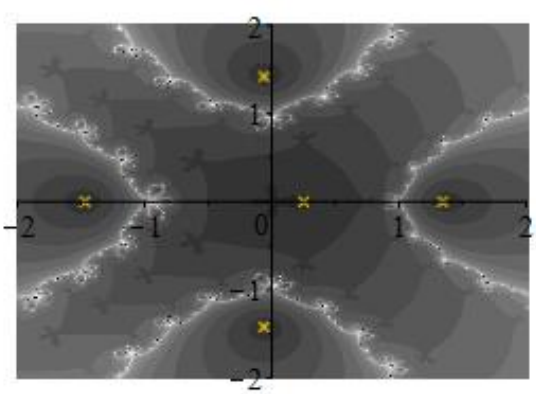

Fig. 3 , Newton-Julia set for $f(z)$.

8. Some Machin-type formulas.

$$\frac{\pi}{4}=\tan^{-1}(x_1)-\tan^{-1}\left(\frac{1}{x_2}\right)+ -\tan^{-1}\left(\frac{1}{x_3}\right)+\tan^{-1}\left(\frac{a^2+b^2-4a-1}{2(a^2+b^2+a-1)}\right) \tag{8}$$

$$\frac{\pi}{2}=\tan^{-1}\left(\frac{1}{x_1}\right)+\tan^{-1}\left(\frac{1}{x_2}\right)+ +\tan^{-1}\left(\frac{1}{x_3}\right)+\tan^{-1}\left(\frac{a^2+b^2+6a-1}{3a^2+3b^2-2a-3}\right) \tag{9}$$

$$\frac{\pi}{4}=\tan^{-1}(x_1)+ +\tan^{-1}\left(\frac{1+x_1-x_1^2+x_1^3-x_1^4}{2}\right) \tag{10}$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2}\right)+ +\tan^{-1}\left(\frac{x_2^4-x_2^3+x_2^2-x_2-1}{2}\right) \tag{11}$$

$$\frac{\pi}{4}=\tan^{-1}(x_1^5)+\tan^{-1}\left(1-\frac{x_1^4}{2}\right) \tag{12}$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2^5}\right)+\tan^{-1}\left(\frac{x_2^4}{2}-1\right) \tag{13}$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_3^5}\right)+\tan^{-1}\left(\frac{x_3^4}{2}-1\right) \tag{14}$$

$$\frac{\pi}{4}=\tan^{-1}(x_1^2)+ +\tan^{-1}\left(\frac{4+x_1+3x_1^2-x_1^3-3x_1^4}{5}\right) \tag{15}$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2^2}\right)+ + \tan^{-1}\left(\frac{3x_2^4+x_2^3-3x_2^2-x_2-4}{5}\right) \quad (16)$$

$$\frac{\pi}{4}=\tan^{-1}(u)+ + \tan^{-1}\left(\frac{1-2u+2u^2-2u^3+2u^4}{2047}\right) \quad (17)$$

$$u=4x_1$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2}\right)+ + \tan^{-1}\left(\frac{3}{2}-\frac{5}{2x_2}+\frac{5}{2x_2^2}-\frac{5}{2x_2^3}+\frac{1}{2x_2^4}\right) \quad (18)$$

$$\frac{\pi}{4}=\tan^{-1}\left(-\frac{1}{x_3}\right)+ + \tan^{-1}\left(x_3^4+x_3^3+x_3^2+x_3-2\right) \quad (19)$$

$$\frac{\pi}{4}=\tan^{-1}\left(-\frac{1}{x_3}\right)+ + \tan^{-1}\left(2+\frac{3}{x_3}+\frac{3}{x_3^2}+\frac{3}{x_3^3}-\frac{1}{x_3^4}\right) \quad (20)$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2^5}\right)+\tan^{-1}\left(1-\frac{1}{2x_2}\right) \quad (21)$$

$$\frac{\pi}{4}=\tan^{-1}\left(x_1+x_1^2\right)+ + \tan^{-1}\left(\frac{3-2x_1+2x_1^2-2x_1^4}{5}\right) \quad (22)$$

$$\frac{\pi}{4}=\tan^{-1}\left(x_1+x_1^5\right)+\tan^{-1}\left(\frac{3-2x_1^4}{5}\right) \quad (23)$$

$$\frac{\pi}{4}=\tan^{-1}\left(x_1^2+x_1^3\right)+ + \tan^{-1}\left(-3+18x_1-12x_1^2+8x_1^3-6x_1^4\right) \quad (24)$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2}+\frac{1}{x_2^3}\right)+ + \tan^{-1}\left(83-\frac{122}{x_2}+\frac{178}{x_2^2}-\frac{262}{x_2^3}+\frac{56}{x_2^4}\right) \quad (25)$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_2^4}+\frac{1}{x_2^5}\right)+\tan^{-1}\left(\frac{3}{5}-\frac{2}{5x_2}\right) \quad (26)$$

$$\frac{\pi}{4}=\tan^{-1}\left(-\frac{1}{x_3}+\frac{1}{x_3^3}\right)+ + \tan^{-1}\left(\frac{3}{5}+\frac{6}{5x_3}+\frac{6}{5x_3^2}-\frac{2}{5x_3^3}\right) \quad (27)$$

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{x_3^4}+\frac{1}{x_3^5}\right)+\tan^{-1}\left(\frac{3}{5}-\frac{2}{5x_3}\right) \quad (28)$$

$$\frac{\pi}{4}=\tan^{-1}\left(1-2x_1\right)+ + \tan^{-1}\left(\frac{1-x_1-x_1^2-x_1^3-x_1^4}{2}\right) \quad (29)$$

$$\frac{\pi}{4}=\tan^{-1}\left(x_2-1\right)+\tan^{-1}\left(7-2x_2^4\right) \quad (30)$$

$$\frac{\pi}{4}=-\tan^{-1}\left(1-\frac{1}{x_2}\right)+ + \tan^{-1}\left(-1+\frac{8}{x_2^3}-\frac{2}{x_2^4}\right) \quad (31)$$

$$\frac{\pi}{4}=\tan^{-1}\left(1-\frac{2}{x_2}\right)+ +\tan^{-1}\left(\frac{1}{2}+\frac{3}{2x_2}+\frac{3}{2x_2^2}+\frac{3}{2x_2^3}-\frac{1}{2x_2^4}\right) \tag{32}$$

$$\frac{\pi}{4}=\tan^{-1}\left(x_1-1\right)+\tan^{-1}\left(7-2x_1^4\right) \tag{33}$$

$$\frac{\pi}{4}=\tan^{-1}\left(-1-\frac{1}{x_3}\right)+ +\tan^{-1}\left(-1-\frac{8}{x_3^3}+\frac{2}{x_3^4}\right) \tag{34}$$

$$\frac{3\pi}{4}=\tan^{-1}\left(2x_3^4-7\right)+\tan^{-1}\left(1-x_3\right) \tag{35}$$

Part 2

9. For x_1, x_2, x_3

$$x_1=\frac{14}{3}-\frac{1}{3}\sqrt{84-\frac{133}{\sqrt{84-\frac{133}{\sqrt{84-...}}}}} \tag{36}$$

$$x_2=\frac{14}{3}-\frac{1}{3}\left\{\frac{19}{12}+\frac{1}{84}\left(\frac{19}{12}+\frac{1}{84}\left(\frac{19}{12}+...\right)^3\right)^3\right\} \tag{37}$$

$$x_3=\frac{14}{3}+\frac{1}{3}\sqrt[3]{133+84\sqrt[3]{133+84\sqrt[3]{133+...}}} \tag{38}$$

We have

$$\frac{\pi}{4}+\tan^{-1}\left(\frac{3}{52}\right)=\tan^{-1}\left(\frac{1}{x_1}\right)+ +\tan^{-1}\left(\frac{1}{x_2}\right)+\tan^{-1}\left(\frac{1}{x_3}\right) \tag{39}$$

10. For x_1, x_2, x_3

$$x_1=\frac{14}{3}-\frac{1}{3}\sqrt{84-\frac{187}{\sqrt{84-\frac{187}{\sqrt{84-...}}}}} \tag{40}$$

$$x_2=\frac{14}{3}-\frac{1}{3}\left\{\frac{187}{84}+\frac{1}{84}\left(\frac{187}{84}+\frac{1}{84}\left(\frac{187}{84}+...\right)^3\right)^3\right\} \tag{41}$$

$$x_3=\frac{14}{3}+\frac{1}{3}\sqrt[3]{187+84\sqrt[3]{187+84\sqrt[3]{187+...}}} \tag{42}$$

We have

$$\frac{\pi}{4}+\tan^{-1}\left(\frac{2}{53}\right)=\tan^{-1}\left(\frac{1}{x_1}\right)+ +\tan^{-1}\left(\frac{1}{x_2}\right)+\tan^{-1}\left(\frac{1}{x_3}\right) \tag{43}$$

11. For x_1, x_2, x_3

$$x_1=\frac{14}{3}-\frac{1}{3}\sqrt{75-\frac{196}{\sqrt{75-\frac{196}{\sqrt{75-...}}}}} \tag{44}$$

$$x_2=\frac{14}{3}-\frac{1}{3}\left\{\frac{196}{75}+\frac{1}{75}\left(\frac{196}{75}+\frac{1}{75}\left(\frac{196}{75}+...\right)^3\right)^3\right\} \tag{45}$$

$$x_3 = \frac{14}{3} + \frac{1}{3}\sqrt[3]{196 + 75\sqrt[3]{196 + 75\sqrt[3]{196 + \ldots}}}$$

(46)

We have

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{x_1}\right) + \tan^{-1}\left(\frac{1}{x_2}\right) + \tan^{-1}\left(\frac{1}{x_3}\right)$$

(47)

12. For x_1, x_2, x_3

$$x_1 = \frac{10}{3} - \frac{1}{3}\sqrt{120 - \frac{470}{\sqrt{120 - \frac{470}{\sqrt{120 - \ldots}}}}} \quad (48)$$

$$x_2 = \frac{10}{3} - \frac{1}{3}\left\{\frac{47}{12} + \frac{1}{120}\left(\frac{47}{12} + \frac{1}{120}\left(\frac{47}{12} + \ldots\right)^3\right)^3\right\}$$

(49)

$$x_3 = \frac{10}{3} + \frac{1}{3}\sqrt[3]{470 + 120\sqrt[3]{470 + 120\sqrt[3]{470 + \ldots}}}$$

(50)

We have

$$\frac{\pi}{2} = \tan^{-1}\left(\frac{1}{x_1}\right) + \tan^{-1}\left(\frac{1}{x_2}\right) + \tan^{-1}\left(\frac{1}{x_3}\right)$$

(51)

Part 3

$$\frac{\pi}{4} = \tan^{-1}\left(\cos\frac{2\pi}{9}\right) + \tan^{-1}\left(\cos\frac{4\pi}{9}\right) + \tan^{-1}\left(\left(\tan\frac{\pi}{18}\right)^2\right) - \tan^{-1}\left(\frac{1}{14}\right) \quad (52)$$

$$\frac{\pi}{4} = \tan^{-1}\left(\sin\frac{\pi}{15}\right) + \tan^{-1}\left(\sin\frac{2\pi}{15}\right) - \tan^{-1}\left(\sin\frac{4\pi}{15}\right) + \tan^{-1}\left(\sin\frac{7\pi}{15}\right) + \tan^{-1}\left(\frac{961 - 552\sqrt{3}}{97}\right) \quad (53)$$

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