Turbulence as structured Route of Energy from Order into Chaos.

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Abstract.

A contemptuous attempt to understand the phenomenon of turbulence dated back to a theory of L. F. RICHARDSON which later became enhanced by A. N. KOLMOGOROV. By the contents of this theory turbulence is considered as a transfer of energy taking place in forms of cascades between eddies of various orders of magnitude. The transfer becomes started by unspecified disturbance acting on eddies in forms of fibres and is further evaluated on account of their stretching. This principally can explain for eddies of various sizes being created in generations while energy is distributed along these generations from the largest eddies down to the smallest. But it is impossible to determine a specific structure of the cascade and its development in details.

In order to step forward this way, this paper will show, when spinning spheres with surface–tension are models for the eddies and a specifically designed disturbance act on them, eddies of large size will decay in a cascade of hierarchically structured generations into smaller ones. The dynamics of this development is due to the balance between the acting force of the disturbance and the reaction due to sphere–tension on each eddy in a self–organisation mode. All eddies within one step of the hierarchy will obtain same size, life–time and rotation–phase, for follower–generations these quantities will be different in a definite way. Each predecessor–generation will double its number of eddies relative to its follower–generation, while each eddy partitions its rotation–energy for its followers and those will get increased their surface–energy by disturbance. Finally the whole cascade will form a structured route of energy from order into chaos similar to many other dynamical systems.

By addendum [12] the introduction of a sphere with surface–tension as an appropriate model for eddies has been justified. Addendum [13] demonstrates that for a quadratic–iterator permanent phase– doubling will lead to a route from order into chaos; due to the topological equivalence between iterator and eddy's decay–cascade in this respect the latter similarly shows a route from order into chaos too.

1. *Introduction*.

L. F. RICHARDSON [1] and A. N. KOLMOGOROV [2 , ..., 7] conceptually related dissipation with other macroscopic quantities of a turbulent flow. They started from the idea that a turbulent flow is fed with energy on large scales, which is transported by decay of eddies through an order of magnitudes to the smallest eddies where finally it is totally transformed into heat. This process is called energy–cascade and starts with the following proportionality:

1.1 $\epsilon \sim (u_\lambda)^3/\lambda \quad \leftarrow \quad \epsilon = Dissipation\text{–}rate,$

$u_\lambda = Tangential\text{–}speed\ of\ smallest\ eddy,$

$\lambda = KOLMOGOROV\text{–}length,$

For an appropriate REYNOLDS–number $/Re/$ it can be written:

1.2 $Re = (l/\lambda)^{4/3} \quad \leftarrow \quad l = Eddy\text{–}extend\ in\ a\ specific\ order\ of\ magnitude$

$\lambda = (\nu^3/\epsilon)^{1/4} \quad \leftarrow \quad \nu = Viscosity\ of\ the\ turbulent\ medium$

Thus the value of $/Re/$ measures the range of various length–scales within the turbulence. The KOLMOGOROV–length represents the extent of a smallest eddy in the turbulence.

According to L. F. RICHARDSON, turbulent flows show a hierarchy of eddies, where the larger ones are built in a preliminary creation–process of the turbulence. Afterward they decay successively in a sequence of instabilities down to a minimal magnitude $/\lambda/$ of eddies. Here they finally are disturbed and their energy is transformed into heat due to viscosity of the turbulent medium. During this hierarchical process, eddies submit most of their energy to their followers, only a small part each time is lost through dissipation. The hierarchy ends as soon as $/l/$ becomes comparable with $/\lambda/$ which results in $/Re \approx 1/$.

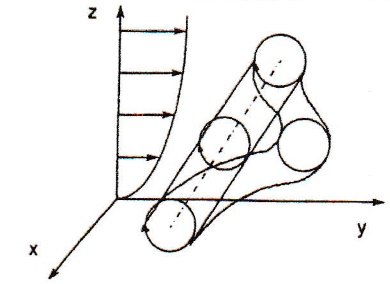
1.3 $\tau = l/u.$

1.4 $\Pi \sim u^2/\tau = u^3/l \sim \epsilon.$

Such an independence of transfer–rate $/\Pi/$ from viscosity $/\nu/$ can be explained – due to RICHARDSON – by the stretching of an eddy.

Turbulence in this sense starts with a picture about eddy–fibres in a shear–flow (H. E. Fiedler[8]):

A.1.1

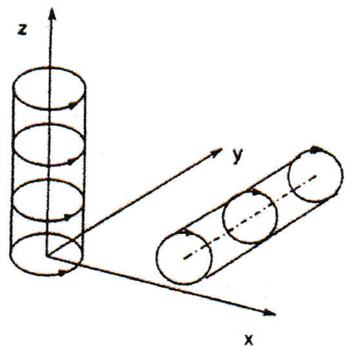


The smallest swelling–out of a fibre will stretch its length, strengthen its angular–speed $/\omega/$ and shrink its cross–section $/A/$ appropriately to HELMHOLTZ's law:

1.5 $\omega \cdot A = const.$

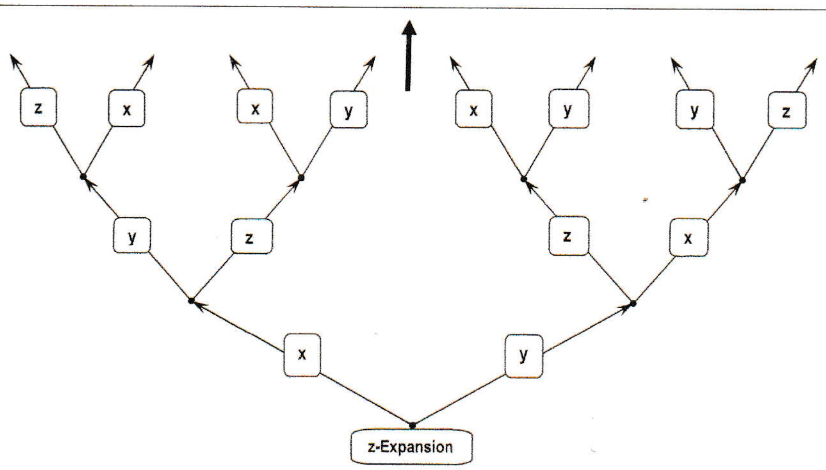
The stretching–mechanism by itself can be explained in a following way:

A.1.2



Given a fibre in z–direction with a rotation in the (x,y)–plain. As soon as it becomes stretched in z–direction, its cross–section in (x,y)–plain and with it the appropriate rotation $/\nabla \times \underline{\omega}/$ and intensity $/\omega^2/$ will be enlarged. Such an increase of intensity – on its side – will cause further stretching of the fibre in the other space–directions. Thus stepping forward this way, the initial swelling–out of the fibre will finally have been resulted into an energy–cascade filling up the complete fluid–space. Such a procedure can be visualized qualitatively by the following graph:

A.1.3



All this together will result in angular–speed $/\omega/$ and with it $/\omega^2/$ to such an extent that the energy–transfer–rate remains quasi constant. Therefore the energy–cascade is assumed to be quasi independent from the viscosity $/\nu/$ of the turbulent medium. For the energy–transfer–rate $/\epsilon/$ across the energy –cascade approximately the following proportionality can be obtained:

1.6 $\epsilon \sim \nu(u_\lambda/\lambda)^2$.

Compared to equation /1.4/ this will result in:

1.7 $\nu(u_\lambda/\lambda)^2 \sim u^3/l = u^2/\tau$.

During the extension of the energy–cascade each eddy partitions its energy $/\sim u^2/$ among the followers, the energy of the followers therefore must be less than that of their predecessor. The value of $/u^2/$ decreases permanently in a propagating energy–cascade and in a similar way $/l/$ does it too. Thus finally – in the case of smallest eddies – the product $/u \cdot l/$ will become comparable with $/\nu/$:

1.8 $Re = u_\lambda \cdot \lambda/\nu \approx 1$.

The values $/\lambda, u_\lambda, \tau_\lambda/$ of the smallest eddies in the turbulence are called KOLMOGOROV–scales, they can be summarized in the following way:

1.9 $\lambda \sim (\nu^3/\epsilon)^{1/4} \sim Re^{-3/4} \cdot l$

$u_\lambda \sim (\nu \cdot \epsilon)^{1/4} \sim Re^{-1/4} \cdot u_l$

$\tau_\lambda \sim (\nu/\epsilon)^{1/2} \sim Re^{-1/2} \cdot \tau_l$.

KOLMOGOROV completed the theory of the energy–cascade, which formally was initiated by RICHARDSON, with three additional hypotheses.
For eddies of $/\lambda < r \ll l/$ statistical isotropy can be assumed. In addition $/\tau_l/$ of large eddies will show in comparison with proper values of medium–eddies $/\tau_r \ll \tau_l/$, the letter will decay much faster. The smallest eddies are in a statistical equilibrium. Under these aspects he came to his hypothesis of the local isotropy:

H.1 *For large REYNOLDS–numbers turbulent motions on smallest scales are statistically isotropic and will expire in statistical equilibrium (universal equilibrium).*

By the next hypothesis KOLMOGOROV expressed his opinion, that:

H.2 *For large REYNOLDS–numbers and length–scales $/r \ll l/$ statistical quantities will only depend on three parameters – the length–scale $/r/$ itself, the energy–transfer–rate $/\Pi/$ and the viscosity $/\nu/$ of the turbulent medium.*

Eddies of length–scales $/l \gg r/$ – from the so–called Inertial–range – will remain nearly untouched by viscosity $/\nu/$. Those eddies will obtain their energy–influx nearly totally from their larger predecessors and will deliver it nearly completely to their smaller followers of universal equilibrium. Thus, for the statistics of these length–scales, energy–transfer is not decisive. In essence of this KOLMOGOROV formulated his final hypothesis:

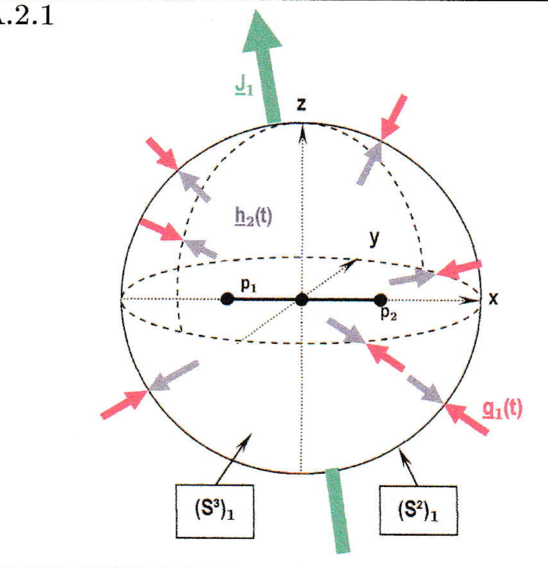
H.3 *For large REYNOLDS–numbers for scales $/l \gg r > \lambda/$ statistical quantities will have universal forms only depending on $/\epsilon/$ and $/r/$.*

2. An Eddy's Decay due to a Disturbance acting on it.

The theory of RICHARDSON enhanced by KOLMOGOROV apparently disclosed some deficiencies for instance with respect to the number and sizes of followers coming into existence as consequences of a predecessor's decay, the individual life–times of the various members in the cascade and last but least a characteristic of the disturbance–signal. This information however is needed in order to determine an appropriate cascade–structure, which then enables a statement about the roper development of the turbulence. The following discussions should be understood to appropriately enhance the former theory in this way (please initially pay attention on [12]).
Inside a fluid an eddy is now considered as sphere $/(S^3)_1/$ with a spin $/\underline{J}_1/$as united angular–

momentum of its particle–set. The form of sphere is chosen because it possesses the smallest surface for an enclosed volume. A time–dependent force $/\underline{q}_1(t)/$ as disturbance may act on the eddy from outside trying to deform $/(S^3)_1/$ into another volume with increased surface. This will cause reaction $/\underline{h}_2(t)/$ parallel to $/\underline{q}_1(t)/$ due to the surface –tension (consequence of the fluid–viscosity $/\nu/$).

A.2.1



The competing forces will influence each other and thus should be considered coupled together in a self–organizing system, appropriately in the following way:

2.01 $d\underline{q}_1/dt = -\gamma \cdot \underline{q}_1 - a \cdot \underline{h}_2 \cdot \underline{q}_1 \;\rightarrow\; dq_1/dt = -\gamma \cdot q_1 - a \cdot h_2 \cdot q_1 \;\leftarrow\; \gamma, \delta = damping\text{–}parameters$

2.02 $d\underline{h}_2/dt = -\delta \cdot \underline{h}_2 + b \cdot (q_1)^2 \;\rightarrow\; dh_2/dt = -\delta \cdot h_2 + b \cdot (q_1)^2 \qquad a, b = coupling\text{–}parameters.$

With respect to $/\underline{q}_1(t)/$ and $/\underline{h}_2(t)/$ one should make use of the so–called adiabatic approximation (see H. HAKEN [10]):

2.03 $\delta \gg \gamma \;\rightarrow\; dh_2/dt \approx 0,$

Due to relation /2.02/ this will result in:

2.04 $h_2 = \delta^{-1} \cdot b \cdot (q_1)^2.$

Equation /2.04/ can be interpreted as: $/h_2/$ must follow $/q_1/$ immediately, $/h_2/$ has become enslaved by $/q_1/$ (H. HAKEN [10]). On the other hand $/h_2/$ will react on $/q_1/$ back again via equation /2.01/ with the consequence:

2.05 $dq_1/dt = -\gamma \cdot q_1 - \delta^{-1} \cdot a \cdot b \cdot (q_1)^3.$

By equation /2.05/ force $/q_1/$ is expressed by the dynamics of a so–called unharmonic oscillator – which depending on the conditions –:

2.06 $[(\gamma > 0)] \;\bigvee\; [(\gamma < 0) \bigwedge (\delta^{-1} \cdot a \cdot b) > 0]$

possesses two qualitatively distinct stability–modes:

2.07 $[p_0 = 0] \;\bigvee\; [p_{1/2} = \pm \langle |\gamma| \cdot \delta \,/ (a \cdot b) \rangle^{1/2}].$

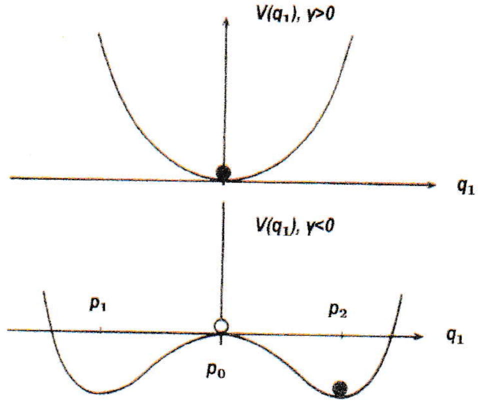
In the first case stable oscillations are performs with respect to the fix–point $/p_0/$, in the second case $/p_0/$ becomes instable and bifurcates symmetrically into new stable–points $/p_{1 \wedge 2}/$, each becoming the centre for subsequent oscillations. Therefore by the bifurcation of $/p_0/$ is expressed, the sphere $/(S^3)_1/$ will be partitioned into smaller follower–spheres $/(S^3)_2 \bigwedge (S^3)_3/$, a process which can be explained as follows.

For $/\underline{q}_1(t)/$ from outside $/(S^2)_1/$ – instead of $/\gamma = constant/$ – a damping of $/\gamma_1(t)/$ should be expected , presumably of the following kind:

2.08 $[\gamma(t_1)>0] \quad \rightarrow \quad [\gamma(t_2)=0] \quad \rightarrow \quad [\gamma(t_3)<0]$.

Potential–curves for $/V(q_1(t_1) \bigwedge V(q_1(t_3)/$ then qualitatively may be visualized in a following way:

A.2.2



Deforming the potential–curve on the way $/(\gamma<0) \rightarrow (\gamma>0)/$ will flatten the neighbourhood of $/p_0/$ steadily, stability accordingly will be reached more slowly and finally will be exchanged by instability at $/p_0/$. During this process, which takes a time $/\Delta t/$, force $/q_1(t)/$ will deform $/(S^2)_1/$ to a shape of surface with higher energy. As soon as the surface–energy has become high enough to build two times the surface of $/½\cdot(S^3)_1/$, this will occur at:

2.09	$(S^3)_1 =$ $(4/3)\cdot\pi\cdot(r_1)^3$	$\rightarrow$	$r_{2\vee 3} = r_1/(2)^{1/3}$ $r_{2\vee 3} \approx 0.79\cdot r_1$	$\rightarrow$	$(S^2)_{2\wedge 3} = 8\cdot\pi\cdot(r_{2\vee 3})^2$ $\approx 15.69\cdot(r_1)^2$	$\rightarrow$	$\Delta O \approx 3.12\cdot(r_1)^2$
					$(S^2)_1 = 4\cdot\pi\cdot(r_1)^2$ $\approx 12.57\cdot(r_1)^2$		
			$(S^3)_{2\wedge 3} = (S^3)_1$				
	$(S^3)_{2\vee 3} =$ $(4/3)\cdot\pi\cdot(r_{2\vee 3})^3$						

Sphere $/(S^3)_1/$ will become partitioned symmetrically at $/p_1 \bigwedge p_2/$ into spheres $/(S^3)_2 \bigwedge (S^3)_3/$ each volume of $/½\cdot(S^3)_1/$.

After the split spin $/\underline{J}_1/$ will have been saved, thus for $/(S^3)_2 \bigwedge (S^3)_3/$ following condition must hold:

2.10 $\underline{J}_1 = \underline{J}_2 + \underline{J}_3 \quad \leftarrow \quad J_2 = J_3$.

Between the rotation–energies $/\epsilon_{2\vee 3}/$ of $/(S^3)_2 \bigvee (S^3)_3/$ and $/\epsilon_1/$ of $/(S^3)_1/$ the following relationship will exist:

2.11	$\epsilon_1 = ½\cdot\theta_1\cdot(\omega_1)^2$	$\leftarrow$	$\theta_1 =$ *momentum of inertia*		
		$\rightarrow$	$\epsilon_1 = ½\cdot\kappa\cdot(r_1)^2\cdot(\omega_1)^2$	$\leftarrow$	$\omega_1 =$ *angular velocity*
	$\theta_1 = \kappa\cdot(r_1)^2$	$\leftarrow$	$\kappa =$ *constant*		
	$\epsilon_{2\vee 3} =$ $½\cdot\theta_{2\vee 3}\cdot(\omega_{2\vee 3})^2$	$\leftarrow$	$\theta_{2\vee 3} =$ *momentum of inertia*		
			$\epsilon_{2\vee 3} =$	$\leftarrow$	$\omega_{2\vee 3} =$ *angular velocity*

	$\theta_{2\vee3} = \kappa \cdot (r_{2\vee3})^2$	$\rightarrow$	$½ \cdot \kappa \cdot (r_{2\vee3})^2 \cdot (\omega_{2\vee3})^2$ $(r_{2\vee3})^2 \approx 0.62 \cdot (r_1)^2$ $\epsilon_{2\vee3} = ½ \cdot \epsilon_1$	$\rightarrow$	$\epsilon_{2\vee3} \approx \kappa \cdot 0.31 \cdot (r_1)^2 \cdot (\omega_{2\vee3})^2$ $\epsilon_{2\vee3} = ¼ \cdot \kappa \cdot (r_1)^2 \cdot (\omega_1)^2$	$\rightarrow$	$\kappa \cdot 0.31 \cdot (r_1)^2 \cdot (\omega_{2\vee3})^2 \approx \kappa \cdot 0.25 \cdot (r_1)^2 \cdot (\omega_1)^2$ $(\omega_{2\vee3})^2 \approx 0.8 \cdot (\omega_1)^2$ $\omega_{2\vee3} \approx 0.9 \cdot \omega_1$ $\omega_{2\wedge3} \approx 1.8 \cdot \omega_1$

This results in:

2.12 $J_{2\vee3} = \theta_{2\vee3} \cdot \omega_{2\vee3} \quad \rightarrow \quad J_{2\vee3} = \kappa \cdot (r_{2\vee3})^2 \cdot \omega_{2\vee3} \quad \rightarrow \quad J_{2\vee3} \approx \kappa \cdot 0.62 \cdot (r_1)^2 \cdot 0.9 \cdot \omega_1 \quad \rightarrow \quad J_{2\vee3} \approx 0.6 \cdot J_1.$

Energy $/\epsilon_{\Delta O}/$ – which has to be transferred by $/q_1(t)/$ in order to enlarge $/(S^2)_1/$ up to an equivalent of $/(S^2)_{2\wedge3}/$ – is proportional to $/\Delta O/$ or $/(r_1)^2/$. Therefore a similar relationship must hold for the proper life–time $/\Gamma((S^3)_1)/$ (time of $(S^3)_1$–existence):

2.13 $\Gamma((S^3)_1) \sim (r_1)^2.$

3. *Disturbance–Model acting on Spheres of Follower–Generations.*

In the preceding discussion a stochastic signal:

3.1 $s'_1(t) = \gamma_1(t) \cdot q_1(t)$

lasting for the time:

3.2 $\Gamma((S^3)_1) = \Gamma_1$

became responsible for the split of sphere $/(S^3)_1/$ into $/(S^3)_2 \wedge (S^3)_3/$. The same procedure is now assumed to take place in a similar way on spheres of a second generation, where the stochastic signal:

3.3 $s'_2(t) = \gamma_2(t) \cdot q_2(t)$

within action–time:

3.4 $\Gamma((S^3)_2 \vee (S^3)_3) = \Gamma_2$

causes the follower–spheres of first generation to became partitioned into spheres of a third generation. Generally speaking, any k–th generation will get its specific resonance–term for the splits of its actual spheres in the following way:

3.5 $s'_k(t) = \gamma_k(t) \cdot q_k(t) \quad \rightarrow \quad \Gamma_k.$

Throughout generations of the splitting–process following conditions must hold:

3.6 $[\Gamma_1 > \Gamma_2 > \ldots\ldots > \Gamma_k] \quad \wedge \quad [\varepsilon(s_1') < \varepsilon(s_2') < \ldots\ldots < \varepsilon(s'_k)],$

($/\varepsilon(s'_j)/$ represent the energy of a signal $/s'_j(t)/$). The first condition is due to the fact that spheres of any predecessor–generation have larger life–times as their followers in next generation. The second condition takes into consideration that the proper signal for any k–th generation will have to invest about 25% more extra–energy than $(k-1)$–th generation for the increase in surface–sum of its spheres.

Finally, in order to obtain a suitable concept of the disturbance in total for appropriate actions on eddies in the current sense, one may consider $/S(t)/$ as a consecutive sequences of stochastic signals $/s'_k(t)/$ whose layout will be obtained on base of the following considerations.

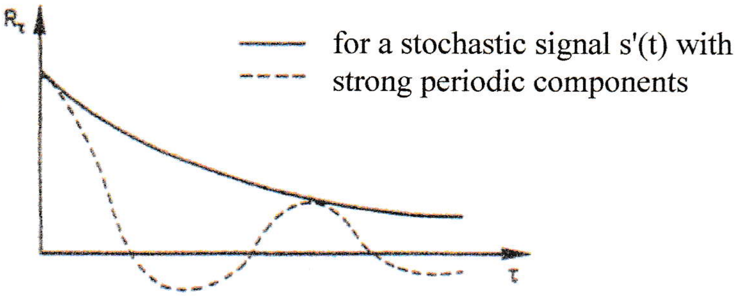
4. Layout of Disturbance on Course of the Route.

In order to obtain a suitable concept of a disturbance–signal appropriate for actions on eddies in the current sense, one may start in a following way. At a specific point in the fluid the overall disturbance– signal $/S(t)/$ is derived from a series of successive auto– or time–correlations due the probabilistic velocity $/v'(t)/$:

4.01 $$S(t) = \langle \{v'(0)\cdot v'(\tau_0)\}^{[\Gamma_0]}, \{v'(\Gamma_0)\cdot v'(\Gamma_0+\tau_1)\}^{[\Gamma_1]}, \{v'({}_{j=0}\textstyle\sum_1 \Gamma_j)\cdot v'({}_{j=0}\textstyle\sum^1 \Gamma_j+\tau_2)\}^{[\Gamma_2]}, \ldots\ldots\ldots\ldots, \{v'({}_{j=0}\textstyle\sum^x \Gamma_j)\cdot v'({}_{j=0}\textstyle\sum^x \Gamma_j+\tau_{x+1})\}^{[\Gamma_{x+1}]}, \ldots\ldots \rangle$$

where $/\{\ldots\}^{[T]}/$ expresses the mean of a quantity within time–period $/T/$(an appropriate value from the list of life–times $/\Gamma_k/$). With each of the partial correlation–functions $/\{v'(t)\cdot v'(t+\tau)\}^{[T]}/$ is associated a stochastic– signal $/s'(t)/$ in form of a probabilistic process with strong periodic components, and can also be expressed on base of proper correlation–coefficients $/R(\tau)/$:

A.4.1



in the following way:

4.02 $$\{(v')^2\}^{[T]}R(\tau) = \{v'(t)\cdot v'(t+\tau)\}^{[T]}.$$

Further discussion is based on a generalization of FOURIER–analysis.

4.1 FOURIER–Expansion of a real function.

A real function $/f(t)/$ with time–period $/1/$:

4.03 $${}_0\int^1 f(t)dt = 0$$

will be generally expanded into the FOURIER–series like the following:

4.04 $$f(t) = {}_{(k=-\infty)}\textstyle\sum^{(k=\infty)} \langle a_k \cdot exp\{2\pi ikt\}\rangle \;\rightarrow \langle [a_0 = 0] \wedge [a_k = a^*_{-k}], (* = \; konjugate\text{–}complex)\rangle.$$

From the amplitudes $/a_k/$ via PARSEVAL–equation the square mean–value of $/f(t)/$ can be found, as follows:

4.05 $${}_0\int^1 f^2(t)dt = {}_{k=1}\textstyle\sum^{\infty} \langle 2\cdot|a_k|^2\rangle.$$

The various terms of the sum (modes) in equation $/4.04/$ are orthogonal to each other:

4.06 $${}_0\int^1 \langle a^*_j \cdot a_k \cdot exp\{2\pi i(k-j)t\}\rangle dt = \delta_{jk}.$$

4.2 FOURIER–Expansion of the stochastic Signal.

Although in real turbulence specific frequencies cannot be found, it becomes possible to associate a certain part of the total fluctuation–energy with a specific frequency in order to perform a harmonic analysis of fluctuations.
The spectral decomposition of a time–dependent, stochastic signal $/s'(t)/$ becomes a generalization of the FOURIER–decomposition of a deterministic, periodic time–functions $/f(t)/$. If an analogous

expansion for a stochastic signal $/\{s'(t)\}^{[T]}=0/$ should be performed, a generalization is necessary because $/s'(t)/$ is not periodic, it contains the complete set of frequencies, not only the discrete ones. In addition $/s'(t)/$ is a random function and repetitions of the same flow–experiment will deliver different results.
Each real stationary process $/\{s'(t)\}^{[T]}=0/$ can expanded approximately in a sum of harmonic oscillations with random and un–correlated amplitudes. Analogous to the equation $/4.04/$ one gets:

4.07 $s'(t)={}_{k=(-n)}\sum^{k=n}\langle[Z(f_{k+1})-Z(f_k)]\cdot[exp\{2\pi i f'_k t\}]\rangle \rightarrow [-\Omega=f_{-n}<f_{-n+1}<\ldots<f_n<f_{n+1}=\Omega].$

For the decomposition of interval $/[-\Omega,\Omega]/$ on frequency–axis following conditions must be fulfilled:

4.08 $f_k<f'_k<f_{k+1}.$

The random quantities $/Z(f_{k+1})-Z(f_k)/$ associated with the interval $/[f_k, f_{k+1}]/$ will approximately comprehend a complex amplitude of all modes contained in $/s'(t)/$ with frequencies between $/f_k/$ and $/f_{k+1}/$. The random quantities $/Z(f_{k+1})-Z(f_k)/$ contain all information about random phases and the random amplitudes as well of all modes from $/s'(t)/$. They can be determined by the random function $/s'(t)/$in a similar way as the FOURIER–coefficients $/a_k/$ from $/f(t)/$. The relations in:

4.09 $a_k=a^*_{-k}$

will find an analogy in:

4.10 $\langle Z(f'')-Z(f')=Z^*(-f')-Z^*(-f'')\rangle \rightarrow \langle[f'<f'']\rangle.$

The random amplitudes are not correlated among each other:

4.11 $$\{[Z^*(f_{k+1})-Z^*(f_k)]\cdot[Z(f_{j+1})-Z(f_j)]\}^{[T]}=\begin{cases}0 \rightarrow (j\neq k)\\ \tfrac{1}{2}E(f'_k)(f_{k+1}-f_k) \rightarrow (j=k)\end{cases}$$

similar to equation $/4.06/$. This way specifically it may be written:

4.12 $\tfrac{1}{2}E(f'_k)(f_{k+1}-f_k)=\{[Z^*(f_{k+1})-Z^*(f_k)]\cdot[Z(f_{k+1})-Z(f_k)]\}^{[T]}=\{[Z(f_{k+1})-Z(f_k)]^2\}^{[T]}.$

This is approximately the mean square of the amplitude–sum of all modes from $/s'(t)/$ whose frequencies are between $/f_k/$ and $/f_{k+1}/$; therefore it is a measure for energy of these modes. Equation $/4.07/$ in the sense of quadratic deviation will become more accurate the closer the $/f_k/$ will be:

4.13 $(\Omega\rightarrow\infty) \rightarrow (max\{f_{k+1}-f_k\}\rightarrow 0) \leftarrow (n\rightarrow\infty).$

For this limiting case it can be written:

4.14 $s'(t)={}_{-\infty}\int^{\infty}\langle exp\{2\pi i f t\}\rangle dZ(f)$

analogous to equation $/4.04/$. Equation /4.11/ will then result in:

4.15 $$\{dZ^*(f)dZ(f_1\}^{[T]}=\begin{cases}0 \rightarrow (f_1\neq f)\\ \tfrac{1}{2}E(f)df \rightarrow (f_1=f)\end{cases}$$

analogous to equation $/4.05/$. From relation $/4.10/$ one will obtain:

4.16 $dZ(f)=dZ^*(-f)$

analogous to equation $/4.09/$. From:

4.17 $$\left[\{dZ^*(f)dZ(f_1\}^{[T]}=\begin{cases}0 \rightarrow (f_1\neq f)\\ \tfrac{1}{2}E(f)df \rightarrow (f_1=f)\end{cases}\right] \wedge [dZ(f)=dZ^*(-f)]$$

it can be written:

4.18 $E(-f) = E(f)$.

Similar to the correlation function $/4.02/$ one finds $/E(f)/$ as an even function. It is the mean energy of all modes from $/s'(t)/$ whose frequencies are from interval $/[-f-df, -f]/$ and $/[f, f+df]/$:

4.19 $\frac{1}{2}E(-f)df + \frac{1}{2}E(f)df = E(f)df \quad \rightarrow \quad [f \geq 0]$.

According to equations $/4.12/$ and $/4.17/$ it obviously can be verified that:

4.20 $E(f) \geq 0$.

Thus $/E(f)/$ becomes the mean energy–density (energy per frequency, spectral–density) at $/f/$ on the frequency–axis when $/(v')^2/$ is measured as the energy of a process $/s'/$. From equations $/4.14/$ and $/4.17/$ can be deduced that the correlation–function $/4.02/$ and the spectral–density $/E(f)/$ are FOURIER–transforms of each other. FOURIER–transformations for $/s'/$ usually are reduced to cosine–transformations and in this case the WIENER–CHINTSCHIN–equations hold:

4.21 $\{(v')^2\}^{[T]}R(\tau) = {}_0\!\int^{\infty}[E(f)cos\{2\pi f\tau\}]df$

4.22 $E(f) = 4{}_0\!\int^{\infty}[\{(v')^2\}^{[T]}R(\tau)cos\{2\pi f\tau\}]d\tau$.

If it becomes $/\tau = 0/$ in equation $/4.22/$ due to $/R(0) = 1/$ a representation can be obtained for the total energy as the sum (integral) of all spectral parts $/E(f)df\,/$ in the form:

4.23 $\{(u')^2\}^{[T]} = {}_0\!\int^{\infty}E(f)df$

analogous to the expression in equation $/4.05/$. Correlations and spectral–densities combined via FOURIER–transformations are representations of the same phenomena.

5. *The Route from Order into Chaos in an Eddy's Split–Cascade.*

As soon as appropriate resonance–terms $/\gamma_j{\cdot}\langle q_1(t)\rangle_j/$ from disturbance–signal $/s(t)/$ will act on the followers $/(S^3)_{2\vee 3}/$and the followers of the followers ..., equivalent conditions are provided for every split of the cascade. These splits will equivalently be performed as the one for $/(S^3)_1/$. Thus the life–time of a sphere in N_th generation of the cascade will be diminished by:

5.1 $\Gamma((S^3)_N) \sim (\langle 0.79\rangle^N{\cdot}r_1)^2 \quad \rightarrow \quad [\Gamma((S^3)_{N+1})/\,\Gamma((S^3)_N)] \approx (0.79)^2 \approx 0.62,$

and the phase–speed will be nearly doubled:

5.2						
	$T_1 = 2{\cdot}\pi/\omega_1$	$\rightarrow$	$T_{2\vee 3}\,/\,T_1 \approx 2.22$	$\rightarrow$	$T_{N+1}/\,T_N \approx 2.22.$
	$T_{2\wedge 3} = 4{\cdot}\pi/\omega_{2\wedge 3}$ $T_{2\wedge 3} \approx 4{\cdot}\pi/(0.9{\cdot}\omega_1)$				

All this together will draw a picture about an eddy's decay as controlled route from order into chaos similar to many other dynamical systems described for instance by O–H. PLEITGEN, H. JÜRGENS, D. SAUPE [11].

6. *Conclusion.*

Due to the theory of L. F. RICHARDSON – successively enhanced by A. N. KOLMOGOROV – turbulence is characterized as production of eddies in a hierarchical order. Only the largest of them are created during initialization of the process, but all will successively decay by series of instabilities into followers on decreasing orders of magnitude. This process transports energy along a cascade nearly without dissipation. Only eddies on lowest hierarchical level will distinctively be influenced by viscous dissipation and finally be destroyed by transformation of their energy into heat.

According to this theory the energy–cascade will be started by disturbances on eddies of highest level which cause them to stretch in all directions. The initial stretching will continually create series of followers on lower hierarchal levels with decreasing portions of energy. This is the concept of cascading so far, but this picture lacks on details about decays, like numbers and sizes of followers coming into existence and individual life–times in any specific case, and on characteristics of appropriate disturbance–signals as well. This information however is needed in order to determine an appropriate cascade–structure, which then enables a statement about the proper development of the turbulence.

In order to enhance the former process with respect to these deficiencies, the model of an eddy in a fluid will be changed. The existing picture of an eddy is replaced by a spinning sphere whose surface is exposed to a self–organizing balance between an outer disturbance–signal and a reaction–force of the sphere due to its surface–tension (on account to the fluid's viscosity). An adiabatic approximation on the forces damping–parameters makes the disturbance to become the leading–force of the system while the reaction–force is enslaved and must follow the disturbance immediately. Due to these facts the behaviour of the self–organizing system at variations of the disturbance could be best described by the dynamics of an un–harmonic oscillator, which is characterized by two different stability– modes. Depending on the value of the disturbance damping–parameter oscillations with respect to a stable fix–point bifurcate into a mode where the former fix–point loses its stability and becomes replaced by two other symmetrically positioned stable fix– points. The bifurcation of the initial stability mode with one fix–point into another one with two fix–points has to be interpreted by a split of the initial sphere into two follower–spheres.

During this splitting–process disturbance transfers energy to the surface of the initial sphere – and thereby deforms it – up to an energy–levels equal to the surface–energies of two follower–spheres each with a half of the predecessor's volume. Thereby the split will save the initial spin and the follower–spins of equal lengths will sum–up for the predecessor's spin. The rotation–energy of the initial sphere will be equally partitioned among the followers which results in individual spin–lengths of about 60% and phase–velocities of about 111% relative to the proper values of the predecessor. Additionally the proper stochastic signal also will invest about 25% more tension–energy into the surfaces of the followers and rotation gets nearly doubled its phase–speed. Split–energy of a sphere (due to enlargement of its surface) and its life–time are supposed to be proportional to the square of sphere– radii.

A disturbance signal within the model's frame is appropriately be assumed as a list of resonance–terms (stochastic–signals), each suitable for a split of a proper sphere. Each term is product of a stochastic function with strong periodic components and an associated time–dependent damping–parameter; it vanishes by integration over an appropriate time–period (life–time of a proper sphere). Due to the letter quality it can be decomposed into a FOURIER–series with complex coefficients. Each coefficient will be derived on base of an individual set of frequency–modes. If the coefficients – which now principally contain all information about amplitudes and phases of their proper modes – are formulated in a suitable way, the FOURIER–series – which they belong to – will result in a function with an associated damping–parameter suitable as split–resonance for spheres of a actual cascade– generation.

Because each follower–generation on the course of the route will find its proper resonance–term for the splits, it can and will go through the same split–procedure with equivalent conditions as its predecessor did. This means, starting from an initial sphere, a series of subsequent follower–splits will occur. Each of them haves rotation–energy, nearly doubles phase–speed, shortens life–time by about a third and increases the tension–energy of surfaces by about a quarter for any follower relative to its predecessor. This way a picture about an eddy's decay can be drawn as a well structured route of energy from order into chaos, similar to those of many other dynamical systems too (please finally pay attention on [13]).

7. *References*.

[1] Richardson L. F., Atmospheric diffusion shown on a distance–neighbour graph, Proc. R. Soc. London A 110. 1926.

[2] Kolmogorov A. N., The local structure of turbulence in incompressible viscous fluids at very large Reynolds numbers, Dokl. Akad. Nauk. SSSR 30, 1941.

[3] Kolmogorov A. N., On the degeneration of isotropic turbulence in an incompressible viscous fluid, Dokl. Akad. Nauk. SSSR 31, 1941.

[4] Kolmogorov A. N., Dissipation of energy in isotropic turbulence, Dokl. Akad. Nauk. SSSR 32, 1941.

[5] Kolmogorov A. N., Equations of turbulent motion in an incompressible fluid, Izv. Akad. Nauk. SSSR ser. Fiz. 6, 1942.

[6] Kolmogorov A. N., On the breakage of drops in a turbulent flow, Dokl. Akad. Nauk. SSSR 662, 1949.

[7] Kolmogorov A. N., A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds of turbulence in a viscous incompressible fluid at high Reynolds numbers, J. Fluid Mech. 13, 1962.

[8] Fiedler H. E., Turbulente Strömungen, Tech. Univ. Berlin, Hermann Föttinger Inst. f. Strömungsmechanik & Tech. Univ. Braunschweig Inst. f. Strömungsmechanik, Vorlesungsskript, 2003.

[9] Zeglin S., Statistische Eigenschaften zweidimensionaler Turbulenz, Westf. Wilhelms–Univ. Münster, Inst. f. theor. Physik, Diplomarbeit, 2005.

[10] Haken, H., Synergetics, An Introduction, Springer, 1983.

[11] Pleitgen O–H., Jürgens H., Saupe D., Chaos and Fractals, New Frontiers of Science, Springer, 1982.

[12] Steinemann U. E., Justification for Sphere with Surface–Tension as Eddy–Model in a Turbulent Fluid, 2019, http://viXra.com/abs/1908.0127.

[13] Steinemann U. E., Final–State–Diagram of Quadratic–Iterator topologically equivalent With an Eddy's Decay–Cascade in turbulent Fluids. 2018, http://viXra.com/abs/1809.0202.