From Bernoulli to Laplace and Beyond

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Abstract. Reviewing Laplace's equation of gravitation from the perspective of D. Bernoulli, known as Poisson-equation, it will be shown that Laplace's equation tacitly assumes the temperature T of the mass system to be approximately $0^\circ K$. For temperatures greater zero, the gravitational field will have to be given an additive correctional field. Now, temperature is intimately related to the heat, and heat is known to be radiated as an electromagnetic field. It is shown to take two things in order to get at the gravitational field in the low temperature limit: the total square energy density of the source in space-time and a (massless) field, which defines interaction as quadratic, Lorentz-invariant, and $U(4)$-symmetric form, that restates the equivalence of inert and gravitational energy/mass in terms of absolute squares. This field not only necessarily must include electromagnetic interaction, it also will be seen to behave like it.

1. Problem Statement

A system of N particles in spacetime in Newtonian mechanics is a system that is to be defined by $3N$ location coordinates q_k as well as a common time coordinate and their associated $3N$ momentum coodinates p_k as a function of time. Mostly these systems are stably confined to a fixed region in space over time like a drop of water or a stone. So, there will be many equations of confinement, and to simplify the mathematical model, Bernoulli changed that model by replacing the particles' position with a spatial mass density $\rho(t) : \mathbb{R}^3 \ni \vec{x} \mapsto \rho(\vec{x}(t)) \geq 0$. Laplace then took over that model and showed that the gravitational force of a mass density ρ could be expressed as Poisson equation $\Delta\Phi = 4\pi G\rho$ of a potential function Φ, the gravitational field and the gravitational constant G, $\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$ being the Laplace operator. That marked the introduction of field as a concept into physics. What made it both bold and dubious, was that it said that the field was to be the sheer equivalent of the mass distribution. It was soon found out that the field was to be an harmonic function of the space coordinates, which led to the famous Laplace demon problem, and another problem then showed to be the lack

of Lorentz covariance, giving evidence that the Laplace field of gravitation cannot be correct.
However, there is much more to it:

Both, Bernoulli and Laplace took it as evident that a (smooth) mass distribution $\rho(x)$ of N particles, which is confined to a bounded region $K \in \mathbb{R}^3$ (for all times t), could be resolved at each given time t into N disjoint bounded regions $K_1, \ldots, K_N$, containing a unique particle, if only the particles would stay apart from eachother. With that, it should be possible to replace ρ with the sum $\sum_k \rho_k$ of smooth, non-negative functions ρ_k of disjoint support and compact support, each (which means, they all vanish outside a bounded set, e.g. K, and if one is greater zero at some point x, then all the others must vanish at this point x). If so, the above Poisson equation could be rewritten as a sum $\sum_k \Delta\Phi_k = \sum_k 4\pi G\rho_k$ of N independent gravitational equations for each and every particle.
And indeed, mathematics proved this to be possible, now known as the partion of unity (see e.g. [1, Ch.16]). That, on one side, means that even if all particles are pointwise in nature, we can approximate these particles through Bernoulli's ingenious replacement of mass position by smooth mass densities. On the downside, that shows that Laplace's theory of gravitation must lack generality, because in it, all the particles of a body are independent from eachother: they just add up individually!
And this is incorrect, because it totally disregards the body's kinetic energy:

The mass m of a body B at rest is to be defined to be equal to the total energy of B. Now, if B was simply the sum of N individual oscillating particles, then the total energy E is to be the square root of $\sum_{1\leq k\leq N} m_k^2c^4 + (cm_kv_k)^2$, where c is the speed of light, m_k are the individual masses, and the v_k are the mean speeds of these masses, so that kinetic energy, a.k.a. "temperature", always will add to the the total mass of B!

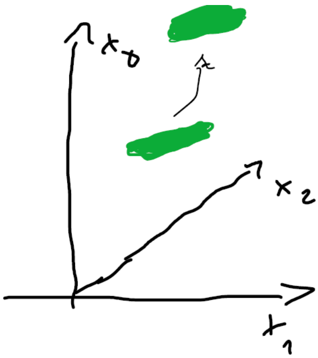
At the same time, this shows, that Bernoulli's notion of expressing the masses in terms of space-time densities $j(t,\vec{x}) = (\rho_0(t,\vec{x}), \rho_0\vec{v}(t,\vec{x}))$ is inappropriate: Instead, j is to become necessarily the 4-vector of the square root density of energy and momentum of the composed system, such that

$$< j, j >:= \|j\| := \int_{\mathbb{R}^4} \left| j_0^2(x) + \cdots + j_3^2(x) \right| d^4x$$

equates (locally) to the square of energy, which then becomes the square of the total energy of B, i.e. up to c^2 is equal to the square of the inert mass m of B. (So, j can be conceived as the macroscopically composed superposition of local quantum states, which approximates the system's particles.)

Could we leave out an integration over time t? - Not at all: Because, given such a 4-vector $j = j(t,\vec{x})$ that extends over a compact space region $K \in \mathbb{R}^3$ of radius $r > 0$, we loose control over the eigentime of the particles within that region. All we know is that the particles' eigentimes must be within the interval of our observing eigentime t_0, plus or minus the bounds $\Delta t = \pm cr$!

In all, the appropriate model for discussing gravity of particle systems is that of time curves $\Omega : \mathbb{R} \ni t \mapsto \Omega(t) := j_t := (j_{0,t}, \ldots, j_{3,t})$, where the $j_{\mu,t}$ are to be smooth functions with compact support in space-time $\mathbb{R}^4$ for each μ and t, such that their absolute squares, $\left|j_{\mu,t}\right|^2$, are the intensities of smooth, local energy-momentum packages of the particles in space and time, as sketched below:



Having $\Omega : t \mapsto j_t \in L^2(\mathbb{R}^4)^4$ in place, we can state:

Proposition 1.1. *The total energy square of a system $\Omega : t \mapsto j_t$ at time t_0, which is at rest at t_0 is given by $E^2 =< j_{t_0}, j_{t_0} >= \sum_\mu \int_{\mathbb{R}^4} \left|j_{\mu,t_0}(x)\right|^2 d^4x$.*

2. Deriving gravity

It now shows up that there is nothing else than this notion of Ω needed to discuss gravity:
If instead of inert masses m_k, the system was made of electric charges, or even hadronic baryons, or whatever could be idealistically thought of to result in massy particles, the energy-momentum distribution is already put as a quadruple j_t of complex-valued states, the absolute squares being their intensities. (We'll shortly see, why this is the case, but for the moment you might look that up from any standard text on quantum field theory.)

So, whatever there might be in a bounded box $B \subset \mathbb{R}^4$ as observed from an external system at some time t_0 assumed to be at rest, $E^2 =< j_{t_0}, j_{t_0} >$ turns out to be c^4 times of the square of its (inert) rest mass!
With this, we then deduce by equivalence principle, that this inert square of mass must be proportional to the square of gravitational mass, and to get at the corresponding gravitational field, we just need to compare with the covariant Maxwell equations, which readily rewrites into:

$$< j_{t_0}, \Box A >= Const < j_{t_0}, j_{t_0} >= Const\ E^2, \tag{2.1}$$

where $\Box := \partial_0^2 - \cdots - \partial_3^2$ is the wave operator, A the electromagnetic 4-vector field, and $Const$ a constant, which in Gaussian units is identically 1 along with c.

Let's now choose that constant differently, to be $Const = -4\pi G$, where G is the positive gravitational constant, such that

$$\sum_{\mu} < j_\mu(x), \Box A_\mu(x) >= -4\pi G E^2. \tag{2.2}$$

Equation 2.2 then states nothing but the equivalence principle: It says that $\Omega : t \mapsto j_t$ has included into the j_t a gravitational interaction potential, which, when squared and summed up, is to be proportional to E^2 and is contracting (due to negative sign of $-4\pi G$).

Theorem 2.1 (U(4)-Invariance). *We now are in the position to explain, why $\Omega : t \mapsto j_t$ suffices to describe gravitational interaction:*
Because equation 2.2 becomes $U(4)$-invariant, with $U(4)$ being the group of unitary 4×4-matrices, just by letting the bra vector $< j_t|$ be the complex adjoint of its ket vector $|j_t >$. (This is also how we get at the non-negative square $E^2 =< j_t, j_t >$.) And, as is basic group theory knowledge, $U(4)$ is reducible and decomposes into a product of subgroups $U(4) = U(2) \times U(2) \times SU(3)$, where in turn $U(2) = U(1) \times SU(2)$ is the product of the phase symmetry group $U(1)$ and the spin group $SU(2)$.

And the fact that the current standard model is a gauge theory based on the symmetry group $U(1) \times SU(2) \times SU(3)$, makes that theory embedded part of the gravity equation 2.2, assigning a well-defined mass to all of the particles of that standard model: The mass of the body is to be defined by squaring and adding up the absolute values of square energy of all of its constituents!

Let's harvest harvest the direct consequences:

3. Gravitational Interaction

An immediate implication of the theorem is:

Corollary 3.1 (Phase Symmetry). *The 4-vector streams $\Omega : t \mapsto j_t$ and the 4-vector potential are $U(1)$-invariant, i.e. phase invariant. In particular, any space-like vector $\Omega : t \mapsto j_t$ is equivalent to its time-like counterpart $i\Omega : t \mapsto ij_t$. Similarly, U(4)-symmetry allows to smoothly rotate elements contained within the forward light cone into ones within the backward light cone, and vice versa. In other words, it would to be an error to restrict consideration of energy-momentum of the dynamic system to the positive-energetic time-cone, only. Instead, we have to symetrically deal with the full set of space-time elements of $\mathbb{R}^4$ outside the light cone $\Gamma := \{(t, \vec{x}) \in \mathbb{R}^4 : t^2 - \vec{x}^2 = 0\}$.*

Now, for $\mu = 0, \ldots, 3$ and $j_{\mu,t}$, which I recall is a smooth function of compact support in space-time $\mathbb{R}^4$, let $\mathcal{F} j_{\mu,t}(\chi) = \int_{\mathbb{R}^4} \frac{1}{(2\pi)^2} e^{-i\chi \cdot x} j_{\mu,t}(x) d^4x$ be the Fourier transform of $j_{\mu,t}$, which exists as a well-defined analytic function, and is inverible by its inverse $\mathcal{F}^{-1}$ to $j_{\mu,t}$ again, so from equation 2.2 we deduce

$$\mathcal{F}A(\chi) = (-4\pi G) \frac{1}{\chi_0^2 - \cdots - \chi_3^3} \mathcal{F} j_t(\chi),$$

that is: $j_t \mapsto A$ is the linear mapping $S^2 j_t$ with S^2 being the Fourier transformation of the multiplication operator $\hat{S^2} := \frac{1}{\chi_0^2 - \cdots \chi_3^3}$.
So, $S^2 j_t := (-4\pi G)A$ is well-defined for each $\Omega :\mapsto j_t$, and therefore $S := (\sum_\mu \gamma_\mu \partial_\mu)(-4\pi G)^{\frac{1}{2}} S^2$ is well-defined for each $\Omega : t \mapsto j_t$, where the γ_μ are the 4×4-Dirac matrices, plus we get that S^2 becomes the square of S.

For the purpose of simplicity, let's drop the external time index from j_t. Again, for each μ, the mapping $\Theta_\mu : j_\mu \mapsto A_\mu$ defines a linear mapping from $j_\mu \in \mathcal{C}_c^\infty(\mathbb{R}^4)$ to a functional which is defined "outside the support $supp(j_\mu)$ of j_μ":
For $x, y \in \mathbb{R}^4$ let $d(x-y) := (x-y)_\mu (x-x)^\mu \in \mathbb{R}$ be the Minkowksi distance of x and y, and with $j_\mu \in \mathcal{C}_c^\infty(\mathbb{R}^4)$ and $x \in \mathbb{R}^4$ let

$$p(x, supp(j)) := \min_{0 \le \mu \le 3} \inf_{y \in supp(j_\mu)} \left| d(x-y) \right| \in [0, \infty),$$

which defines a seminorm on $\mathbb{R}^4$. With it, given $j = (j_0, \ldots, j_3)$ as above, let $\Xi(j) := \{x \in \mathbb{R}^4 \mid p(x, supp(j)) > 0\}$, which is open in $\mathbb{R}^4$. Then $\Theta = (\Theta_0, \ldots, \Theta_3)$ maps j to a quadrupel of functionals on $\mathcal{C}_c^\infty(\Xi(j))$ (as shown subsequently).

Let's define the functional spaces above and see what the seemingly undefined term $< j, A >=< j, S^2 j >$ gives in terms of distributions:

Let $K \subset \mathbb{R}^4$ be the (compact) closure of a non-empty, open, and bounded subset $K^o \subset \mathbb{R}^4$, and let $\Xi(K)$ as above be the set of all $x \in \mathbb{R}^4$ with $p(x, K) > 0$, which is an open, non-empty subset of $\mathbb{R}^4$. $\Xi(K)$ itself is the union of a sequence $X_1, X_2, \ldots$ of compact regions of $\mathbb{R}^4$, which as K are the closures of nontrivial, open sets $X_l^o \subset \mathbb{R}^4$. Given such a compact region X, the set of all infinitely differentiable (complex-valued) functions with support in X is a vector space $\mathcal{C}_c^\infty(X)$, which becomes a complete locally convex, separable space, when equipping it with the sequence of supremum norms for all its n-th order partial derivatives (where $n \ge 0$ is understood), see e.g. [1]. Then the space $\mathcal{C}_c^\infty(X)^4 = \mathcal{C}_c^\infty(X) \oplus \cdots \oplus \mathcal{C}_c^\infty(X)$ of quadruples $(j_1, \ldots, j_4)$ is a (separable, complete) locally convex space, and so is its dual, $\mathcal{C}_c'^\infty(X)^4$, the space of continuous linear functionals on $\mathcal{C}_c^\infty(X)^4$ (see again: [1]). This then defines $\mathcal{C}_c'^\infty(\Xi(K))^4$ as the union $\bigcup_{l \in \mathbb{N}} \mathcal{C}_c'^\infty(X_l)^4$, giving it the finest locally convex topology, for which the embeddings $\iota : \mathcal{C}_c'^\infty(X_l)^4 \to \mathcal{C}_c'^\infty(\Xi(K))^4$ are continuous, which is called LF-space (see again: [1, Ch.13]).

Proposition 3.2. *S and S^2 are well-defined as linear mappings on $\mathcal{C}_c^\infty(K)^4$ into $\mathcal{C}_c'^\infty(\Xi(K))^4$, and $< j, Sj >=< j, S^2 j >= 0$ holds for each $j \in \mathcal{C}_c^\infty(K)^4$.*

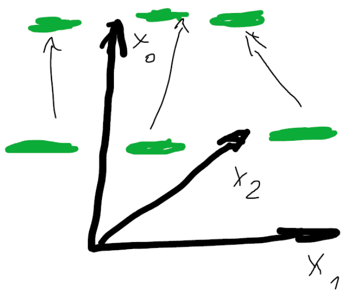
Proof. Without loss of generality, let's assume $-4\pi G \equiv 1$. Let $\delta : \mathcal{C}(\mathbb{R}^4) \ni f \mapsto f(0) \in \mathbb{C}$ be the Dirac-distribution (in 4 dimensions). Then $\Box f = \delta$ is solved by $f(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{ix \cdot \xi} \frac{-1}{\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2} d^4\xi$, so for $x \in \Xi(K)$ and $j \in \mathcal{C}_c^\infty(K)^4$,

$$S^2 j(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} e^{(ix-y) \cdot \xi} \frac{-1}{(x_0 - y_0)^2 - \cdots - (x_3 - y_3)^2} j(y) d^4y d^4\xi$$

is a well-defined complex functional on $\mathcal{C}_c^\infty(\Xi(K))^4$, since for $g \in \mathcal{C}_c^\infty(\Xi(K))^4$ $g(x) \cdot \int f(x-y)j(y)d^4y$ is integrable in x, due to $\inf_{x \in supp(g)} p(x,K) > 0$. And, since j is infinitely differentiable, $S^2 j$ is infinitely differentable on $\Omega(K)$. (Because the 4 components j_k of j satisfy $\int |j_k| \, d^4y \leq Vol(K) \sup_{y \in K} |j_k(y)|$, S^2 even defines a continuous mapping from $\mathcal{C}_c^\infty(K)^4$ into $\mathcal{C}_c'^\infty(\Xi(K))^4$.)
Along with S^2, all its partial derivatives are well-defined too.
Hence, $S = (\sum_{0 \leq \mu \leq 3} \gamma_\mu \partial_\mu) S^2$ is a well-defined mapping from $\mathcal{C}_c^\infty(K)^4$ to $\mathcal{C}_c'^\infty(\Xi(K))^4$.
Lastly, $< j, S^2 j >=< j, Sj >= 0$ follows from the fact that every $j_\mu \in \mathcal{C}_c^\infty(K)$ is equal to zero outside of K, so in particular vanishes on $\Xi(K)$. □

Remark 3.3. Physically, what the proposition tells, is that the field does not interact with its own source.

With it, let $\Omega : t \mapsto j_t = \sum_{1 \leq k \leq N} j_k(t)$ be the sum of N time-curves of smooth vector functions $t \mapsto j_1(t), \ldots, j_N(t) \in \mathcal{C}_c^\infty(\mathbb{R}^4)^4$ of disjoint support and of compact support at each instance of time as illustrated below:



That's what an external observer would e.g. see, as he looks at our solar system: at each time $t = x_0$, he sees planets and sun as chunks of energy-momentum distributions spatially staying apart of eachother. Dropping the external parameter t t again, equation 2.2 holds for the sum of energy momentum distributions $j = \sum_k j_k$, and as such it includes the interaction between all the N chunks j_k (at "retarded" times: note however, that the composed system is distributed over space-time and the observer has no information on what particle point comes first). If instead the N chunks were independently moving from eachother, we would see different distributions of energy-momentum $j_{free,1}, \ldots, j_{free,n}$, each moving in a straight line. What we want is an interaction defining field $V(j_{free,1}, \ldots, j_{free,N})$, which captures that interaction, i.e. such that:

$$< \sum_{1 \leq k \leq N} j_k, \sum_{1 \leq k \leq N} j_k >=< \sum_k j_{free,k}, \sum_k j_{free,k} > + V(j_{free,1}, \ldots, j_{free,N}).$$

This in mind, let's put $j_k = j_{free,k} + iSj_{free,k}$. Then, letting S^* be the adjoint of S,

$$\begin{aligned}< \sum_k j_k, \sum_k j_k >=< \sum_k j_{free,k}, \sum_k j_{free,k} > + \sum_k < S^* j_{free,k}, Sj_{free,k} > \\ + \sum_{1\le k,l \le N} < S^* j_{free,k}, Sj_{free,l} > \\ =< \sum_k j_{free,k}, \sum_k j_{free,k} > + \sum_k < j_{free,k}, S^2 j_{free,k} > \\ + 2 \sum_{1\le k<l\le N} Re < j_{free,k}, S^2 j_{free,l} >,\end{aligned}$$

where the mixed, imaginary products of $j_{free,k}$ and $iSj_{free,l}$ cancel out due to complex conjugation. By the proposition above, $\sum_k < j_{free,k}, S^2 j_{free,k} >= 0$, so we get that the square E^2 of the total energy of the interacting N-part system equals the square E^2_{free} of the total energy of the N non-interacting systems plus a sum V of mixed, real-valued terms $Re < j_{free,k}, S^2 j_{free,l} >$, $(k < l)$, so for $|E_{free}| \gg |V|$:

$$\begin{aligned}|E| = +|E_{free}| \sqrt{1 + \frac{V}{E_{free}}} \approx |E_{free}| + \frac{V}{2|E_{free}|} = \\ = |E_{free}| + \frac{1}{|E_{free}|} \sum_{1\le k<l\le N} Re < j_{free,k}, S^2 j_{free,l} > .\end{aligned}$$

For $N = 2$ and restricting to real-valued j_1, j_2, we have

$$\begin{aligned}\frac{1}{|E_{free}|} Re < j_{free,1}, S^2 j_{free,2} > \\ = \frac{1}{|E_{free}|} \int j_1(x) \cdot (S^2 j_2)(x) d^4x = \int j_1(x) \Phi(x) d^4x\end{aligned}$$

with $\Phi(x) := \frac{1}{|E_{free}|} S^2 j_{free,2}(x)$, which then is the vector field of gravitational interaction, of which the first component converges to the classical gravitational field as $c \to \infty$, and the other components converge to zero.

Remark 3.4. Note the slight, but important shift of dimensioning of the potential field Φ: Φ is no longer the field of an external source, which is taken by unit mass (or charge) through division by the source mass/charge, but the division is by the total energy of the (free) composed system instead, and this factor is retrieved not by definition, but by calculating the square root. Also note that the gravitational potential Φ now is the additive correction to the sum of free rest masses of its two interacting parts, these free rest masses are the dominating energetic contents of the overall interacting system. Finally, although we extracted Φ from that system, it is still integral part of $\Omega : t \mapsto j_t$ for temperature limit $T \to 0$.

Doing the same for $N > 2$, would lead to a vector field Φ which depends on N space-time quadruples $x_1, \ldots, x_N \in \mathbb{R}^4$, however. And by refining the partitioning, letting the number N steadily increase, one ends up with Feynman path integration.

References

[1] François Trèves, *Topological Vector Spaces, Distributions, And Kernels*, Academic Press, 1967.

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